Relations between Campanato spaces and the duals of atomic Hardy spaces

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1 Introduction

Let $\mathbb{R}^d$ be the d-dimensional Euclidean space. In this paper we denote by $B(x,r)$ the open ball centered at $x \in \mathbb{R}^d$ and of radius $r \in (0,\infty)$. For a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and a ball B, let

$$f_B = \fint_B f = \fint_B f(y)\,dy = \frac{1}{|B|}\int_B f(y)\,dy,$$

where $|B|$ is the Lebesgue measure of B.

For $p \in [1,\infty)$ and $\lambda \in [-d/p, 1]$, the Campanato space $\mathcal{L}_{p,\lambda}(\mathbb{R}^d)$ is defined as the set of all functions f such that the following functional is finite

$$\|f\|_{\mathcal{L}_{p,\lambda}} = \sup_{B=B(x,r)} \frac{1}{r^\lambda}\left(\fint_B |f(y)-f_B|^p\,dy\right)^{1/p}, \tag{1.1}$$

where the supremum is taken over all balls $B = B(x,r)$. Then $\|f\|_{\mathcal{L}_{p,\lambda}}$ is a norm modulo constant functions and thereby $\mathcal{L}_{p,\lambda}(\mathbb{R}^d)$ is a Banach space. Similarly, for

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$p \in [1,\infty)$ and $\lambda \in [-d/p, 0)$, the Morrey space $L_{p,\lambda}(\mathbb{R}^d)$ is defined as the set of all functions f such that the following functional is finite

$$\|f\|_{L_{p,\lambda}} = \sup_{B=B(x,r)} \frac{1}{r^\lambda} \left(\fint_B |f(y)|^p \, dy \right)^{1/p}, \tag{1.2}$$

where the supremum is taken over all balls $B = B(x,r)$. Then $\|f\|_{L_{p,\lambda}}$ is a norm and thereby $L_{p,\lambda}(\mathbb{R}^d)$ is a Banach space. If $p = 1$ and $\lambda = 0$, then $\mathcal{L}_{p,\lambda}(\mathbb{R}^d) = \mathrm{BMO}(\mathbb{R}^d)$. If $p = 1$ and $\lambda = \alpha$ $(0 < \alpha \le 1)$, then $\mathcal{L}_{p,\lambda}(\mathbb{R}^d)$ coincides with $\mathrm{Lip}_\alpha(\mathbb{R}^d)$. If $-d/p \le \lambda < 0$, then $\mathcal{L}_{p,\lambda}(\mathbb{R}^d)$ is the same as the Morrey space $L_{p,\lambda}(\mathbb{R}^d)$ modulo constant functions. Moreover, if $\lambda = -d/p$, then $L_{p,\lambda}(\mathbb{R}^d) = L^p(\mathbb{R}^d)$.

It is known that, if $p, q \in (1,\infty)$ and $1/p + 1/q = 1$, then $\mathcal{L}_{p,\lambda}(\mathbb{R}^d)$ is the dual of some atomic Hardy space $H^{[\lambda,q]}(\mathbb{R}^d)$ for all $\lambda \in (-d/p, 1)$, i.e.,

$$\mathcal{L}_{p,\lambda}(\mathbb{R}^d) = \left(H^{[\lambda,q]}(\mathbb{R}^d)\right)^* \quad \text{for all } \lambda \in (-d/p, 1).$$

In the case $p = 1$, it is also known that

$$\mathcal{L}_{1,\lambda}(\mathbb{R}^d) = \left(H^{[\lambda,\infty]}(\mathbb{R}^d)\right)^*, \quad \text{if } \lambda \in [0,1), \tag{1.3}$$

since $\mathcal{L}_{1,\lambda}(\mathbb{R}^d) = \mathcal{L}_{p,\lambda}(\mathbb{R}^d)$ and $H^{[\lambda,\infty]}(\mathbb{R}^d) = H^{[\lambda,q]}(\mathbb{R}^d)$, for any $p, q \in (1,\infty)$, by the John-Nirenberg theorem and [2, Theorem 3.3], respectively. However, in the case $p = 1$ and $\lambda \in (-d, 0)$, it is unknown whether (1.3) is true or not, while it is known that

$$\mathcal{L}_{1,\lambda}(\mathbb{R}^d) \subset \left(H^{[\lambda,\infty]}(\mathbb{R}^d)\right)^*, \quad \text{if } \lambda \in (-d, 0). \tag{1.4}$$

