Simple, New Type and Best Possible Integral Inequalities and Hardy-type Inequalities

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Abstract: In this note, we give simple, new type and best possible integral inequalities for Sobolev spaces by using the theory of reproducing kernels. Further, we refer to Hardy-type inequalities by elementary method with some general and valuable coefficients for the first order Sobolev Hilbert spaces. Furthermore, up to date information is introduced simply, that is, A. Yamada and Q. Guan's recent results.

Key Words: Extension and restriction of functions, Sobolev space, Fourier transform, inequality, reproducing kernel, norm inequality, Hardy type inequality. A. Yamada's recent results, Q. Guan's recent results.

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1 Introduction and restriction of reproducing kernel Hilbert spaces

In order to show the basic background of this note, we recall the restriction and extension of reproducing kernel Hilbert spaces.

We consider a positive definite quadratic form function (reproducing kernel) $K : E \times E \to \mathbb{C}$. We consider restriction of K to $E_0 \times E_0$, where E_0 is a subset of E. Of course, the restriction is again a positive definite quadratic form function on the subset $E_0 \times E_0$. We shall consider the relation between the two reproducing kernel Hilbert spaces.

Theorem A. ([19], pages 78-80). *Let E_0 be a subset of E. Then the Hilbert space that $K|E_0 \times E_0 : E_0 \times E_0 \to \mathbb{C}$ defines is given by:*

$$H_{K|E_0\times E_0}(E_0) = \{f \in \mathcal{F}(E_0) \,:\, f = \tilde{f}|E_0 \text{ for some } \tilde{f} \in H_K(E)\}. \tag{1.1}$$

Furthermore, the norm is expressed in terms of the one of $H_K(E)$:

$$\|f\|_{H_{K|E_0\times E_0}(E_0)} = \min\{\|\tilde{f}\|_{H_K(E)} \,:\, \tilde{f} \in H_K(E),\, f = \tilde{f}|E_0\}. \tag{1.2}$$

In Theorem A, note that the inequality, for any function $f \in H_K(E)$

$$\|f\|_{H_{K|E_0\times E_0}(E_0)} \leq \|f\|_{H_K(E)}, \tag{1.3}$$

that is, the restriction map is a bounded linear operator.

In addition, for the minimum extension formula we have the general formula in Theorem A,

$$f(p) = (f|E_0(\cdot), K(\cdot, p))_{H_{K|E_0\times E_0}(E_0)},$$

for the minimum norm extension f of $f|E_0$. See the proof of Proposition 2.5 in [19] (pages 79-80), in particular, (2.48).

With these strong motivations, we gave the realization of restricted reproducing kernels in [20] for some Sobolev Hilbert spaces.

The space $H_S(\mathbb{R})$ is comprising of absolutely continuous functions f on $\mathbb{R}$ with the norm

$$\|f\|_{H_S(\mathbb{R})} \equiv \sqrt{\int_{\mathbb{R}} (|f(x)|^2 + |f'(x)|^2) dx}. \tag{1.4}$$

The Hilbert space $H_S(\mathbb{R})$ admits the reproducing kernel (Green function)

$$K(x, y) \equiv \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{1+\xi^2} \exp(i(x-y)\xi) d\xi = \frac{1}{2} e^{-|x-y|} \quad (x, y \in \mathbb{R}). \tag{1.5}$$

Its restriction to the closed interval $[a, b]$ is the reproducing kernel Hilbert space $H_S[a, b] = W^{1,2}[a, b]$ as a set of functions, and the norm is given by

$$\|f\|_{H_S[a,b]} \equiv \sqrt{\left(\int_a^b (|f(x)|^2 + |f'(x)|^2)\, dx\right) + |f(a)|^2 + |f(b)|^2}. \tag{1.6}$$

([19], pages 10-16).

In particular, we obtain the best possible inequality:

$$\int_{\mathbb{R}} (|f(x)|^2 + |f'(x)|^2)dx \geq \left(\int_a^b (|f(x)|^2 + |f'(x)|^2)\, dx \right) + |f(a)|^2 + |f(b)|^2. \tag{1.7}$$

We obtained several realizations of restricted reproducing kernel Hilbert spaces as in (1.6), however, they are, in general, involved. See [17], [19]. The formula (1.6) is a simple result, however, the realization of the restricted reproducing kernel spaces is, in general, complicated in this sense.

For the Sobolev Hilbert space $W^{2,2}(\mathbb{R})$ defined to be the completion of $C_c^\infty(\mathbb{R})$ with respect to the norm:

$$\|f\|_{W^{2,2}(\mathbb{R})} = \sqrt{\|f''\|_{L^2(\mathbb{R})}{}^2 + 2\|f'\|_{L^2(\mathbb{R})}{}^2 + \|f\|_{L^2(\mathbb{R})}{}^2},$$

we have the reproducing kernel

$$G(s,t) \equiv \frac{1}{4} e^{-|s-t|}(1 + |s-t|) \quad (s, t \in \mathbb{R})$$

([19], pages 21-22).

