ERGODIC PROPERTIES OF RANDOM PIECEWISE CONVEX MAPS

HISAYOSHI TOYOKAWA

Faculty of Engineering, Kitami Institute of Technology
Kitami, Hokkaido, 090-8507, Japan
e-mail address: `h_toyokawa@mail.kitami-it.ac.jp`

Abstract. In this note, we briefly sketch our current state on the investigation, for random piecewise convex maps, towards ergodic properties: the existence of σ-finite invariant measures absolutely continuous with respect to the Lebesgue measure, their ergodicity, and further limit theorems.

1. Introduction

Let $X = [0,1]$, $\mathcal{B}$ be the Borel σ-algebra and λ be the Lebesgue probability measure on $(X, \mathcal{B})$. A piecewise convex map in the sense of Lasota–Yorke [5] is a map $\tau : X \to X$ such that there is a finite partition $0 = a_0 < a_1 < \cdots < a_N = 1$ satisfying

(1) for each $k = 1, \ldots, N$ the restriction $\tau_k \coloneqq \tau|_{[a_{k-1}, a_k)}$ is continuous and convex in the sense that $\tau_k(tx + (1-t)y) \leq t\tau_k(x) + (1-t)\tau_k(y)$ for any $x, y \in [a_{k-1}, a_k)$ and $0 \leq t \leq 1$;
(2) $\tau(a_{k-1}) = 0$ for each $k = 1, \ldots, N$;
(3) $\tau'(0) > 1$ and $\tau'(a_{k-1}) > 0$ for each $k = 2, \ldots, N$.

Lasota and Yorke showed in [5] the existence of a unique λ-absolutely continuous invariant probability measure μ for τ, where μ is called invariant for τ if $\mu(\tau^{-1}A) = \mu(A)$ holds for any $A \in \mathcal{B}$. Moreover, they showed that the density of μ is bounded and non-increasing, and that μ is exact: $\bigcap_{n \geq 0} \tau^{-n}\mathcal{B} = \{\emptyset, X\} \pmod{\mu}$, which implies asymptotic stability of the system.

In the piecewise C^1 setting, Inoue relaxed their condition (3) to that τ can admit an indifferent fixed point at 0 (i.e., $\tau'(0) = 1$) and critical points (i.e., $\tau'(a_{k-1}) = 0$ for some $k = 2, \ldots, N$). He proved that a unique λ-absolutely continuous exact invariant probability measure is still valid under certain assumption (see [2]); otherwise any σ-finite, λ-absolutely continuous ergodic invariant measure is an infinite measure and such an invariant measure exists (see [3]), which is unique up to multiplicative constants. Here, a measure μ is called ergodic if any $E \in \mathcal{B}$ with $\tau^{-1}E = E \pmod{\mu}$ satisfies either $E = \emptyset$ or $X \pmod{\mu}$.

Motivated by their previous researches, we consider random piecewise convex maps as follows. We introduce two probability spaces $(\mathbb{A}, \nu_{\mathbb{A}})$ and $(\mathbb{B}, \nu_{\mathbb{B}})$ as parameter spaces. For each $\alpha \in \mathbb{A}$ and $\beta \in \mathbb{B}$, a map $T_{\alpha,\beta}$ is assigned which is of the form

$$T_{\alpha,\beta}x = \begin{cases} \tau_\alpha x & (x \in [0, \frac{1}{2}]), \\ S_\beta x & ((\frac{1}{2}, 1]) \end{cases}$$

where $\tau_\alpha : [0, \frac{1}{2}] \to X$ and $S_\beta : (\frac{1}{2}, 1] \to X$ are injective and continuous maps with the following conditions:

(0) The map $\mathbb{A} \times \mathbb{B} \times X \ni (\alpha, \beta, x) \mapsto T_{\alpha,\beta}x \in X$ is measurable with respect to each variables;

for $\nu_{\mathbb{A}}$-almost every $\alpha \in \mathbb{A}$ and $\nu_{\mathbb{B}}$-almost every $\beta \in \mathbb{B}$,

(1) τ_α and S_β are C^1-functions and S_β can be extended to a continuous function on $[\frac{1}{2}, 1]$ with $\tau_\alpha(0) = 0$, $\tau_\alpha(\frac{1}{2}) = 1$ and $S_\beta(\frac{1}{2}) = 0$;

(2) τ'_α and S'_β are non-decreasing with $\tau'_\alpha(0) \geq 1$, $\tau'_\alpha(x) > 1$ for $x \in (0, \frac{1}{2})$, $S'_\beta(\frac{1}{2}) \geq 0$ and $S'_\beta(x) > 0$ for $x \in (\frac{1}{2}, 1)$.