In this paper, we prove that $\mathcal{L}_{1,\lambda}(\mathbb{R}^d)$ is weak*-dense in $\left(H^{[\lambda,\infty]}(\mathbb{R}^d)\right)^*$ if $\lambda \in (-d, 0)$. Moreover, we prove that

$$\|f\|_{\mathcal{L}_{1,\lambda}} \sim \|f\|_{(H^{[\lambda,\infty]})^*} \quad \text{for all} \quad f \in \mathcal{L}_{1,\lambda}(\mathbb{R}^d).$$

Unfortunately, it is an open problem whether $\mathcal{L}_{1,\lambda}(\mathbb{R}^d)$ is a proper subspace of $\left(H^{[\lambda,\infty]}(\mathbb{R}^d)\right)^*$ if $\lambda \in (-d, 0)$, while we conjecture that it is proper, since $L^1(\mathbb{R}^d)$ is a proper subspace of the dual of $L^\infty(\mathbb{R}^d)$.

In this paper, more generally, we consider $\mathcal{L}_{p,\phi}(\mathbb{R}^d)$ with variable growth function $\phi : \mathbb{R}^d \times (0,\infty) \to (0,\infty)$. If $\phi(x,r) = r^\lambda$, then $\mathcal{L}_{p,\phi}(\mathbb{R}^d) = \mathcal{L}_{p,\lambda}(\mathbb{R}^d)$. We also investigate for the Morrey space $L_{p,\phi}(\mathbb{R}^d)$ with variable growth function $\phi : \mathbb{R}^d \times (0,\infty) \to (0,\infty)$.

In the next section we state the definitions and known results. Then we give the main results in the last section.

2 Definitions and known results

For a function $\phi : \mathbb{R}^d \times (0,\infty) \to (0,\infty)$ and a ball $B = B(x,r)$, we write $\phi(B)$ instead of $\phi(x,r)$. The function spaces $\mathcal{L}_{p,\phi}(\mathbb{R}^d)$ and $L_{p,\phi}(\mathbb{R}^d)$ are defined as follows:

Definition 2.1. For $p \in [1,\infty)$ and $\phi : \mathbb{R}^d \times (0,\infty) \to (0,\infty)$, let $\mathcal{L}_{p,\phi}(\mathbb{R}^d)$ and $L_{p,\phi}(\mathbb{R}^d)$ be the sets of all functions f such that the following functionals are finite, respectively:

$$\|f\|_{\mathcal{L}_{p,\phi}} = \sup_B \frac{1}{\phi(B)} \left(\fint_B |f(y) - f_B|^p \, dy \right)^{1/p},$$
$$\|f\|_{L_{p,\phi}} = \sup_B \frac{1}{\phi(B)} \left(\fint_B |f(y)|^p \, dy \right)^{1/p},$$

where the suprema are taken over all balls B in $\mathbb{R}^d$.

Then the functional $\|f\|_{\mathcal{L}_{p,\phi}}$ is a norm modulo constant functions and thereby $\mathcal{L}_{p,\phi}(\mathbb{R}^d)$ is a Banach space. The functional $\|f\|_{L_{p,\phi}}$ is a norm and thereby $L_{p,\phi}(\mathbb{R}^d)$ is a Banach space.

For a ball $B = B(x,r)$ and a positive constant k we denote $B(x,kr)$ by kB. For a measurable set $G \subset \mathbb{R}^d$, we denote by $|G|$ and χ_G the Lebesgue measure of G and the characteristic function of G, respectively.