For simplicity, we shall consider functions in real valued functions.

We looked the reproducing kernel Hilbert space $W_S([a,b]), (a < b)$ admitting the restricted reproducing kernel $G(s,t)$ to the interval $[a,b]$:

$$\begin{aligned} \|f\|^2_{W_S([a,b])} &= \|f''\|^2_{L^2([a,b])} + 2\|f'\|^2_{L^2([a,b])} + \|f\|^2_{L^2([a,b])} \\ &+2(f(a)^2 - f(a)f'(a) + f'(a)^2) + 2(f(b)^2 - f(b)f'(b) + f'(b)^2). \end{aligned} \tag{1.8}$$

Let

$$K(s,t) \equiv \int_0^\infty \frac{\cos(s\,u)\cos(t\,u)}{u^2+1}\, du = \frac{\pi}{4} (\exp(-|s-t|) + \exp(-s-t)) \tag{1.9}$$

for $s, t > 0$. Then $H_K(0,\infty) = W^{1,2}(0,\infty)$ as a set of functions and the norm is given by:

$$\|f\|_{H_K(0,\infty)} = \sqrt{\frac{2}{\pi} \int_0^\infty (|f'(u)|^2 + |f(u)|^2)\, du} \tag{1.10}$$

([19], pages 12-13). From the restriction of the kernel $K(s,t)$ to $[a,b], a>0$, we have the realization of the norm

$$\|f\|^2_{H_K[a,b]} = \frac{2}{\pi}\frac{1-\exp(-2a)}{1+\exp(-2a)}|f(a)|^2 \tag{1.11}$$

$$+\frac{2}{\pi}\int_a^b (|f'(u)|^2+|f(u)|^2)\,du + \frac{2}{\pi}|f(b)|^2.$$

Let

$$K(s,t) \equiv \int_0^\infty \frac{\sin(s\,u)\sin(t\,u)}{u^2+1}\,du = \frac{\pi}{4}\left(\exp(-|s-t|)-\exp(-s-t)\right)$$

for $s,t>0$.

Then we have

$$H_K(0,\infty) = \{f\in \mathrm{AC}(0,\infty)\,:\, f(0)=0\} \tag{1.12}$$

as a set of functions and the norm is given by

$$\|f\|_{H_K(0,\infty)} = \sqrt{\frac{2}{\pi}\int_0^\infty (|f'(u)|^2+|f(u)|^2)\,du} \tag{1.13}$$

([19], pages 13-14). For the restriction of the kernel $K(s,t)$ to $[a,b], a>0$, we have the realization of the norm

$$\|f\|^2_{H_K[a,b]} = \frac{2}{\pi}\frac{1+\exp(-2a)}{1-\exp(-2a)}|f(a)|^2 \tag{1.14}$$

$$+\frac{2}{\pi}\int_a^b (|f'(u)|^2+|f(u)|^2)\,du + \frac{2}{\pi}|f(b)|^2.$$

Let

$$K(s,t) \equiv \min(s,t) \quad (s,t>0). \tag{1.15}$$

Then we have

$$H_K(0,\infty) = \left\{f\in W^{1,2}(0,\infty)\,:\, \lim_{\varepsilon\downarrow 0} f(\varepsilon)=0\right\} \tag{1.16}$$

as a set of functions and the norm is given by

$$\|f\|_{H_K(0,\infty)} = \sqrt{\int_0^\infty |f'(u)|^2 \, du} \tag{1.17}$$

([19], pages 14-15). For the restriction of the kernel $K(s,t)$ to $[a,b], a > 0$, we have the realization of the norm

$$\|f\|^2_{H_K[a,b]} = \frac{1}{a}|f(a)|^2 + \int_a^b |f'(u)|^2 \, du. \tag{1.18}$$

Furthermore, we recall the following two reproducing kernel Hilbert spaces, see pages 104-105 in [19]:

Let $H_K[0,\infty)$ be the set of all absolutely continuous functions f on $(0,\infty)$ such that f and its derivative f' satisfy

$$\lim_{x \downarrow 0} f(x) = 0 \text{ and } \int_0^\infty |f'(x)|^2 e^x \, dx < \infty.$$

Then, $H_K[0,\infty)$ is a reproducing kernel Hilbert space admitting the reproducing kernel kernel $K(x,y) = 1 - e^{-\min(x,y)}$.