A *random piecewise convex map* $\{T_{\alpha,\beta}; \nu_{\mathbb{A}}, \nu_{\mathbb{B}} : \alpha \in \mathbb{A}, \beta \in \mathbb{B}\}$ is given by the transition probability

$$\mathbb{P}(x, A) = \int_{\mathbb{A}\times\mathbb{B}} 1_A(T_{\alpha,\beta}x)\, d\nu_{\mathbb{A}}(\alpha) d\nu_{\mathbb{B}}(\beta)$$

for each $x \in X$ and $A \in \mathcal{B}$, where $T_{\alpha,\beta}$ satisfies the above conditions.

2. Main results

Recall that for a transition probability $\mathbb{P}(x, A)$, a measure μ is called *invariant* if $\int_X \mathbb{P}(x, A) d\mu(x) = \mu(A)$ for any $A \in \mathcal{B}$. Moreover, μ is called *ergodic* if any set $E \in \mathcal{B}$ with $\mathbb{P}(x, E) = 1_E(x)$ μ-almost every $x \in X$ satisfies either $E = \emptyset$ or $X \pmod{\mu}$.

Theorem 2.1 ([4])**.** *For a random piecewise convex map* $\{T_{\alpha,\beta}; \nu_{\mathbb{A}}, \nu_{\mathbb{B}} : \alpha \in \mathbb{A}, \beta \in \mathbb{B}\}$*, if*

$$\int_{\mathbb{A}} \frac{1}{\frac{1}{2} - \tau_\alpha^{-1}(\frac{1}{2})} d\nu_{\mathbb{A}}(\alpha) < \infty,$$

then there exists a unique (up to multiple constants) λ-equivalent, σ-finite invariant measure which is ergodic such that

(Decreasing) the density function $\frac{d\mu}{d\lambda}$ is non-increasing;

(Upper bound) for any $\varepsilon > 0$, there is $C > 0$ such that $\frac{d\mu}{d\lambda} \leq C$ on $[\varepsilon, 1]$.

Moreover, if we assume $\operatorname{ess\,sup}_{\alpha\in\mathbb{A}} \tau'_\alpha(\frac{1}{2}) < \infty$*, then we have*

(Lower bound) there is $c > 0$ such that $\frac{d\mu}{d\lambda} \geq c$.

Remark 2.1. In [4], we prove that the invariant measure μ in Theorem 2.1 is indeed conservative. Moreover, the asymptotic size of μ close to 0 is also established so that we can tell whether μ is finite or not.

In what follows, we will see two examples of our theorem. The first example is random intermittent maps with a critical point, which is a modified version of random LSV (Liverani–Saussol–Vaienti) maps. The second one is random piecewise linear maps which have no indifferent (on average) fixed point but exhibit critical-type intermittency (cf. [1]). We can establish for this further statistical properties. For the precise information and motivation, see [7,8] and references therein.

2.1. **Random intermittent maps with a critical point.** Let $\mathbb{A} = [\alpha_1, \alpha_2]$ for some $0 < \alpha_1 \le \alpha_2$, $\mathbb{B} \subset (1, \infty)$ be a compact set, and $\nu_\mathbb{A}$ and $\nu_\mathbb{B}$ be probability measures. For each $\alpha \in \mathbb{A}$ and $\beta \in \mathbb{B}$, set $\tau_\alpha = x + 2^\alpha x^{1+\alpha}$ and $S_\beta = (2x-1)^\beta$ and consider a random piecewise convex map $\{T_{\alpha,\beta}; \nu_\mathbb{A}, \nu_\mathbb{B} : \alpha \in \mathbb{A}, \beta \in \mathbb{B}\}$. Note that 0 is an indifferent fixed point for each $\alpha \in \mathbb{A}$ and $\frac{1}{2}$ is a critical point for each $\beta \in \mathbb{B}$. Then by Theorem 2.1 there always exits a λ-equivalent, ergodic, σ-finite invariant measure μ. Furthermore, $\mu(X) = \infty$ if $\nu_\mathbb{B}\{\beta \in \mathbb{B} : \alpha_1\beta \ge 1\} > 0$ and $\mu(X) < \infty$ if $\alpha_1 \cdot \max_\mathbb{B} \beta < 1$. Roughly speaking, this means that the most expanding branch τ_α on the left and the most contracting branch S_β on the right determine the statistical law of the system. We remark that when $\mathbb{B} = \{1\}$ and $S_\beta = 2x - 1$, that is, for random LSV maps, it holds that $\mu(X) = \infty$ if and only if $\alpha_1 \ge 1$.