We say that a function $\theta : \mathbb{R}^d \times (0,\infty) \to (0,\infty)$ satisfies the doubling condition (resp. nearness condition) if there exists a positive constant C such that, for all $x, y \in \mathbb{R}^d$ and $r, s \in (0,\infty)$,

$$\frac{1}{C} \le \frac{\theta(x,r)}{\theta(x,s)} \le C, \quad \text{if } \frac{1}{2} \le \frac{r}{s} \le 2 \tag{DC}$$
$$\left(\text{resp. } \frac{1}{C} \le \frac{\theta(x,r)}{\theta(y,r)} \le C, \quad \text{if } |x-y| \le r \right). \tag{NC}$$

We say that θ is almost increasing (resp. almost decreasing) if there exists a positive constant C such that, for all $x \in \mathbb{R}^d$ and $r, s \in (0,\infty)$,

$$\theta(x,r) \le C\theta(x,s), \quad \text{if } r < s \tag{AI}$$
$$(\text{resp. } C\theta(x,r) \ge \theta(x,s), \quad \text{if } r < s). \tag{AD}$$

In this paper we consider the following class of ϕ:

Definition 2.2. For $p \in [1,\infty)$, let $\mathcal{G}_p$ be the set of all functions $\phi : \mathbb{R}^d \times (0,\infty) \to (0,\infty)$ such that $r \mapsto r^{d/p}\phi(x,r)$ is almost increasing and that $r \mapsto \phi(x,r)/r$ is almost decreasing. That is, there exists a positive constant C such that, for all $x \in \mathbb{R}^d$ and $r, s \in (0,\infty)$,

$$r^{d/p}\phi(x,r) \le Cs^{d/p}\phi(x,s), \quad C\phi(x,r)/r \ge \phi(x,s)/s, \quad \text{if } r < s.$$

Let $\mathcal{G}^{\rm inc}$ be the set of all functions $\phi : \mathbb{R}^d \times (0,\infty) \to (0,\infty)$ such that ϕ is almost increasing and that $r \mapsto \phi(x,r)/r$ is almost decreasing. Let $\mathcal{G}_p^{\rm dec}$ be the set of all functions $\phi : \mathbb{R}^d \times (0,\infty) \to (0,\infty)$ such that $r \mapsto r^{d/p}\phi(x,r)$ is almost increasing and that ϕ is almost decreasing.

Then $\mathcal{G}^{\rm inc} \cup \mathcal{G}_p^{\rm dec} \subset \mathcal{G}_p \subset \mathcal{G}_1$ for $p \in [1,\infty)$. If $\phi \in \mathcal{G}_1$, then ϕ satisfies the doubling condition (DC).

For a function $\lambda(\cdot) : \mathbb{R}^d \to (-\infty,\infty)$, let

$$\lambda_- = \inf_{x\in\mathbb{R}^d} \lambda(x), \quad \lambda_+ = \sup_{x\in\mathbb{R}^d} \lambda(x).$$

We say that a function $\lambda(\cdot) : \mathbb{R}^d \to (-\infty,\infty)$ is log-Hölder continuous if there exists a positive constant $C_{\lambda(\cdot)}$ such that

$$|\lambda(x) - \lambda(y)| \le \frac{C_{\lambda(\cdot)}}{\log(e/|x-y|)} \quad \text{for} \quad 0 < |x-y| < 1.$$

For a log-Hölder continuous function $\lambda(\cdot) : \mathbb{R}^d \to (-\infty,\infty)$ such that $\lambda_-, \lambda_+ \in (-\infty,\infty)$, and for a constant λ_* in $(-\infty,\infty)$, let

$$\phi(x,r) = \begin{cases} r^{\lambda(x)}, & 0 < r < 1/2, \\ r^{\lambda_*}, & 1/2 \le r < \infty. \end{cases}$$

Then ϕ satisfies (DC) and (NC), see [3, Proposition 3.3]. Moreover, if $\lambda_-, \lambda_+, \lambda_* \in [-d/p, 1]$, then $\phi \in \mathcal{G}_p$. If $\lambda_-, \lambda_+, \lambda_* \in [-d/p, 0]$, then $\phi \in \mathcal{G}_p^{\rm dec}$. If $\lambda_-, \lambda_+, \lambda_* \in [0,1]$, then $\phi \in \mathcal{G}^{\rm inc}$.

Remark 2.1. Let $p \in [1,\infty)$. If $\phi \in \mathcal{G}_p$ satisfies (NC), then $C^\infty_{\rm comp}(\mathbb{R}^d) \subset \mathcal{L}_{p,\phi}(\mathbb{R}^d)$, see [5, Proposition 6.4]. If $\phi \in \mathcal{G}_p^{\rm dec}$ satisfies (NC), then $L^\infty_{\rm comp}(\mathbb{R}^d) \subset L_{p,\phi}(\mathbb{R}^d)$, see [4, 6].

