Understanding the limit sets generated by general iterated function systems

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Abstract

We introduce a definition of iterated functions systems which is more general than the ones introduced before, and present a theorem on the existence and uniqueness of the family of the limit sets generated by general IFSs, instead of the limit set introduced before, under the "natural" assumption. In addition, we also give some examples of general IFSs to discuss the necessity of the assumption and the importance of the properties in the main result.

1 Introduction

Many researchers have developed studies on the limit set by generalized iterated function systems (for short, generalized IFSs) in many directions. In fact, there are at least three lines in these studies. The studies in the first line ensure the existence of the limit set generated by non-autonomous IFSs (and some generalizations; for example, see [9] and [1]). These papers also give theorems on the estimates of the dimensions and measures of the limit sets. The studies in the second line discuss the existence of the limit set (V-variable fractals) generated by generalized IFSs (V-variable IFSs). These studies also give theorems on estimates of the Hausdorff dimension of the limit sets using probabilistic techniques (for example, see [2], [3] and [10]). The studies in the third line present estimates on the dimensions of the limit set (the Moran set) generated by "nice" structures (Moran structures; for example, see [8], [5] and [4]).

However, these studies propose different definitions and assumptions to deduce a "nice" structure of the limit set generated by each generalized IFS. The papers in the first line consider non-autonomous IFSs on compact sets to obtain the results (in particular, the existence of the limit sets). The papers in the second line consider V-variable IFSs with assumptions which allow us to restrict the domains of the IFSs to a bounded set. The papers in the third line consider Moran structures and the Moran set with assumptions which allow us to obtain the results. Therefore, we do not clearly know how these studies are related to each other.

As the first approach to address the issue, we now consider more general IFSs (henceforth, denoted by general IFSs) and give a theorem on the existence and uniqueness (in some sense) of the limit set generated by general IFSs. Recall the Hutchinson method: we interpret the limit set as the fixed point of a "natural" operator on the set of compact subsets (see [7]). An advantage of this method is to obtain not only the existence but also the uniqueness of the limit set. We focus on this advantage and extend the method to a generalized setting. More precisely, we first introduce a definition of IFSs which is more general than the ones introduced before. We next consider the set of families of compact subsets instead of the set of compact subsets. Then, under a "natural" assumption, we obtain the fixed point of a "natural" operator on the set of families of compact subsets (main result). By this generalization, we finally deduce the existence and uniqueness of the family of the limit sets. After the main result, we also present the connection between the family of limit sets and the limit set mentioned in the first paragraph (henceforth, denoted by the original limit set). Hence, the aim of this paper is to introduce more general IFSs and to present a theorem on the existence and uniqueness of the family of the limit sets generated by general IFSs, instead of the original limit set.

It is worth mentioning that the family of the limit sets generated by general IFSs is characterized not only by the (generalized) self-similarity, but also by the "summable" condition (for details, see Theorem 2.5). Recall that the limit set generated by autonomous IFSs is characterized by the unique compact subset with self-similarity. We also give some examples of general IFSs to discuss the necessity of the assumption and the importance of the properties in the main result (see Examples 3.1 $\sim$ 3.4).

The rest of the paper is organized as follows. In Section 2, we present the main result and related remarks. In Section 3, we consider simple examples for general IFSs to discuss about the "natural" assumption.

2 General iterated function systems

In this section, we give the main result and related remarks. In Subsection 2.1, we first introduce some notions (trees, subtrees and branches) to define general IFSs which are main objects in this paper. In Subsection 2.2, we next define general IFSs and present the main result in this paper. In Subsection 2.3, we give some remarks on the main result.

2.1 Definition of trees

Let I be a non-empty and countable set endowed with the discrete topology. We set $I^* := \{\phi\} \cup \bigcup_{n\in\mathbb{N}} I^n$, where ϕ is the empty word. We write $\omega \in I^m$ $(m \in \mathbb{N})$ as $\omega_1 \cdots \omega_m$ $(\omega_k \in I, k = 1, \dots, m)$ and $\omega \in I^{\mathbb{N}}$ as $\omega_1\omega_2\cdots$ $(\omega_k \in I, k \in \mathbb{N})$ respectively.