Likewise let $H_K(-\infty, 0]$ be the set of all absolutely continuous functions f on $(-\infty, 0)$ such that f and its derivative f' satisfy

$$\lim_{x \uparrow 0} f(x) = 0 \text{ and } \int_{-\infty}^0 |f'(x)|^2 e^{-x} \, dx < \infty.$$

Then $H_K(-\infty, 0]$ is also a reproducing kernel Hilbert space admitting the reproducing kernel $K(x,y) = 1 - e^{\max(x,y)}$.

Then, we obtain the similar results for the restriction reproducing kernel Hilbert spaces as follows.

For any interval $[a,b], a > 0$

$$\|f\|^2_{H_K[a,b]} = \frac{f(a)^2}{1 - \exp(-a)} + \int_a^b |f'(x)|^2 e^x \, dx. \tag{1.19}$$

For any interval $[a,b], b < 0$,

$$\|f\|^2_{H_K[a,b]} = \frac{f(b)^2}{1 - \exp b} + \int_a^b |f'(x)|^2 e^{-x} \, dx. \tag{1.20}$$

The function

$$k_a(x,y) = \log\frac{\min(x,y)}{a}, a > 0$$

is the reproducing kernel for the one dimensional Sobolev space with finite norms

$$\int_a^\infty f'(x)^2 x dx$$

satisfying

$$f(a) = 0,$$

([16]). We can see that the restriction of the kernel to $[b,c], 0 < a < b < c$ admits the norm square

$$\frac{f(b)^2}{\log(b/a)} + \int_b^c f'(x)^2 x dx$$

and so we obtain the norm inequality

$$\frac{f(b)^2}{\log(b/a)} + \int_b^c f'(x)^2 x dx \leq \int_a^\infty f'(x)^2 x dx. \tag{1.21}$$

We have many type Sobolev Hilbert spaces. For example,
for $\omega^2 = \gamma^2 - \alpha^2 > 0$, the kernel

$$K(s,t) = \frac{\exp(-\alpha|s-t|)}{4\alpha\gamma^2}\cos(\omega|s-t|) + \frac{\alpha}{\omega}\sin(\omega|s-t|)$$

is the reproducing kernel for the Sobolev Hilbert space admitting the norm

$$||u||^2 = 4\alpha\gamma^2 u(a)^2 + 4\alpha u'(a)^2$$

$$+ \int_a^b \left(u''(t) + 2\alpha^2 u'(t) + \gamma^2 u(t)\right)^2 dt$$

(E. Parzen, [12]).

See also [1] and the recent paper A. Yamada ([21]).

2 Results

From the above results, we obtain the simple best possible integral inequalities

Theorem 2.1.

For any function $f \in H_K[a,b], a > 0$ in (1.18), we have

$$\int_a^b |f'(x)|^2 \, dx \geq \frac{1}{b}|f(b)|^2 - \frac{1}{a}|f(a)|^2. \tag{2.1}$$

For any function $f \in H_K[a,b]$ in (1.6), we have

$$\int_a^b \left(|f'(x)|^2 + |f(x)|^2\right) \, dx \geq \pm \left(|f(b)|^2 - |f(a)|^2\right). \tag{2.2}$$

For any function $f \in H_K[a,b], a > 0$ in (1.11) or in (1.13), we have

$$\int_a^b (|f'(x)|^2 + |f(x)|^2) \, dx \tag{2.3}$$

$$\geq \left(\frac{1+\exp(-2b)}{1-\exp(-2b)}|f(b)|^2 - \frac{1+\exp(-2a)}{1-\exp(-2a)}|f(a)|^2\right), |f(a)|^2 - |f(b)|^2$$

For any function $f \in H_K[a,b]$ in (1.8), we have

$$\int_a^b \left(|f''(x)|^2 + 2|f'(x)|^2 + |f(x)|^2\right) dx \tag{2.4}$$

$$\geq \pm \left(2(f(b)^2 - f(b)f'(b) + f'(b)^2) - 2(f(a)^2 - f(a)f'(a) + f'(a)^2)\right).$$

For any interval $[a,b], a > 0$

$$\int_0^a |f'(x)|^2 e^x \, dx \geq \frac{f(a)^2}{1-\exp(-a)}. \tag{2.5}$$

For any interval $[a,b], b < 0$,

$$\int_b^0 |f'(x)|^2 e^{-x} \, dx \geq \frac{f(b)^2}{1-\exp b}. \tag{2.6}$$

For any interval $[b,c], 0 < a \leq b < c$, we obtain inequality

$$\frac{f(c)^2}{\log(c/a)} - \frac{f(b)^2}{\log(b/a)} \leq \int_b^c f'(x)^2 x dx. \tag{2.7}$$

For example, for (2.2) in (1.6), for $a < b < c$, by considering the extension of $H_K[a,b]$ to $H_K[a,c]$

$$\|f\|^2_{H_K[a,c]} \geq \|f\|^2_{H_K[b,c]},$$

that is,

$$\int_a^c \left(|f(x)|^2 + |f'(x)|^2\right) dx + f(a)^2 + f(c)^2$$

$$\geq \int_b^c \left(|f(x)|^2 + |f'(x)|^2\right) dx + f(b)^2 + f(c)^2$$

and the desired result in $+$ sign.

