# Isolated eigenvalues of the Perron-Frobenius operators for random beta-maps

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## 1 Introduction

Let $\beta > 0$. The beta-map $\tau_\beta : [0,1] \to [0,1]$ is defined by

$$\tau_\beta(x) = \beta x - [\beta x]$$

for $x \in [0,1]$, where $[y]$ denotes the integer part of $y \in \mathbb{R}$. For $\beta > 1$ the beta-map is known as a map generating expansions of real numbers with the base $\beta$ and has been investigated after [12] at the intersection of ergodic theory and number theory (e.g., [5], [11]). In this paper, we consider an i.i.d. random dynamical system generated by beta-maps and investigate isolated eigenvalues of its sample-averaged (or annealed) Perron-Frobenius operator. As a main result, we give a formula for a certain analytic function, each of whose zero is the inverse of an isolated eigenvalue of the sample-averaged Perron-Frobenius operator. This formula is a natural generalization of deterministic cases given in [4] and [8].

## 2 Deterministic beta-maps

### 2.1 Beta-expansions

For $\beta > 1$ it is well-known that the map $\tau_\beta$ gives an expansion of $x \in [0,1]$ as follows. Since

$$x = \frac{[\beta x]}{\beta} + \frac{\tau_\beta(x)}{\beta}$$

for $x \in [0,1]$, we have

$$\tau_\beta^n(x) = \frac{[\beta \tau_\beta^n(x)]}{\beta} + \frac{\tau_\beta^{n+1}(x)}{\beta}$$

for $n \geq 0$. Using the above equations inductively, we obtain

$$x = \sum_{n=1}^{N} \frac{[\beta \tau_\beta^{n-1}(x)]}{\beta^n} + \frac{\tau_\beta^N(x)}{\beta^N}$$

for $N \geq 1$. Set $a_n(\beta, x) = [\beta \tau_\beta^{n-1}(x)]$ for $n \geq 1$. Taking $N \to +\infty$ on the right side of the above equation provides the greedy expansion of $x$:

$$x = \sum_{n=1}^{\infty} \frac{a_n(\beta, x)}{\beta^n}.$$

## 2.2 Perron-Frobenius operators

Let us denote by $l$ the Lebesgue measure on $[0,1]$ and by $(L^1(l), ||\cdot||_1)$ the Banach space of integrable functions with respect to the Lebesgue measure $l$. For a positive constant $\beta > 0$, we define the Perron-Frobenius operator for the corresponding map $\tau_\beta$ by

$$\mathcal{L}_\beta f(x) = \frac{1}{\beta} \sum_{y; x = \tau_\beta(y)} f(y)$$

for $x \in [0,1]$ and $f \in L^1(l)$. This operator is linear and bounded on $L^1(l)$ with $||\mathcal{L}_\beta f||_1 = ||f||_1$ for $f \in L^1(l)$. For a function $f : [0,1] \to \mathbb{C}$ denote by $\bigvee f$ the total variation of $f$. We define

$$|f|_{BV} = \inf \left\{ \bigvee f^* ; f^* \text{is a version of } f \right\}.$$

Set $BV = \{f \in L^1(l) \ ; \ |f|_{BV} < \infty\}$ and

$$||f||_{BV} = ||f||_1 + |f|_{BV}$$

for $f \in BV$. Then $(BV, ||\cdot||_{BV})$ is a Banach space and the Perron-Frobenius operator is linear and bounded on $(BV, ||\cdot||_{BV})$ (e.g., [1], [3], [9], [10]). If $\beta > 1$, we know that the map $\tau_\beta$ is uniformly expanding and the corresponding Perron-Frobenius operator $\mathcal{L}_\beta$ is quasi-compact, i.e., for any spectral value $\lambda \in \mathbb{C}$ whose modulus is greater than $1/\beta$ is an isolated eigenvalue with finite multiplicity, which plays an important role in investigating the ergodic properties of $\tau_\beta$ (e.g., [1], [3], [9], [10]).

Denote by $BV^*$ the set of complex-valued linear functionals on $BV$. Let us define the dual operator of the Perron-Frobenius operator $\mathcal{L}_\beta^* : BV^* \to BV^*$ by

$$\mathcal{L}_\beta^*(\nu(f)) = \nu(\mathcal{L}_\beta f)$$

for $\nu \in BV^*$ and $f \in BV$.