Next we state the definitions of $H^{[\phi,q]}(\mathbb{R}^d)$ and $B^{[\phi,q]}(\mathbb{R}^d)$.

Definition 2.3 ($[\phi,q]$-atom and $[\phi,q]$-block)**.** Let $\phi : \mathbb{R}^d \times (0,\infty) \to (0,\infty)$ and $1 < q \le \infty$. A function a on $\mathbb{R}^d$ is called a $[\phi,q]$-atom if there exists a ball B, which is called the corresponding ball of a, such that

(i) $\operatorname{supp} a \subset B$,

(ii) $\|a\|_{L^q} \le \dfrac{1}{|B|^{1/q'}\phi(B)}$,

(iii) $\displaystyle\int_{\mathbb{R}^d} a(x)\,dx = 0$,

where $\|a\|_{L^q}$ is the L^q norm of a and $1/q+1/q'=1$. Denote by $A[\phi,q]$ the set of all $[\phi,q]$-atoms. A function a on $\mathbb{R}^d$ is called a $[\phi,q]$-block if there exists a ball B such that the above (i) and (ii) hold. Denote by $B[\phi,q]$ the set of all $[\phi,q]$-blocks.

If a is a $[\phi,q]$-atom and B is its corresponding ball, then

$$\begin{aligned}\left|\int_{\mathbb{R}^d} a(x)g(x)\,dx\right| &= \left|\int_B a(x)(g(x)-g_B)\,dx\right| \\ &\le \|a\|_{L^q}\left(\int_B |g(x)-g_B|^{q'}\,dx\right)^{1/q'} \\ &\le \frac{1}{\phi(B)}\left(\frac{1}{|B|}\int_B |g(x)-g_B|^{q'}\,dx\right)^{1/q'} \le \|g\|_{\mathcal{L}_{q',\phi}}. \qquad (2.1)\end{aligned}$$

That is, the mapping $g \mapsto \int ag$ is a bounded linear functional on $\mathcal{L}_{q',\phi}(\mathbb{R}^d)$ with norm not exceeding 1. Similarly, if a is a $[\phi,q]$-block and B is its corresponding ball, then the mapping $g \mapsto \int ag$ is also a bounded linear functional on $L_{q',\phi}(\mathbb{R}^d)$ with norm not exceeding 1.

Definition 2.4 ($H^{[\phi,q]}(\mathbb{R}^d)$ and $B^{[\phi,q]}(\mathbb{R}^d)$)**.** Let $\phi:\mathbb{R}^d\times(0,\infty)\to(0,\infty)$, $q\in(1,\infty]$ and $1/q+1/q'=1$.

(i) Assume that $\mathcal{L}_{q',\phi}(\mathbb{R}^d)\ne\{0\}$. Define the space $H^{[\phi,q]}(\mathbb{R}^d)\subset\left(\mathcal{L}_{q',\phi}(\mathbb{R}^d)\right)^*$ as follows:

$f\in H^{[\phi,q]}(\mathbb{R}^d)$ if and only if there exist sequences $\{a_j\}\subset A[\phi,q]$ and positive numbers $\{\lambda_j\}$ such that

$$f=\sum_j \lambda_j a_j \text{ in } \left(\mathcal{L}_{q',\phi}(\mathbb{R}^d)\right)^* \quad\text{and}\quad \sum_j \lambda_j<\infty, \qquad (2.2)$$

(ii) Assume that $L_{q',\phi}(\mathbb{R}^d)\ne\{0\}$. Define the space $B^{[\phi,q]}(\mathbb{R}^d)\subset\left(L_{q',\phi}(\mathbb{R}^d)\right)^*$ as follows:

$f\in B^{[\phi,q]}(\mathbb{R}^d)$ if and only if there exist sequences $\{a_j\}\subset B[\phi,q]$ and positive numbers $\{\lambda_j\}$ such that

$$f=\sum_j \lambda_j a_j \text{ in } \left(L_{q',\phi}(\mathbb{R}^d)\right)^* \quad\text{and}\quad \sum_j \lambda_j<\infty. \qquad (2.3)$$

Define

$$\|f\|_{H^{[\phi,q]}} = \inf \sum_j \lambda_j \quad \text{and} \quad \|f\|_{B^{[\phi,q]}} = \inf \sum_j \lambda_j,$$

where the infima are taken over all expressions as in (2.2) and in (2.3), respectively.