Meanwhile, with $c < a < b$ from

$$\|f\|^2_{H_K[c,b]} \geq \|f\|^2_{H_K[c,a]},$$

we have the desired result with $-$ sign.

Other results may be given similarly.

We obtained the new type fundamental norm inequalities for Sobolev Hilbert spaces, and we can apply the inequalities in this way, for example

$$\frac{1}{b}|f(b)|^2 \leq \frac{1}{a}|f(a)|^2 + \int_a^b |f'(x)|^2 \, dx.$$

3 Elementary norm inequalities

We shall consider elementary norm inequalities in connection with the above results. For this purpose we consider the integral

$$\|f\|^2_{H[a,b;A,B]} \equiv \int_a^b (A(x)f(x) + B(x)f'(x))^2 \, dx. \tag{3.1}$$

Here, we can consider continuously differentiable functions $A(x)$ and $B(x)$ on $[a,b]$ and for the Sobolev space $H_S[a,b]$ of the first order. Then, by partial integral formula, we have

$$\|f\|^2_{H[a,b;A,B]} \tag{3.2}$$

$$= \int_a^b \alpha(x)|f(x)|^2 dx + \int_a^b B^2(x)|f'(x)|^2 dx$$
$$+ \left[A(x)B(x)f^2(x)\right]_a^b .$$

Here,

$$\alpha(x) = A^2(x) - A'(x)B(x) - A(x)B'(x).$$

3.1 $A(x) = A$ constant case

In this case, with A, X_B: constants and $\alpha(x) \geq 0$, by setting

$$B_X(x) = xA - \frac{1}{A}\int_a^x \alpha(\xi)d\xi + X_B,$$

we have the desired inequality

$$\int_a^b \alpha(x)|f(x)|^2 dx + \int_a^b B_X(x)^2|f'(x)|^2 dx$$
$$\geq A\left[B_X(a)f^2(a) - B_X(b)f^2(b)\right].$$

3.1.1 A: constant and $\alpha(x) = 0$ case

Then, with a constant X_B we have

$$B(x) = Ax + X_B,$$

and we have the inequality

$$\int_a^b (Ax + X_B)^2 |f'(x)|^2 dx$$
$$\geq A\left[(Aa + X_B)f^2(a) - (Ab + X_B)f^2(b)\right].$$

Example: For $\alpha(x) = \sin x$ with $a = 0, b = \pi$

Then, for the function

$$B_X(x) = xA - \frac{1}{A}(\cos x - 1) + X_B,$$

we have the inequality

$$\int_0^\pi \sin(x)|f(x)|^2 dx + B_X(x)^2 |f'(x)|^2 dx$$

$$\geq A\left[X_B f^2(0) - B_X(\pi) f^2(\pi)\right].$$

3.2 $B(x) = B$ constant case

In this case we have, for $\alpha(x) \geq 0, B \neq 0$ and constant X_A, by

$$A_X(x) = \alpha(x)^{1/2} \frac{1 + \exp\left(\frac{2}{B}\int_a^x \alpha(\xi)^{1/2} d\xi + X_A\right)}{1 - \exp\left(\frac{2}{B}\int_a^x \alpha(\xi)^{1/2} d\xi + X_A\right)},$$

we have the inequality

$$\int_a^b \alpha(x)|f(x)|^2 dx + \int_a^b B^2 |f'(x)|^2 dx$$

$$\geq B\left[A_X(a) f^2(a) - A_X(b) f^2(b)\right].$$

3.2.1 B: constant and $\alpha(x) = 0$ case

Then, we have

$$A_X(x) = -\frac{1}{(1/B)x + X_A},$$

with a constant X_A and we have the inequality

$$\int_a^b |f'(x)|^2 dx$$

$$\geq \frac{1}{B}\left[A_X(a) f^2(a) - A_X(b) f^2(b)\right].$$

3.3 $A(x), B(x)$ constants case A, B

Then, $\alpha(x) = A^2$ and we have the inequality

$$\int_a^b A^2|f(x)|^2dx + \int_a^b B^2|f'(x)|^2dx$$
$$\geq AB\left(f^2(a) - f^2(b)\right).$$

4 Hardy type's inequalities

The very famous and important Hardy's inequalities are stated typically in the following way:

Hardy's inequalities.