Definition 2.1. We say that $\mathbb{T}$ is a tree on I if $\mathbb{T}$ is a compact subset of $I^{\mathbb{N}}$, where we endow $I^{\mathbb{N}}$ with the product topology.

We set $\mathbb{T}_n := P_{[1,n]}(\mathbb{T})$ $(n \in \mathbb{N})$ and $\mathbb{T}_* := \{\phi\} \cup \bigcup_{n\in\mathbb{N}} \mathbb{T}_n$, where for $m, n \in \mathbb{N}$ with $m \le n$, and $\omega = \omega_1\omega_2\cdots \in I^{\mathbb{N}} \cup I^l$ (with $n \le l$), we set $P_{[m,n]}(\omega) := \omega_m \cdots \omega_n \in I^{n-m+1}$. Similarly, we also set $P_{[m,\infty)}(\omega) := \omega_m\omega_{m+1}\cdots \in I^{\mathbb{N}}$ for each $m \in \mathbb{N}$ and $\omega \in I^{\mathbb{N}}$. We denote by $|\omega|$ the length of $\omega \in \mathbb{T}_* \cup \mathbb{T}$, that is, for each $\omega \in \mathbb{T}_* \cup \mathbb{T}$, we set

$$|\omega| := \begin{cases} 0 & \text{if } \omega = \phi \\ n & \text{if } \omega \in \mathbb{T}_n \ (n \in \mathbb{N}) \\ \infty & \text{if } \omega \in \mathbb{T} \end{cases}.$$

We next introduce the definition of subtrees of a tree.

Definition 2.2. Let $\mathbb{T}$ be a tree on I, and $\omega \in \mathbb{T}_*$. The subtree $\mathbb{T}^\omega$ of $\mathbb{T}$ conditioned by ω is defined by

$$\mathbb{T}^\omega := \begin{cases} \mathbb{T} & \text{if } \ \omega = \phi \\ \{P_{[|\omega|+1,\infty)}(\tau) \mid \tau \in \mathbb{T},\ P_{[1,|\omega|]}(\tau) = \omega\ \} & \text{if } \ \omega \in \bigcup_{n\in\mathbb{N}} \mathbb{T}_n \end{cases}.$$

We endow $\mathbb{T}^\omega \subset I^{\mathbb{N}}$ with the induced topology.

We set $\mathbb{T}^\omega_n := P_{[1,n]}(\mathbb{T}^\omega)$ $(n \in \mathbb{N})$ and $\mathbb{T}^\omega_* := \{\phi\} \cup \bigcup_{n\in\mathbb{N}} \mathbb{T}^\omega_n$. Note that $\mathbb{T}^\omega \subset I^{\mathbb{N}}$ satisfies the definition of trees for each $\omega \in \mathbb{T}_*$.

Before we conclude this subsection, we introduce some additional notions to state the main result.

Definition 2.3. Let $\mathbb{T}$ be a tree on I, and $\omega \in \mathbb{T}_*$. The branch $N(\omega) \subset I$ (of $\mathbb{T}$) at $\omega \in \mathbb{T}_*$ is defined by $N(\omega) := \mathbb{T}^\omega_1$. In addition, for $\omega \in I^*$ and $\omega' \in I^* \cup I^{\mathbb{N}}$, $\omega\omega'$ is defined by

$$\omega\omega' := \begin{cases} \omega_1 \cdots \omega_{|\omega|}\omega'_1 \cdots \omega'_{|\omega'|} \in I^* & \text{if } \omega' \in I^* \\ \omega_1 \cdots \omega_{|\omega|}\omega'_1\omega'_2 \cdots \quad \in I^{\mathbb{N}} & \text{if } \omega' \in I^{\mathbb{N}} \end{cases}.$$

Note that for each tree $\mathbb{T}$ and $\omega \in \mathbb{T}_*$, $N(\omega) \subsetneq P_{[|\omega|,\,|\omega|]}(\mathbb{T})$ in general and we have $\#N(\omega) < \infty$.