Then $\|f\|_{H^{[\phi,q]}}$ and $\|f\|_{B^{[\phi,q]}}$ are norms and $H^{[\phi,q]}(\mathbb{R}^d)$ and $B^{[\phi,q]}(\mathbb{R}^d)$ are Banach spaces, respectively.

Remark 2.2. If $\phi \in \mathcal{G}_{q'}^{\mathrm{dec}}$ satisfies (NC), then the sum $f = \sum_j \lambda_j a_j$ in (2.2) or (2.3) converges a.e. and f is in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$, see [4, 6]

We denote by $H_0^{[\phi,q]}(\mathbb{R}^d)$ (resp. $B_0^{[\phi,q]}(\mathbb{R}^d)$) the space of all finite linear combinations of $[\phi, q]$-atoms (resp. $[\phi, q]$-blocks). Then $H_0^{[\phi,q]}(\mathbb{R}^d)$ (resp. $B_0^{[\phi,q]}(\mathbb{R}^d)$) is dense in $H^{[\phi,q]}(\mathbb{R}^d)$ (resp. $B^{[\phi,q]}(\mathbb{R}^d)$).

The following results are known.

Theorem 2.1 ([2])**.** *Let $\phi : \mathbb{R}^d \times (0,\infty) \to (0,\infty)$, $p \in [1,\infty)$, $q \in (1,\infty]$ and $1/p + 1/q = 1$. Assume that $\mathcal{L}_{p,\phi}(\mathbb{R}^d) \neq \{0\}$. If $q = \infty$, then assume also that ϕ is in $\mathcal{G}^{\mathrm{inc}}$ and satisfies* (NC)*. Then*

$$\left(H^{[\phi,q]}(\mathbb{R}^d)\right)^* = \mathcal{L}_{p,\phi}(\mathbb{R}^d).$$

More precisely, given $g \in \mathcal{L}_{p,\phi}(\mathbb{R}^d)$, the mapping

$$L_g : f \mapsto \int_{\mathbb{R}^d} f(x)g(x)\,dx \quad \text{for} \quad f \in H_0^{[\phi,q]}(\mathbb{R}^d), \tag{2.4}$$

can be extended on the entire $H^{[\phi,q]}(\mathbb{R}^d)$. Conversely, for every bounded linear functional L on $H^{[\phi,q]}(\mathbb{R}^d)$, there exists $g \in \mathcal{L}_{p,\phi}(\mathbb{R}^d)$ such that L is realized as L_g in (2.4)*. The linear functional norm of L_g is equivalent to $\|g\|_{\mathcal{L}_{p,\phi}}$.*

Remark 2.3. If $\phi \in \mathcal{G}^{\mathrm{inc}}$ satisfies (NC), then $\mathcal{L}_{1,\phi}(\mathbb{R}^d) = \mathcal{L}_{p,\phi}(\mathbb{R}^d)$ and $H^{[\phi,\infty]}(\mathbb{R}^d) = H^{[\phi,q]}(\mathbb{R}^d)$ for any $p, q \in (1,\infty)$, see [2, Theorems 3.1 and 3.3].

Theorem 2.2 ([2, 4])**.** *Let $\phi : \mathbb{R}^d \times (0,\infty) \to (0,\infty)$, $p, q \in (1,\infty)$ and $1/p+1/q = 1$. Assume that $L_{p,\phi}(\mathbb{R}^d) \neq \{0\}$. Then*

$$\left(B^{[\phi,q]}(\mathbb{R}^d)\right)^* = L_{p,\phi}(\mathbb{R}^d).$$

More precisely, given $g \in L_{p,\phi}(\mathbb{R}^d)$, the mapping

$$L_g : f \mapsto \int_{\mathbb{R}^d} f(x)g(x)\,dx \quad \text{for} \quad f \in B_0^{[\phi,q]}(\mathbb{R}^d), \tag{2.5}$$

can be extended on the entire $B^{[\phi,q]}(\mathbb{R}^d)$. Conversely, for every bounded linear functional L on $B^{[\phi,q]}(\mathbb{R}^d)$, there exists $g \in L_{p,\phi}(\mathbb{R}^d)$ such that L is realized as L_g in (2.5)*. The linear functional norm of L_g is equal to $\|g\|_{L_{p,\phi}}$.*