## 2.3 Isolated eigenvalues of $\mathcal{L}_\beta$

Let $\beta > 1$. We define the power series $\phi_\beta(z)$ associated to the coefficient sequence $\{a_n(\beta, 1)\}_{n=1}^{\infty}$ by

$$\phi_\beta(z) = \sum_{n=1}^{\infty} \frac{a_n(\beta, 1)}{\beta^n} z^n.$$

We can easily see that the convergence radius of $\phi_\beta(z)$ is at least $\beta$. For the analytic function $1-\phi_\beta(z)$ defined on $\{z \in \mathbb{C}; |z| < \beta\}$, we have the following:

**Theorem 2.1.** *A complex number $\lambda$ with $1/\beta < |\lambda| \le 1$ is an isolated eigenvalue of $\mathcal{L}_\beta$ if and only if $\lambda^{-1}$ is a zero of $1-\phi_\beta(z)$.*

The proof of the above theorem is deduced from an explicit formula for the analytic continuation of the dynamical zeta function of $\tau_\beta$ in [8] (see also [4]) and application of the results in [2] (or [6]) to our setting, which yields that each zero of its continuation is the inverse of an isolated eigenvalue of $\mathcal{L}_\beta$, and vice versa.

The main result of this paper is a generalization of the above theorem to random dynamical systems generated by beta-maps. In the setting of random dynamical systems, however, there are few results for dynamical zeta functions, including their analytic continuation. In this paper, we use a new approach derived from a formula for an eigenfunctional of the Perron-Frobenius operator, described in deterministic cases in the following. A key ingredient is the following equation:

**Lemma 2.2.** *Let $n \ge 0$ be a non-negative integer. Then we have*

$$\mathcal{L}_\beta \mathbf{1}_{[0,\tau_\beta^n(x)]} = \frac{a_{n+1}(\beta,x)}{\beta}\mathbf{1}_{[0,1]} + \frac{1}{\beta}\mathbf{1}_{[0,\tau_\beta^{n+1}(x)]},$$

*where $\mathbf{1}_A$ denote the indicator function of $A$.*

The above lemma yields the following formula:

**Theorem 2.3.** *Let $\lambda \in \mathbb{C}$ be an isolated eigenvalue of the Perron-Frobenius operator $\mathcal{L}_\beta$ with $|\lambda| > 1/\beta$ and let $\nu \in BV^*$ be a non-zero eigenfunctional corresponding to $\lambda$. Then for $x \in [0,1]$ we have*

$$\nu(\mathbf{1}_{[0,x]}) = \nu(\mathbf{1}_{[0,1]}) \sum_{n=1}^{\infty} \frac{a_n(\beta,x)}{\beta^n \lambda^n}. \tag{2.1}$$

*In addition, $\nu(\mathbf{1}_{[0,1]}) \neq 0$ and the geometric multiplicity of $\mathcal{L}_\beta$ is 1.*

By taking $x = 1$ in the equation (2.1), we obtain the following result.

**Theorem 2.4.** *Let $\lambda \in \mathbb{C}$ with $|\lambda| > 1/\beta$ be an isolated eigenvalue of $\mathcal{L}_\beta$. Then*

$$\sum_{n=1}^{\infty} \frac{a_n(\beta,1)}{\beta^n \lambda^n} = 1.$$

*That is, $\lambda^{-1}$ is a zero of an analytic function $1-\phi_\beta(z)$.*

The above theorem states that any isolated eigenvalues of $\mathcal{L}_\beta$ is the inverse of a zero of $1-\phi_\beta(z)$. In fact, as in the construction of an eigenfunction of $\mathcal{L}_\beta$ in [13], we have that the inverse of any zero of $1-\phi_\beta(z)$ is actually an isolated eigenvalue of $\mathcal{L}_\beta$.

**Theorem 2.5.** *Let $\lambda^{-1}$ be a zero of $1 - \phi_\beta(z)$ with $1 \leq |\lambda^{-1}| < \beta$. Set*

$$h = C \sum_{m=0}^{\infty} \frac{\mathbf{1}_{[0,\tau_\beta^m(1)]}}{\beta^m \lambda^m},$$

*where $C$ is a non-zero constant. Then $h$ is a non-zero function of bounded variation satisfying $\mathcal{L}_\beta h = \lambda h$. In particular, $\lambda$ is an eigenvalue of $\mathcal{L}_\beta$.*

By Theorem 2.4 and Theorem 2.5, we reprove Theorem 2.1 not using the theory of dynamical zeta functions. Our main result in this paper is a generalization of Theorem 2.1 to random beta-maps, whose proof is given by an analogue of Theorem 2.4 and 2.3 to random cases.