2.2 General iterated function systems and the main result

Definition 2.4. Let I be a non-empty and countable set endowed with the discrete topology and (X, ρ) a complete metric space. We say that a pair $(\{f_i\}_{i\in I}, \mathbb{T})$ is a general iterated function systems (for short, a general IFS) on (X, ρ) with the uniform contraction constant $c \in (0, 1)$ if

(i) $\mathbb{T}$ is a tree on a set I and

(ii) $f_i \colon X \to X$ $(i \in I)$ is a family of contractive mappings on X with the uniform contraction constant c, that is, for all $i \in I$ and $x, y \in X$,

$$\rho(f_i(x), f_i(y)) \le c\, \rho(x, y).$$

Note that, for each $i \in I$, there exists the unique fixed point z_i of f_i since X is complete and f_i is contractive on X.

Before we state the main result, we introduce the Hausdorff distance and present its property. Let $\mathcal{K}(X)$ be the set of non-empty compact subsets in (X, ρ). For $\epsilon > 0$ and $A \subset X$, we set $A_\epsilon := \{x \in X \mid \inf_{a\in A} \rho(a, x) \le \epsilon\}$. Let ρ_H be the Hausdorff distance on $\mathcal{K}(X)$ defined by

$$\rho_H(A, B) := \inf\{\epsilon > 0 \mid A \subset B_\epsilon, B \subset A_\epsilon\} \quad (A, B \in \mathcal{K}(X)).$$

Note that since (X, ρ) is complete, $(\mathcal{K}(X), \rho_H)$ is also complete (For example, see [7]).

We now give the main result in this paper.

Theorem 2.5. Let $(\{f_i\}_{i\in I}, \mathbb{T})$ be a general IFS on a complete metric space (X, ρ) with $c \in (0, 1)$, and z_i the unique fixed point of f_i $(i \in I)$. Suppose that there exists $x \in X$ such that

$$\alpha := \limsup_{n\to\infty} \sqrt[n]{\max_{i\in P_{[n,n]}(\mathbb{T})} \rho(x, z_i)} < \frac{1}{c}. \tag{2.1}$$

Then, there exists the unique $\{L_\omega\}_{\omega\in\mathbb{T}^*} \in \mathcal{K}(X)^{\mathbb{T}^*}$ such that for each $x \in X$ and $r \in \{r > 0 \mid c \vee \alpha c < r < 1\}$,

$$L_\omega = \bigcup_{i\in N(\omega)} f_i(L_{\omega i}) \quad \text{for all} \quad \omega \in \mathbb{T}^*, \quad \text{and} \quad \sum_{n=0}^{\infty} \left(\frac{c}{r}\right)^n \cdot \max_{\omega\in\mathbb{T}_n} \rho_H(\{x\}, L_\omega) < \infty. \tag{2.2}$$

Remark 2.6. Fix a general IFS and assume that $Z := \{z_i \in X \mid i \in I\}$ is unbounded. If the general IFS satisfies the assumption (2.1) in Theorem 2.5 for some $x \in X$, then it satisfies the assumption for all $x \in X$ and the constant $\alpha \geq 0$ in Theorem 2.5 does not depend on $x \in X$.

On the other hand, fix a general IFS and assume that Z is bounded. Then, even if the general IFS satisfies the assumption for some $x \in X$, the constant $\alpha \geq 0$ unfortunately depends on $x \in X$. However, in this case, we can obtain the following stronger result than Theorem 2.5: there exists the unique $\{L_\omega\}_{\omega \in \mathbb{T}_*} \in \mathcal{K}(X)^{\mathbb{T}^*}$ such that

$$L_\omega = \bigcup_{i \in N(\omega)} f_i(L_{\omega i}) \quad \text{for all} \quad \omega \in \mathbb{T}^*, \quad \text{and} \quad \{L_\omega\}_{\omega \in \mathbb{T}_*} \text{ is uniformly bounded.}$$

The above property does not inculude α (and r). Note that $\{L_\omega\}_{\omega \in \mathbb{T}_*}$ in Theorem 2.5 is not uniformly bounded in general (see Example 3.2).

Remark 2.7. In Theorem 2.5, we can regard the constant r as the convergence rate. Indeed, under the assumption (2.1), we can deduce that for all for all $A \in \mathcal{K}(X)$, $\omega \in \mathbb{T}_*$,

$$\bigcup_{\omega' \in \mathbb{T}_n^\omega} f_{\omega'_1} \circ \cdots \circ f_{\omega'_n}(A) \to L_\omega \quad (n \to \infty)$$

exponentially fast with the rate r, in sense of the Hausdorff distance.