For $0 < b < \infty$ and for smooth functions f, vanishing around the origin

$$\frac{1}{4}\int_0^b |f(x)|^2x^{-2}dx \leq \int_0^b |f'(x)|^2dx.$$

Furthermore, for $\rho(s) \geq 0$ and $\rho'(s) \leq 0$ and for $-\infty < \alpha < 1$

$$\frac{(1-\alpha)^2}{4}\int_0^b s^{\alpha-2}|f(s)|^2\rho(s)ds \leq \int_0^b s^{\alpha}|f'(s)|^2\rho(s)ds.$$

See, for example, [2], Section 5.3 with many applications to partial differential operators. It will be very interesting that one book ([11, 10]) even was published with Hardy's inequalities.

From our method (3.1) we can obtain a general inequality with some natural and general weights of Hardy' type

Theorem 3.1. *For any function $\alpha(x) \leq 0$ on $[a, b]$, for a non-vanishing and continuously differentiable function $A(x)$ on $[a, b]$ and a constant C such that the integral*

$$B(x) = \frac{C}{A(x)} + \frac{1}{A(x)}\int_a^x (A^2(\xi) - \alpha(\xi))d\xi$$

exists and the function $B(x)$ is integrable, we obtain the Hardy type inequality

$$\int_a^b B^2(x)|f'(x)|^2 dx \geq \int_a^b (-\alpha(x))|f(x)|^2 dx$$
$$+ \left[A(x)B(x)f^2(x)\right]_a^b .$$

Meanwhile, for any continuously differentiable function $B(x)$ satisfying that for a continuously differentiable function $A(x)$

$$\alpha(x) := A^2(x) - (A(x)B(x))' \leq 0,$$

we have the above Hardy's type inequality.

Corollary 3.1. *In Section 3.1, for a function $\alpha(x)$ satisfying $\alpha(x) \leq 0$, we obtain the Hardy type inequality*

$$\int_a^b B_X(x)^2 |f'(x)|^2 dx$$
$$\geq \int_a^b (-\alpha(x))|f(x)|^2 dx + A\left[B_X(a)f^2(a) - B_X(b)f^2(b)\right],$$

for the function

$$B_X(x) = xA - \frac{1}{A}\int_a^x \alpha(\xi)d\xi + X_B.$$

Corollary 3.2. *In Section 3.1, for a constant A and for any function $B(x)$ satisfying on $[a, b]$*

$$0 \neq A \leq \min B'(x)$$

for the function

$$\alpha(x) = A^2 - AB'(x) \leq 0,$$

we obtain the Hardy type inequality

$$\int_a^b B(x)^2 |f'(x)|^2 dx$$
$$\geq \int_a^b (-\alpha(x))|f(x)|^2 dx + A\left[B(a)f^2(a) - B(b)f^2(b)\right].$$

Example: For $\alpha(x) = -\sin x$ with $a = 0, b = \pi$

Then, for the function

$$B_X(x) = xA - \frac{1}{A}(1 - \cos x) + X_B,$$

we have the inequality

$$\int_0^\pi B_X(x)^2 |f'(x)|^2 dx$$

$$\geq \int_0^\pi |f(x)|^2 \sin x dx + A \left[X_B f^2(0) - B_X(\pi) f^2(\pi) \right].$$

Example: For $\alpha(x) = -x^s, s > -1$ with $a = 0, b > 0$ and $A \equiv 1$, $C = 0$

.

Then, we have the inequality

$$\frac{1}{(s+1)^2} \int_0^b |f'(x)|^2 x^{2s+2} dx$$

$$\geq \int_0^b |f(x)|^2 x^s dx + \left[\frac{1}{s+1} x^{s+1} f^2(x) \right]_0^b.$$

For any real number s ($\neq -1$), for $0 < a < b$ we have the inequality

$$\frac{1}{(s+1)^2} \int_a^b |f'(x)|^2 \left(x^{s+1} - a^{s+1} \right)^2 dx$$

$$\geq \int_a^b |f(x)|^2 x^s dx + \left[\frac{1}{s+1} \left(x^{s+1} - a^{s+1} \right) f^2(x) \right]_a^b.$$

For $s = -1$ we have

$$\int_0^b |f'(x)|^2 \left(\log \frac{x}{a} \right)^2 dx$$

$$\geq \int_0^b |f(x)|^2 \frac{dx}{x} + \left[\log \frac{x}{a} f^2(x)\right]_0^b.$$

Example: For $\alpha(x) = -\log x$ with $a = 1 < b$ and $A \equiv 1$, $C = 0$

.

Then, we have the inequality

$$\int_1^b |f'(x)|^2 (x \log x - x + 1)^2 dx$$

$$\geq \int_1^b |f(x)|^2 \log x dx + \left[(x \log x - x + 1) f^2(x)\right]_1^b.$$

Example: For $\alpha(x) = -\log x$ with $0 < a, b < 1$ and $A \equiv 1$, $C = 0$

.