# 3 Random beta-maps

## 3.1 Setting

Let $\mathcal{A}$ be a set of finite positive integers $\{1, \dots, n\}$ or all positive integers $\{1, 2, \dots\}$. Denote by $2^{\mathcal{A}}$ the power set of $\mathcal{A}$. Let $\hat{\mathbb{P}}$ be a Bernoulli measure on $\mathcal{A}$. For the product space $(\Omega, \mathcal{F}, \mathbb{P}) = (\mathcal{A}, 2^{\mathcal{A}}, \hat{\mathbb{P}})^{\mathbb{N}}$, let $\theta : \Omega \to \Omega$ be the left shift on it, defined by $\theta((\omega_i)_{i=1}^\infty) = (\omega_{i+1})_{i=1}^\infty$ for $(\omega_i)_{i=1}^\infty$. Let $\hat{\beta} : \mathcal{A} \to (0, \infty)$ be a measurable function and define the function $\beta : \Omega \to (0, \infty)$ by $\beta((\omega_i)_{i=1}^\infty) = \hat{\beta}(\omega_1)$ for $(\omega_i)_{i=1}^\infty \in \Omega$. Then the function $\beta$ is a positive measurable function on $(\Omega, \mathcal{F})$ and the random variables $\{\beta \circ \theta^n\}_{n=0}^\infty$ are i.i.d. on $(\Omega, \mathcal{F}, \mathbb{P})$. We define the map $\tau : \Omega \times [0, 1] \to [0, 1]$ by $(\omega, x) \mapsto \tau_{\beta(\omega)}(x)$. Then the random dynamical system can be represented by the skew-product map $R(\omega, x) = (\theta\omega, \tau(\omega, x))$ for $(\omega, x) \in \Omega \times [0, 1]$. For simplicity, we write $\tau_\omega(\cdot) = \tau(\omega, \cdot)$ for $\omega \in \Omega$. By setting $\tau_\omega^0 = id$ and $\tau_\omega^n = \tau_{\theta^{n-1}\omega} \circ \dots \tau_\omega$ for $\omega \in \Omega$ and $n \geq 1$, where $id$ denotes the identity map on $[0, 1]$, we can write the $n$-th iteration of the skew product map as

$$R^n(\omega, x) = (\theta^n\omega, \tau_\omega^n(x))$$

for $(\omega, x) \in \Omega \times [0, 1]$.

## 3.2 Random beta-expansions

As a generalization of beta-expansions generated by a single beta-map, the random dynamical systems as defined above generate multiple base expansions of $x \in [0, 1]$ as follows. Set $\beta_\omega^{(0)} = 1$ and

$$\beta_\omega^{(n)} = \prod_{i=0}^{n-1} \beta(\theta^i\omega)$$

for $\omega \in \Omega$ and $n \geq 1$. We write $\beta_\omega = \beta_\omega^{(1)} = \beta(\omega)$ for simplicity. Define the digit function by $d_n(\omega, x) = [\beta_{\theta^{n-1}\omega} \tau_\omega^{n-1}(x)]$ for $(\omega, x) \in \Omega \times [0, 1]$ and $n \geq 1$. Then by the definition of the map $\tau$, we have

$$x = \frac{[\beta_\omega x]}{\beta_\omega} + \frac{\tau_\omega(x)}{\beta_\omega}$$

for $(\omega, x) \in \Omega \times [0,1]$. By the fact that

$$\tau_\omega^n(x) = \frac{[\beta_{\theta^n\omega}\tau_\omega^n(x)]}{\beta_{\theta^n\omega}} + \frac{\tau_\omega^{n+1}(x)}{\beta_{\theta^n\omega}}$$

for $n \geq 0$, we obtain

$$\begin{aligned} x &= \frac{[\beta_\omega x]}{\beta_\omega} + \frac{\tau_\omega(x)}{\beta_\omega} \\ &= \frac{[\beta_\omega x]}{\beta_\omega} + \frac{[\beta_{\theta\omega}\tau_\omega(x)]}{\beta_\omega\beta_{\theta\omega}} + \frac{\tau_\omega^2(x)}{\beta_\omega\beta_{\theta\omega}} \\ &= \cdots \\ &= \sum_{n=1}^{N} \frac{d_n(\omega, x)}{\beta_\omega^{(n)}} + \frac{\tau_\omega^N(x)}{\beta_\omega^{(N)}} \end{aligned}$$

for $N \geq 2$. Note that $\tau_\omega^n(x) \in [0,1]$ for $n \geq 0$. Then under the assumption that

$$(\beta_\omega^{(n)})^{-1} \to 0$$

as $n \to \infty$, we have

$$x = \sum_{n=1}^{\infty} \frac{d_n(\omega, x)}{\beta_\omega^{(n)}}.$$

One of the sufficient conditions which guarantees $(\beta_\omega^{(n)})^{-1} \to 0$ as $n \to \infty$ for $\mathbb{P}$-a.e. $\omega \in \Omega$ is the following.