2.3 Connections with the main result and related results

Remark 2.8. Under the assumption in Theorem 2.5, we can define a "natural" generalization of the projection map (or the coding map) for general IFSs, without the boundedness or compactness of the space (X, ρ). Indeed, under the assumption (2.1), we can obtain the following: for fixed $\omega = \omega_1\omega_2 \cdots \in \mathbb{T}$, the sequence $\{f_{\omega_1} \circ \cdots \circ f_{\omega_n}(A)\}_{n \in \mathbb{N}}$ ($A \in \mathcal{K}(X)$) of compact subsets converges to a single set (it does not depend on A) as n tends to infinity in sense of the Hausdorff distance. In particular, by the general theory of the Hausdorff distance, if a sequence $\{f_{\omega_1} \circ \cdots \circ f_{\omega_n}(A)\}_{n \in \mathbb{N}}$ is non-increasing (with respect to the inclusion order) for some $A \in \mathcal{K}(X)$, then the intersection $\cap_{m \in \mathbb{N}} f_{\omega_1} \circ \cdots \circ f_{\omega_n}(A)$ is a single set and is the limit point of the sequence $\{f_{\omega_1} \circ \cdots \circ f_{\omega_n}(A)\}_{m \in \mathbb{N}}$ in sense of the Hausdorff distance.

The above result leads us to a connection between L_ϕ in Theorem 2.5 and the original limit set. Indeed, under the assumption (2.1), we can deduce that L_ϕ is equal to the image of the projection map for the general IFSs. Note that the unique limit set generated by autonomous IFSs is equal to the image of the coding map. In Rempe-Gillen's and Urbański's paper [9], the projection map on compact metric space (X, ρ) is defined by the intersection of non-increasing compact subsets $\{f_{\omega_1} \circ \cdots \circ f_{\omega_m}(X)\}_{m \in \mathbb{N}}$ ($\omega = \omega_1\omega_2 \cdots$) and the limit set is defined by the image of the projection map.

Remark 2.9. Theorem 2.5 gives a generalized result on the paper [2]. Indeed, under the assumption (2.1), we deduce that

$$\sup_{i \in I} \rho(x, z_i) \leq \frac{1}{(1-c)} \sup_{i \in I} \rho(f_i(x), x) < \infty$$

and it follows that $\alpha := \limsup_{n \to \infty} \sqrt[n]{\max_{i \in P_{[n,n]}(\mathbb{T})} \rho(x, z_i)} \leq 1 < 1/c$. This is one of the assumptions under which we define the limit set in the paper [2].

In addition, the family $\{L_\omega\}_{\omega\in\mathbb{T}_*}$ of limit sets for general IFSs $(\{f_i\}_{i\in I},\mathbb{T})$ in Theorem 2.5 is compatible with some conditions in the definition of the Moran structure and the compact subset L_ϕ therefore equals to the Moran sets. Recall that, in the papers [8] and [4], the limit set (the Moran set) is generated by the Moran structure.

3 Examples of General IFSs and the family of limit sets

In this section, we consider simple examples for general IFSs to check the necessity of the assumption and the importance of the properties in the main result. In Examples 3.1 and 3.2, there is the unique original limit set with a "nice" property since the general IFS satisfies the "natural" assumption. In Example 3.3, there are no limit sets with a "nice" property since the general IFS does not satisfy the "natural" assumption. In Example 3.4, there is the unique original limit set with a "nice" property, while the general IFS does not satisfy the "natural" assumption and "nice" property is indeed complicated. Note that the examples in this section also appear in [6].

Henceforth, we set $X := \mathbb{R}$ (with the Euclidean norm $|\cdot|$), $I := \mathbb{N}$, and $\mathbb{T} := \{12\cdots\} \subset I^{\mathbb{N}}$. In this case, $\mathbb{T}$ is a single set, $\mathbb{T}_* = \{1\cdots n \mid n \in \mathbb{N}\}$ and $\mathbb{T}^{1\cdots n} := \{(n+1)(n+2)\cdots\}$. Note that, under this setting, we can reduce the general IFSs to the non-autonomous iterated function systems.