Then, we have the inequality

$$\int_a^b |f'(x)|^2 \left(x(\log x - 1) - a(\log a - 1)\right)^2 dx$$

$$\geq \int_a^b |f(x)|^2 (-\log x) dx + \left[\left(x(\log x - 1) - a(\log a - 1)\right) f^2(x)\right]_a^b.$$

5 For the second order Sobolev spaces

We consider the integral

$$\|f\|^2_{H[a,b;A,B,C]} \equiv \int_a^b \left(Af(x) + Bf'(x) + Cf''(x)\right)^2 dx. \tag{5.1}$$

Here, we can consider constants A, B, C and for the Sobolev space $H_S[a,b]$ of the second order. Then, by partial integral formula, we have

$$\|f\|^2_{H[a,b;A,B,C]} \tag{5.2}$$
$$= \int_a^b \left(A^2|f(x)|^2dx + \beta(x)|f'(x)|^2 + C^2|f''(x)|^2\right) dx$$
$$+ \left[ABf^2(x) + BCf'(x)^2 + 2CAf(x)f'(x)\right]_a^b.$$

Here,

$$\beta = B^2 - 2AC.$$

Therefore, for $\beta(x) \geq 0$, we have the inequality

$$\int_a^b \left(A^2|f(x)|^2dx + \beta|f'(x)|^2 + C^2|f''(x)|^2\right) dx$$
$$\geq \left[ABf^2(a) + BCf'(a)^2 + 2CAf(a)f'(a)\right]$$
$$- \left[ABf^2(b) + BCf'(b)^2 + 2CAf(b)f'(b)\right].$$

6 Up-to-date information

6.1 In connection with Yamada's recent results

For the simple statements, we assume the basic results on some general integral transforms on the framework of Hilbert spaces using the theory of reproducing kernels, Section 2.5 in [19] and the recent results by A. Yamada [22].

Let $\mathcal{H}$ be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. Let $\mathbf{h} : E \to \mathcal{H}$ be a fixed $\mathbf{h}$ valued mapping on E. Then, we shall consider the linear mapping L from $\mathbf{f} \in \mathcal{H}$ into $\mathcal{F}(E)$ defined by the following:

$$L\mathbf{f}(p) \equiv \langle \mathbf{f}, \mathbf{h}(p) \rangle_{\mathcal{H}}. \tag{6.1}$$

The key to consider this fundamental linear mapping is to form the two variables complex-valued function; that is, a positive definite quadratic form function:

$$K(p,q) \equiv \langle \mathbf{h}(q), \mathbf{h}(p) \rangle_{\mathcal{H}} \tag{6.2}$$

defined on $E \times E$. We denote by $\mathcal{R}(L)$ the linear function space of comprising all complex-valued functions of the images of $\mathbf{h}$ by L defined on E. In the image space $\mathcal{R}(L)$, we introduce the norm by

$$\|f\|_{\mathcal{R}(L)} = \inf\{\|\mathbf{f}\|_{\mathcal{H}} : \mathbf{f} \in \mathcal{H}, \quad f = L\mathbf{f}\}, \tag{6.3}$$

then the image space forms a Hilbert space and we obtain the fundamental theorem:

Theorem 2.37 ([19], 137). *The image space* $\{L\mathbf{f}\}_{\mathbf{f}\in\mathcal{H}}$ *by* (6.1) *of* $\mathcal{H}$ *is characterized as the reproducing kernel Hilbert space* $H_K(E)$ *determined by* K *in* (6.2) *and we have the inequality*

$$\|L\mathbf{f}\|_{H_K(E)} \leq \|\mathbf{f}\|_{\mathcal{H}} \quad (\mathbf{f} \in \mathcal{H}). \tag{6.4}$$

Furthermore, for any $f \in H_K(E)$, *there exists a uniquely determined vector* $\mathbf{f}^* \in \mathcal{H}$ *satisfying*:

$$f \equiv \langle \mathbf{f}^*, \mathbf{h}(\cdot)\rangle_{\mathcal{H}} \quad \text{on} \quad E \quad \text{and} \quad \|f\|_{H_K(E)} = \|\mathbf{f}^*\|_{\mathcal{H}}. \tag{6.5}$$

In this general situation, we examine the basic properties of the linear mapping (6.1), in particular, its inversion formula. However, its situation is very involved. For the inversion formula, A. Yamada [22], in particular, examined the linear mapping with the assumption that $\mathcal{H}$ is itself a reproducing kernel Hilbert space and he obtained very beautiful results. Furthermore, he examined the linear mapping (6.1) itself from the viewpoint of delicate situations.

However, many linear mappings (6.1) with physical meanings are not for any reproducing kernel Hilbert space as an input space. However, we will be interested in the linear mapping (6.1) with some smooth function spaces. Here we note that there exists a general method for this problem by considering input function spaces as a reproducing kernel Hilbert space. Note that any reproducing kernel Hilbert space may be considered as the image of a Hilbert space $\mathcal{H}$, by considering a decomposition (6.2) of a reproducing kernel in some way conversely.