Remark 2.4. In the above theorems, if $p = 1$ and $q = \infty$, then we can prove that

$$\big(H^{[\phi,\infty]}(\mathbb{R}^d)\big)^* \supset \mathcal{L}_{1,\phi}(\mathbb{R}^d), \quad \text{and} \quad \big(B^{[\phi,\infty]}(\mathbb{R}^d)\big)^* \supset L_{1,\phi}(\mathbb{R}^d)$$

by the same way as in their proofs, respectively.

At the end of this section we state a theorem in the functional analysis. Let X be a normed space. A subspace G of X^* is called total if $g(x) = 0$ for all $g \in G$ implies $x = 0$. In this case the space G is called a total space of functionals on X. A subspace G of X^* is called to be X-dense in X^* if, for any $g \in X^*$, there exists a sequence $\{g_n\}$ in G such that $\lim_n g_n(x) = g(x)$ for each $x \in X$.

Theorem 2.3 ([1, V.7.41 (page 439)])**.** *Let X be a normed space. Then G is X-dense in X^* if and only if G is a total space of functionals on X.*

3 Main results

In this section we give the main results.

Theorem 3.1. *For $g \in \mathcal{L}_{1,\phi}(\mathbb{R}^d)$, let L_g be defined by* (2.4)*. If $\phi \in \mathcal{G}_1$ satisfies* (NC)*, then $\mathcal{L}_{1,\phi}(\mathbb{R}^d)$ is $H_0^{[\phi,\infty]}$-dense in $\big(H_0^{[\phi,\infty]}(\mathbb{R}^d)\big)^* = \big(H^{[\phi,\infty]}(\mathbb{R}^d)\big)^*$ and*

$$\frac{1}{4}\|g\|_{\mathcal{L}_{1,\phi}} \le \|L_g\|_{(H^{[\phi,\infty]})^*} \le \|g\|_{\mathcal{L}_{1,\phi}} \quad \text{for all} \quad g \in \mathcal{L}_{1,\phi}(\mathbb{R}^d). \tag{3.1}$$

Moreover, if $\phi \in \mathcal{G}_1^{\mathrm{dec}}$, then $\mathcal{L}_{1,\phi}(\mathbb{R}^d)$ is $H^{[\phi,\infty]}$-dense, i.e., weak-dense, in $\big(H^{[\phi,\infty]}(\mathbb{R}^d)\big)^*$.*

Remark 3.1. In general, let Z be a dense subspace of a Banach space X. Then Z^* and X^* are isometrically isomorphic, but that the isometric isomorphism between them is not a homeomorphism with the Z topology of Z^* and the X topology of X^* unless $X = Z$, see [1, V.7.40 (page 438)]. In the case that $\phi \in \mathcal{G}_1 \setminus (\mathcal{G}_1^{\mathrm{dec}} \cup \mathcal{G}^{\mathrm{inc}})$, it is unknown whether $\mathcal{L}_{1,\phi}(\mathbb{R}^d)$ is $H^{[\phi,\infty]}$-dense (weak*-dense) in $\big(H^{[\phi,\infty]}(\mathbb{R}^d)\big)^*$ or not.

For the Morrey space we have the following theorem.

Theorem 3.2. *For $g \in L_{1,\phi}(\mathbb{R}^d)$, let L_g be defined by* (2.5)*. If $\phi \in \mathcal{G}_1^{\mathrm{dec}}$ satisfies* (NC)*, then $L_{1,\phi}(\mathbb{R}^d)$ is $B^{[\phi,\infty]}$-dense, i.e., weak*-dense, in $\big(B^{[\phi,\infty]}(\mathbb{R}^d)\big)^*$ and*

$$\|L_g\|_{(B^{[\phi,\infty]})^*} = \|g\|_{L_{1,\phi}} \quad \text{for all} \quad g \in L_{1,\phi}(\mathbb{R}^d). \tag{3.2}$$

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