Example 3.1. We set $f_n(x) := (x + (1-2^{-n}))/2$ $(x \in X)$ and consider a general IFSs $(\{f_n\}_{n\in I},\mathbb{T})$ on X with $1/2$. In this case, $1-2^{-n} \in X$ is the unique fixed point of f_n for each $n \in I = \mathbb{N}$ and $(\{f_n\}_{n\in I},\mathbb{T})$ satisfies the assumption (2.1) in Theorem 2.5 (indeed, the set $\{1-2^{-n}\}_{n\in\mathbb{N}}$ of the fixed points is bounded (see Remark 2.6), or we can directly deduce that $\alpha \le 1 < 2 = (1/2)^{-1}$ for each $x \in X$).

Then, the non-autonomous iteration has the limit point. Indeed, by the induction with respect to $k \in \mathbb{N}$, we deduce that

$$f_m \circ \cdots \circ f_{m+k-1}(y) = 1 - \frac{2}{3}\cdot\frac{1}{2^m} + \frac{y-1}{2^k} + \frac{1}{2^m}\cdot\frac{2}{3}\cdot\frac{1}{4^k}$$

for each $y \in X$ and $m,k \in \mathbb{N}$. It follows that

$$x_m := \lim_{k\to\infty} f_m \circ \cdots \circ f_{m+k-1}(y) = 1 - \frac{2}{3}\cdot\frac{1}{2^m} \quad \text{and} \quad x_m = f_m(x_{m+1})$$

for each $y \in X$ and $m \in \mathbb{N}$. On the other hand, for $y_1 \in X$, we set

$$y_m := f_{m-1}^{-1}\circ\cdots\circ f_1^{-1}(y_1) = 2^{m-1}\left(y_1 - \frac{2}{3}\right) + 1 - \frac{2}{3}\cdot\frac{1}{2^m} \quad \text{and} \quad L_{1\cdots m} := \{y_m\} \quad (m \in \mathbb{N}).$$

Then, $f_m(y_{m+1}) = y_m$ (this is equivalent to $f_m(L_{1\cdots(m+1)}) = L_{1\cdots m}$) for all $m \in \mathbb{N}$ and $\{x_m\}_{m\in\mathbb{N}} = \{y_m\}_{m\in\mathbb{N}}$ if $y_1 = 2/3$.

By these arguments, the sequence with the generalized self-similarity $f_m(y_{m+1}) = y_m$ $(m \in \mathbb{N})$ is not unique, but $\{x_m\}_{m\in\mathbb{N}}$ (the case $y_1 = 2/3$) is the unique bounded sequence with the generalized self-similarity.

Example 3.2. We set $f_n(x) := (x+n)/2$ $(x \in X)$ and consider a general IFSs $(\{f_n\}_{n\in I},\mathbb{T})$ on X with $1/2$. In this case, $n \in X$ is the unique fixed point of f_n for each $n \in I = \mathbb{N}$ and $(\{f_n\}_{n\in I},\mathbb{T})$ satisfies the assumption (2.1) in Theorem 2.5 (indeed, we deduce that $\alpha = 1 < 2 = (1/2)^{-1}$ for all $x \in X$).

Then, the non-autonomous iteration has the limit point. Indeed, by the induction with respect to $k \in \mathbb{N}$, we deduce that

$$f_m \circ \cdots \circ f_{m+k-1}(y) = m + 1 + \frac{y - m - 2}{2^k} - \frac{k-1}{2^k}$$

for each $y \in X$ and $m, k \in \mathbb{N}$. It follows that

$$x_m := \lim_{k \to \infty} f_m \circ \cdots \circ f_{m+k-1}(y) = m + 1 \quad \text{and} \quad x_m = f_m(x_{m+1})$$

for all $y \in X$ and $m \in \mathbb{N}$. On the other hand, for $y_1 \in X$, we set

$$y_m := f_{m-1}^{-1} \circ \cdots \circ f_1^{-1}(y_1) = 2^{m-1}(y_1 - 2) + m + 1 \quad \text{and} \quad L_{1\cdots m} := \{y_m\} \quad (m \in \mathbb{N}).$$

Then, $f_m(y_{m+1}) = y_m$ (this is equivalent to $f_m(L_{1\cdots(m+1)}) = L_{1\cdots m}$) for all $m \in \mathbb{N}$ and $\{x_m\}_{m\in\mathbb{N}} = \{y_m\}_{m\in\mathbb{N}}$ if $y_1 = 2$.