We shall show the details with a typical example.

As a typical case, we consider the simple heat equation

$$u_t(x,t) = u_{xx}(x,t) \quad \text{on} \quad \mathbb{R} \times T_+ \quad (T_+ \equiv \{t > 0\}) \tag{6.6}$$

satisfying the initial condition

$$u_F(\cdot, 0) = F \in L^2(\mathbb{R}) \quad \text{on} \quad \mathbb{R}. \tag{6.7}$$

Using the Fourier transform, we obtain a representation of the solution $u_F(x,t)$

$$u_F(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} F(\xi) \exp\left(-\frac{(x-\xi)^2}{4t}\right) d\xi \tag{6.8}$$

at least in the formal sense. Apart from a classical and educative arguments about the properties of this transform, for any fixed $t > 0$, we examine the integral transform $F \mapsto u_F$ and we shall characterize the image function $u_F(x,t)$.

Let us write

$$k(x;t) \equiv \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right) \quad (x \in \mathbb{R},\ t > 0). \tag{6.9}$$

We define the kernel

$$K(x,x';t) \equiv \int_{\mathbb{R}} k(x-\xi;t)k(x'-\xi;t)d\xi = \frac{1}{2\sqrt{2\pi t}} \exp\left(-\frac{x^2}{8t} - \frac{x'^2}{8t} + \frac{xx'}{4t}\right) \tag{6.10}$$

according to (6.2).

The typical results of Section 2.5.3, in [19] are the following theorems and their corollary, which identify the RKHS $H_K(\mathbb{R})$:

Theorem 2.40: *Let $t > 0$. A function f takes the form $u_F(\cdot,t)$ for some $F \in L^2(\mathbb{R})$ if and only if f admits analytic extension $\tilde{f}$ to $\mathbb{C}$ and satisfies*

$$\sqrt{\frac{1}{\sqrt{2\pi t}} \iint_{\mathbb{C}} |\tilde{f}(x+iy)|^2 \exp\left(\frac{-y^2}{2t}\right) dxdy} < \infty. \tag{6.11}$$

In this case, $f \in H_K(\mathbb{R})$, where K is given by (6.10), and the norm is given by:

$$\|f\|_{H_K(\mathbb{R})} = \sqrt{\frac{1}{\sqrt{2\pi t}} \iint_{\mathbb{C}} |\tilde{f}(x+iy)|^2 \exp\left(-\frac{y^2}{2t}\right) dxdy}.$$

Theorem 2.41. *Let $t > 0$. In the integral transform $F \mapsto u_F(\cdot,t)$ of $L^2(\mathbb{R})$ functions F, the images $u_F(\cdot,t)$ extend analytically onto $\mathbb{C}$ to a* *function, which we still write $u_F(\cdot,t)$. Furthermore, we have the isometrical identities*

$$\int_{\mathbb{R}} |F(\xi)|^2 d\xi = \frac{1}{\sqrt{2\pi t}} \iint_{\mathbb{C}} |u_F(z,t)|^2 \exp\left(-\frac{y^2}{2t}\right) dxdy = \sum_{j=0}^{\infty} \frac{(2t)^j}{j!} \|u_F(\cdot,t)\|_{L^2(\mathbb{R})}{}^2 \tag{6.12}$$

for any fixed $t > 0$.

Corollary 2.7. *If a C^∞-function $f : \mathbb{R}^n \to \mathbb{C}$ has a finite integral on the most right-hand side in* (6.12)*, then f is extended analytically onto $\mathbb{C}$ and*

$$\sum_{j=0}^{\infty} \frac{(2t)^j}{j!} \int_{\mathbb{R}} |\partial_x^j f(x)|^2 dx = \frac{1}{\sqrt{2\pi t}} \iint_{\mathbb{C}} |f(x+iy)|^2 \exp\left(-\frac{y^2}{2t}\right) dxdy. \tag{6.13}$$

Now, as a simple smooth function space, we shall consider a reproducing kernel Hilbert space. The space $H_S(\mathbb{R})$ is comprising of absolutely continuous functions f on $\mathbb{R}$ with the norm

$$\|f\|_{H_S(\mathbb{R})} \equiv \sqrt{\int_{\mathbb{R}} (|f(x)|^2 + |f'(x)|^2) dx}. \tag{6.14}$$

The Hilbert space $H_S(\mathbb{R})$ admits the reproducing kernel (Green function)

$$K(x,y) \equiv \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{1+\xi^2} \exp(i(x-y)\xi) d\xi = \frac{1}{2} e^{-|x-y|} \quad (x, y \in \mathbb{R}). \tag{6.15}$$

The representation (6.15) means that the functions $f(x)$ of $H_S(\mathbb{R})$ are represented in the form