**Lemma 3.1.** *If*

$$\int_\Omega \frac{d\mathbb{P}(\omega)}{\beta(\omega)} < 1, \tag{3.1}$$

*then for $\mathbb{P}$-a.e. $\omega \in \Omega$ we have*

$$(\beta_\omega^{(n)})^{-1} \to 0$$

*as $n \to \infty$.*

## 3.3 Sample-averaged Perron-Frobenius operators

Let us define the sample averaged (or annealed) Perron-Frobenius operator $\mathcal{L} : L^1(l) \to L^1(l)$ by

$$\mathcal{L}f = \int_\Omega (\mathcal{L}_{\beta(\omega)} f) d\mathbb{P}$$

for $f \in L^1(l)$. We note that this operator is well-defined by the inequality

$$||\mathcal{L}_{\beta(\omega)} f||_1 \leq \int_0^1 \int_\Omega |\mathcal{L}_{\beta(\omega)} f| d\mathbb{P} dl \leq \int_\Omega \int_0^1 |\mathcal{L}_{\beta(\omega)} f| dl d\mathbb{P} \leq ||f||_1$$

for $f \in L^1(l)$, which also ensures that the operator $\mathcal{L}$ is bounded. The fact that the operator $\mathcal{L}$ is linear follows from its definition. Assume that

$$\int_\Omega \frac{d\mathbb{P}(\omega)}{\beta(\omega)} = \int_{\hat{\Omega}} \frac{d\hat{\mathbb{P}}(a)}{\hat{\beta}(a)} < 1.$$

By applying Lemma 6.7 in [7] to our setting, we have that $\mathcal{L}$ is well-defined as a linear bounded operator on $BV$ and it is quasi-compact, i.e., any spectral value $\lambda \in \mathbb{C}$ whose modulus is greater than $\int_\Omega d\mathbb{P}(\omega)/\beta(\omega)$ is an isolated eigenvalue with finite multiplicity. In addition, 1 is actually this eigenvalue of $\mathcal{L}$, which ensures the existence of an $R$-invariant probability measure $\mu$ absolutely continuous with respect to $\mathbb{P} \times l$. Let $\mathcal{U}$ be the set of all functions of the form $\sum_{i=1}^n a_i f_i$, where $n \geq 1$, $a_i \in \mathbb{C}$ and $f_i$ is an indicator function of some interval in $[0,1]$ for $1 \leq i \leq n$. Let $\mathcal{F}$ be the closure of $\mathcal{U}$ in the sense of the topology derived from the norm $||\cdot||_{BV}$ in $BV$. Then we can see that $\mathcal{L}\mathcal{F} \subset \mathcal{F}$ and $\mathcal{L}$ is a linear bounded operator, which is quasi-compact on $(\mathcal{F}, ||\cdot||_{BV})$. Since $\mathbf{1}_{[0,1]} \in \mathcal{F}$ we know that 1 is also an isolated eigenvalue of $\mathcal{L}$ on $\mathcal{F}$.

# 4 Main results

Let $\Phi(z)$ be the formal power series defined by

$$\Phi(z) = \sum_{n=1}^{\infty} \left( \int_\Omega \frac{d_n(\omega, 1)}{\beta_\omega^{(n)}} d\mathbb{P} \right) z^n.$$

Since

$$\frac{d_n(\omega, 1)}{\beta_\omega^{(m)}} \leq \frac{1}{\beta_\omega^{(n-1)}}$$

for $\omega \in \Omega$ and $n \geq 1$, we have that the convergence radius of $\Phi(z)$ is at least $\left( \int_\Omega \frac{d\mathbb{P}(\omega)}{\beta(\omega)} \right)^{-1} > 1$. The main result in this paper is the following.

**Theorem 4.1.** *A complex number $\lambda$ with $\int_\Omega \frac{d\mathbb{P}(\omega)}{\beta(\omega)} < |\lambda| \leq 1$ is an isolated eigenvalue of $\mathcal{L}$ defined on $\mathcal{F}$ if and only if $\lambda^{-1}$ is a zero of $1 - \Phi(z)$.*

*Remark* 4.2. (1) The above theorem states that the mean value of $n$-terms of random beta expansions of 1 determines all isolated eigenvalues of $\mathcal{L}$ outside the closed disk whose radius is $\int_\Omega \frac{d\mathbb{P}(\omega)}{\beta(\omega)}$, which can be seen as a natural generalization of Theorem 2.1.

(2) The proof of Theorem 4.1 is given by a generalization of Theorem 2.3 and that of Theorem 2.4 to random beta-maps, which also yields an explicit formula for any eigenfunction and that for a value of any eigenfunctional applied to the indicator function of some interval corresponding to an isolated eigenvalue of $\mathcal{L}$.

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