By these arguments, the sequence with the generalized self-similarity $f_m(y_{m+1}) = y_m$ $(m \in \mathbb{N})$ is not unique, but $\{x_m\}_{m\in\mathbb{N}}$ (the case $y_1 = 2$) is the unique sequence with the generalized self-similarity that does not diverge exponentially fast with the rate 2. Note that the set $\{x_m \in X \mid m \in \mathbb{N}\}$ is unbounded and this property does not hold if X is bounded.

Example 3.3. We set $f_n(x) := (x + 2^n)/2$ $(x \in \mathbb{R})$ and consider a general IFSs X on $(\mathbb{R}, |\cdot|)$ with 1/2. In this case, $2^n \in X$ is the unique fixed point of f_n for each $n \in I = \mathbb{N}$ and $(\{f_n\}_{n\in I}, \mathbb{T})$ does not satisfy assumption (2.1) in Theorem 2.5 (indeed, we deduce that $\alpha = 2 \not< (1/2)^{-1}$ for all $x \in X$).

Then, the non-autonomous iteration does not have a limit point. Indeed, by the induction with respect to $k \in \mathbb{N}$, we deduce that

$$f_m \circ \cdots \circ f_{m+k-1}(y) = \frac{y}{2^k} + 2^{m-1} \cdot k$$

for each $y \in X$ and $m, k \in \mathbb{N}$. It follows that $f_m \circ \cdots \circ f_{m+k-1}(y)$ does not converge as k tends to infinity for all $m \in \mathbb{N}$ and $y \in X$. In particular, we can not obtain the conclusion in Remarks 2.7 and 2.8 without the assumption in Theorem 2.5.

On the other hand, for $y_1 \in X$ and $m \in \mathbb{N}$, we set

$$y_m := f_{m-1}^{-1} \circ \cdots \circ f_1^{-1}(y_1) = 2^{m-1}(y_1 - m + 1) \quad \text{and} \quad L_{1\cdots m} := \{y_m\}.$$

Then, $f_{m+1}(y_{m+1}) = y_m$ (this is equivalent to $f_{m+1}(L_{1\cdots(m+1)}) = L_{1\cdots m}$) for all $m \in \mathbb{N}$. By these arguments, for any $y_1 \in X$, the sequence $\{y_m\}_{m\in\mathbb{N}}$ with the generalized self-similarity $f_m(y_{m+1}) = y_m$ $(m \in \mathbb{N})$ diverges exponentially fast with the rate 2.

Finally, we also give an example that cannot apply the main theorem but is important when we understand the limit sets generated by general IFSs.

Example 3.4. We set $\beta > 1$ and $f_n(x) := (x + 2^n/n^{\beta+1})/2$ $(x \in X)$, and consider a general IFSs $(\{f_n\}_{n\in I}, \mathbb{T})$ on X with 1/2. In this case, $2^n/n^{\beta+1} \in X$ is the unique fixed point of f_n for each $n \in I = \mathbb{N}$ and $(\{f_n\}_{n\in I}, \mathbb{T})$ does not satisfy the assumption (2.1) in Theorem 2.5 (indeed, we deduce that $\alpha = 2 \not< (1/2)^{-1}$ for all $x \in X$).

However, the non-autonomous iteration has the limit point. Indeed, by the induction with respect to $k \in \mathbb{N}$, we deduce that

$$f_m \circ \cdots \circ f_{m+k-1}(y) = \frac{y}{2^k} + 2^{m-1} \sum_{l=0}^{k-1} \frac{1}{(m+l)^{\beta+1}}$$

for each $y \in X$ and $m, k \in \mathbb{N}$. Since $\sum_{l=1}^{\infty} l^{-(\beta+1)} < \infty$, it follows that

$$x_m := \lim_{k\to\infty} f_m \circ \cdots \circ f_{m+k-1}(y) = 2^{m-1} \sum_{l=m}^{\infty} l^{-(\beta+1)} \quad \text{and} \quad x_m = f_m(x_{m+1})$$

for all $y \in X$ and $m \in \mathbb{N}$. On the other hand, for $y_1 \in X$, we set

$$y_m := f_{m-1}^{-1} \circ \cdots \circ f_1^{-1}(y_1) = 2^{m-1} \left(y_1 - \sum_{l=1}^{m-1} l^{-(\beta+1)} \right) \quad \text{and} \quad L_{1\cdots m} := \{y_m\} \quad (m \in \mathbb{N}).$$