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{1+\xi^2} \exp(ix\xi) F(\xi) d\xi \tag{6.16}$$

with the functions $F(\xi)$ satisfying

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{1+\xi^2} |F(\xi)|^2 d\xi < \infty \tag{6.17}$$

and the norm is represented by

$$\|f\|_{H_S(\mathbb{R})} = \sqrt{\frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{1+\xi^2} |F(\xi)|^2 d\xi}.$$

By combining two integral transforms (6.8) and (6.15), we obtain the integral transform of the Sobolev space $H_S(\mathbb{R})$

$$u_F(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} F(\xi)\exp i\xi x \frac{\exp(-\xi^2 t)}{1+\xi^2} d\xi \tag{6.18}$$

for the space satisfying (6.16). Then, by the fundamental theorem (2.37) we obtain the corresponding theorem to Theorem (2.41):

Theorem 6.1. *Let $t > 0$. In the integral transform $F \mapsto u_F(\cdot,t)$ of the Sobolev space $H_S(\mathbb{R})$ functions F, the images $u_F(\cdot,t)$ extend analytically onto $\mathbb{C}$ to a function, which we still write $u_F(\cdot,t)$. Furthermore, we have the isometrical identities*

$$\int_{\mathbb{R}} |F(\xi)|^2 d\xi$$

$$= \frac{1}{\sqrt{2\pi t}} \iint_{\mathbb{C}} \left(|u_F(z,t)|^2 + \frac{\partial}{\partial z} u_F(z,t)|^2 \right) \exp\left(-\frac{y^2}{2t}\right) dxdy$$

$$= \sum_{j=0}^{\infty} \frac{(2t)^j}{j!} \left(\|u_F(\cdot,t)\|_{L^2(\mathbb{R})}{}^2 + \|\frac{\partial}{\partial \cdot} u_F(\cdot,t)\|_{L^2(\mathbb{R})}{}^2 \right)$$

for any fixed $t > 0$.

Meanwhile, in the Yamada's situation, we will consider the integral transform

$$\int_{\mathbb{R}} \left(F(\xi)k(x-\xi,t) + \frac{\partial}{\partial \xi}F(\xi)\frac{\partial}{\partial \xi}k(x-\xi,t) \right) d\xi$$

for the Sobolev space $H_S(\mathbb{R})$ and it does not have a physical sense.

6.2 Q, Guan's recent extensions

For two functions φ and ψ of $H_2(D)$ for any regular domain D, the analytic Hardy 2 space (Szegö space), we obtain the generalized isoperimetric inequality

$$\frac{1}{\pi}\iint_D |\varphi(z)\psi(z)|^2 dxdy \le \frac{1}{2\pi}\int_{\partial D} |\varphi(z)|^2 |dz| \frac{1}{2\pi}\int_{\partial D} |\psi(z)|^2 |dz|, \tag{6.19}$$

and so, we obtain the bounded linear operator from the Szegö space to the Bergman space

$$\iint_D |f(z)|^2 dxdy \leq \frac{l(\partial D)}{4\pi} \int_{\partial D} |f(z)|^2 |dz|, \tag{6.20}$$

for the length $l(\partial D)$ of the boundary ∂D([13]).

Note that the inequality (6.20) is a curious one in the sense that the Bergman norm is for analytic differentials, but the Szegö norm for half order differentials. In connection with this inequality, we recall the interesting best possible norm inequality:

$$\iint_D |f'(z)|^2 dxdy \leq \frac{1}{2} \int_{\partial D} \frac{|f'(z)dz|^2}{idW(z,t)} \tag{6.21}$$

$$= \frac{1}{2\pi} \int_{\partial D} |f'(z)|^2 \left(\frac{\partial G(z,t)}{\partial \nu_z} \right)^{-1} |dz|,$$

that means the relation between the Bergman norms and the weighted Szegö norm for (exact) differentials. Here, for the conjugate harmonic function $G^*(z,t)$ of the Green function $G(z,t)$ of D, $W(z,t) = G(z,t) + iG^*(z,t)$, and $idW(z,t)$ is a single-valued meromorphic differential and positive along the boundary ∂D and $\partial/\partial \nu_z$ is the inner normal derivative with respect to D. $\partial G(z,t)/\partial \nu_z$ is a positive continuous function on ∂D.

This inequality is not so simple to derive and for its proof we must examine deeply the relations among the Hardy reproducing kernel, its conjugate kernel and the Bergman kernel ([14]).

The conjugate Hardy space which is given by the left hand side of (6.3) was surprisingly generalized as the Ohsawa-Saitoh-Hardy space on n-dimensional complex manifolds by Q. Guan and Z. Yuan ([8]) through ([5, 6, 7]) with many concrete and deep results. Furthermore, surprisingly enough, further extensions were given recently in [9].

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