2.2. **Random piecewise linear maps with low slopes.** Let $\mathbb{B} = \mathbb{N}$ and $p_\beta = \nu_\mathbb{B}(\{\beta\})$ be a point measure on $\mathbb{B}$. We set $\tau_\alpha = 2x$ (independent of the choice of $\alpha \in \mathbb{A}$) and $S_\beta x = \frac{2x-1}{2^\beta}$ and consider a random piecewise convex map $\{T_\beta; p_\beta : \beta \in \mathbb{N}\}$. Note that this system has no indifferent (even on average) fixed point nor critical point, but it does have strong contracting property (as $\beta \to \infty$) on average depending on p_β. As in Theorem 2.1, the invariant measure μ always exists for any p_β. If we choose $p_\beta = 2^{-\beta}$ for $\beta \in \mathbb{N}$ then $\mu(X) < \infty$, since the probability of the choice of low slopes (i.e., large β) is exponentially small. However, when we set $p_\beta = C\beta^{-t}$ for $\beta \in \mathbb{N}$ where $t > 1$ and C is a normalizing constant, one can see that $\mu(X) = \infty$ if and only if $t \in (1, 2]$. Since we are interested in statistical properties of infinite measure preserving (random) systems, hereafter we assume $t \in (1, 2]$ and set $s := t - 1 \in (0, 1]$. Since each map T_β is piecewise linear, we can obtain exact formula of the density function of μ and consequently, we get

$$\mu\left(\left(2^{-(n+1)}, 2^{-n}\right]\right) \propto n^{-s}. \tag{2.1}$$

Then, by the equation (2.1), one can calculate the wandering rate and the asymptotic entrance density for the reference set $[\frac{1}{2}, 1]$ (see [6] for the definitions) corresponding to the skew-product transformation associated with the random map. That is, one can apply Thaler–Zweimüller's distributional limit theorem in [6] to the system. Let $\{T^{(n)}\}_n$ be a random dynamics given by $T^{(n)} = f_n \circ f_{n-1} \circ \cdots \circ f_1$, where $\{f_n\}_n$ denotes an i.i.d. sequence of random maps such that $f_n = T_\beta$ with probability p_β for any $n \ge 1$. Then one gets the Darling–Kac law as follows. Below $\xrightarrow[N\to\infty]{\mathrm{d}}$ means the convergence in distribution and $\mathcal{M}_s$ denotes the normalized Mittag-Leffler distribution of order s, which is characterized by $\mathbb{E}[e^{z\mathcal{M}_s}] = \sum_{k=0}^\infty \frac{\Gamma(1+s)^k z^k}{\Gamma(1+ks)}$.

Theorem 2.2 ([8]). *Fix $s \in (0, 1)$. For any random variable Θ with values in $[\frac{1}{2}, 1]$, which is independent of the random map $\{f_n\}_n$ and whose distribution is absolutely continuous with respect to*

λ and for any $E \subset [\frac{1}{2}, 1]$, it holds that

$$\frac{1}{N^s} \sum_{n=0}^{N-1} 1_{\{T^{(n)}(\Theta) \in E\}} \xrightarrow[N\to\infty]{\mathrm{d}} C_s \mu(E) \mathcal{M}_s,$$

where C_s is a positive constant depending only on s.

Remark 2.2. (1) Since the wandering rate and the asymptotic entrance density are estimated in the proof of the above theorem, one can also establish the Dynkin–Lamperti arcsine law for waiting times in [6] for this system.

(2) Even when $s = 1$, the Darling–Kac law and the Dynkin–Lamperti arcsine law above are still valid under a small modification.

(3) For the case when $\mu(X) < \infty$ (i.e., $t > 2$), it seems not hard to see polynomial decay of (annealed) correlation for appropriate observables. We will investigate them (and more general random piecewise convex maps as in §2.1).

References

[1] N. Abbasi, M. Gharaei and A. J. Homburg, Iterated function systems of logistic maps: synchronization and intermittency, Nonlinearity, **31** (2018): 3880–3913.

[2] T. Inoue, Asymptotic stability of densities for piecewise convex maps, *Ann. Polon. Math.*, **57** (1992), 83–90.

[3] T. Inoue, Weakly attracting repellors for piecewise convex maps, *Japan J. Indust. Appl. Math.*, **9** (1992), 413–430.

[4] T. Inoue and H. Toyokawa, Invariant measures for random piecewise convex maps, *Nonlinearity*, **37** (2024), 055016.

[5] A. Lasota and J. A. Yorke, Exact dynamical systems and the Frobenius-Perron operator, *Trans. Amer. Math. Soc.*, **273** (1982), 375–384.

[6] M. Thaler and R. Zweimüller, Distributional limit theorems in infinite ergodic theory, *Probab. Theory Related Fields*, **135** (2006): 15–52.

[7] H. Toyokawa, On the existence of a σ-finite acim for a random iteration of intermittent Markov maps with uniformly contractive part, *Stoch. and Dyn.*, **21**, (2021), 14 pages.

[8] H. Toyokawa, Conservative and ergodic σ-finite invariant measures for Markov operators and Darling–Kac law, *Hokkaido Math. J.*, accepted.

[9] 豊川 永喜, 井上友喜, 区分的に凸なランダム写像の不変測度について, 数理解析研究所講究録 **2217** (2022), 127–136.