Then, $f_m(y_{m+1}) = y_m$ (this is equivalent to $f_m(L_{1\cdots(m+1)}) = L_{1\cdots m}$) for all $m \in \mathbb{N}$ and $\{x_m\}_{m\in\mathbb{N}} = \{y_m\}_{m\in\mathbb{N}}$ if $y_1 = \sum_{l=1}^{\infty} l^{-(\beta+1)}$.

By these arguments, the sequence with the generalized self-similarity $f_m(y_{m+1}) = y_m$ $(m \in \mathbb{N})$ is not unique, but $\{x_m\}_{m\in\mathbb{N}}$ (the case $y_1 = \sum_{l=1}^{\infty} l^{-(\beta+1)}$) is the unique sequence with the following property:

$$\lim_{m\to\infty} \frac{x_m}{c^m} = \lim_{m\to\infty} \frac{1}{2} \left(\frac{2}{c}\right)^m \sum_{l=m}^{\infty} l^{-(\beta+1)} = \begin{cases} 0 & \text{if} \quad c \geq 2 \\ \infty & \text{if} \quad c < 2 \end{cases}.$$

Note that the sequence $\{x_m\}_{m\in\mathbb{N}}$ in Example 3.2 also holds the above property and the sequence $\{x_m\}_{m\in\mathbb{N}}$ in Example 3.3 does not hold the above property.

We next prove that we cannot obtain the conclusion in Remark 2.7. To this end, let $r \in (0, 1)$, $m \in \mathbb{N}$ and $y \in X$. Assume that there exists a constant $\tilde{C}(m, y) > 0$ such that

$$|x_m - f_m \circ \cdots \circ f_{m+k-1}(y)| \leq \tilde{C}_m(y) \cdot r^k$$

for each $k \in \mathbb{N}$. Note that

$$\frac{y}{2^k} + \frac{1}{(m+k)^{\beta+1}} \leq \frac{y}{2^k} + 2^{m-1} \sum_{l=m+k}^{\infty} \frac{1}{l^{\beta+1}} = |x_m - f_m \circ \cdots \circ f_{m+k-1}(y)|.$$

Thus, we obtain that $1/(m+k)^{\beta+1} \leq \max\{|y|, \tilde{C}_m(y)\} \cdot \max\{1/2, r\}^k$, which deduce the contradiction. Therefore, we cannot obtain the conclusion in Remark 2.7.

In addition, we also deduce that for all for all $y \in X$, $m \in \mathbb{N}$,

$$f_m \circ \cdots \circ f_{m+k-1}(y) \to x_m \quad (k \to \infty)$$

with with a polynomial rate $k^{-\beta}$ of convergence. To this end, note that

$$|x_m - f_m \circ \cdots \circ f_{m+k-1}(y)| = \frac{y}{2^k} + 2^{m-1} \sum_{l=m+k}^{\infty} \frac{1}{l^{\beta+1}}.$$

Since $1/l^{\beta+1} < 1/(l-1)^{\beta} - l^{\beta}$ for each $l \in \mathbb{N}$ with $l \geq 2$, we have

$$\begin{aligned} |x_m - f_m \circ \cdots \circ f_{m+k-1}(y)| &\leq \frac{y}{2^k} + 2^{m-1} \sum_{l=m+k}^{\infty} \left(\frac{1}{(l-1)^{\beta}} - \frac{1}{l^{\beta}} \right) \\ &= \frac{y}{2^k} + \frac{2^{m-1}}{(m+k-1)^{\beta}} \leq C(m, y) \cdot \frac{1}{k^{\beta}} \end{aligned}$$

for all $y \in X$ and $m, k \in \mathbb{N}$, where $C(m, y)$ is the constant which depends on $m \in \mathbb{N}$ and $y \in X$ (and which does not depend on $k \in \mathbb{N}$).

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