Rigged Hilbert space treatment for the bra-ket formalism of positive-definite metric non-Hermitian systems

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abstract

The rigged Hilbert space (RHS) treatment for the Dirac's bra-ket formalism in a non-Hermitian system with a positive-definite metric is reviewed. We propose an RHS related to the positive-definite metric induced from a positive operator η, called η-RHS. The nuclear spectral theorem for the η-RHS exhibits spectral expansions of the bra and kets by the generalized eigenvectors of an η-quasi Hermitian operator. We establish a complete bi-orthogonal system that endows the transformation theory between the Hermitian and non-Hermitian systems using the spectral expansions. The η-quasi Hermitian operator can be extended to the general operator on the bra-ket space while preserving its symmetric structure.

1 Introduction

A rigged Hilbert space (RHS), also called a Gel'fand's triplet, was introduced mathematically to correlating the distribution theory with the Hilbert space theory, by I. M. Gel'fand and his collaborators[1, 2]. This space has been utilized to formalize the Dirac's bra-ket notations in quantum mechanics because the mathematical formalism of them is considered insufficient within the context of von Neumann's Hilbert space theory[3]–[26]. Nowadays, the RHS is considered as an underlying space for describing quantum mechanics using the bra-ket notation. Actually, accurate elegant formulations for quantum mechanics based on the RHS are proposed for several systems, e.g., harmonic oscillator[9], resonance state (Gamow vectors)[10], and scattering problem[16]. Note that the RHS has also been applied to the fundamental study of complex eigenvalues of the operators treated in the context of statistical physics [11]–[14][21, 22].

The RHS is given the following triplet of topological vector space[17],

$$\Phi \subset \mathcal{H} \subset \Phi', \Phi^{\times}, \tag{1}$$

where $\mathcal{H} = (\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ is a complex Hilbert space, Φ is a dense linear subspace of $\mathcal{H}$ equipping a topology τ_Φ such that (Φ, τ_Φ) is a nuclear space. Φ' and $\Phi^{\times}$ are sets of continuous linear and *anti*-linear functionals on (Φ, τ_Φ), namely, dual and anti-dual spaces of (Φ, τ_Φ), respectively. The inner product $\langle \cdot, \cdot \rangle_\Phi$ on Φ becomes separately continuous on (Φ, τ_Φ), where $\langle \phi, \psi \rangle_\Phi \equiv \langle \phi, \psi \rangle_\mathcal{H}$ for $\phi, \psi \in \Phi$. (Note that by the Rieszs' familiar theorem, $\mathcal{H}$ can be identified with its dual $\mathcal{H}'$, $\mathcal{H} = \mathcal{H}'$; then, the relation $\mathcal{H}' \subset \Phi'$ holds.) In the RHS (1), the bra-ket vectors can be constructed using the following procedure[17] (see also Sec. 2). Given a quantum system, observables of the system that are described as Hermitian (self-adjoint) operators in $\mathcal{H}$ realizes the space Φ, and for the RHS, the bra and ket vectors are defined as the elements of Φ' and $\Phi^\times$, respectively. Then, the nuclear spectral theorem of the RHS provides the eigenequations of the bra and ket vectors individually and the spectral expansions of the bra and ket vectors of the observables. In the sense, the Dirac's bra-ket formalism is obtained completely.

Recently, the mathematical treatment for non-Hermitian quantum systems, such as, a parity-time ($\mathcal{PT}$)-symmetrical system, has been developed based on the RHS[19, 20, 23, 25, 26]. In a $\mathcal{PT}$-symmetrical system, a non-Hermitian Hamiltonian H is assumed to satisfy the following symmetry:

$$H = \eta H \eta^{-1}, \tag{2}$$

where η is given as the composition operator of the parity and time transformations[27, 28]. When the left-hand side of (2) is the adjoint $H^\dagger$ instead of H i.e., $H^\dagger = \eta H \eta^{-1}$ and the operator η is assumed to be a positive operator, such an operator H is referred to as an η-quasi Hermitian, and the operator describes the system with a positive definite-metric[29, 30, 31].

In this short article, we review the study of the RHS formalizing the Dirac's bra-ket notation for a non-Hermitian system with a positive definite metric[26] after introducing the bra-ket formalism mathematically constructed on the basis of the RHS.

2 Definitions, Statements, and Bra-ket formalism

2.1 Definitions and Statements

2-A. A *countable Hilbert space* (CHS) is a topological vector space constructed by the following steps.

- **(i)** Let a vector space Φ and let a system of inner products, $\{\langle \cdot, \cdot \rangle_n \ ; \ n = 1, 2, \cdots\}$, such that the induced norms $\| \cdot \|_n = \sqrt{\langle \cdot, \cdot \rangle_n}$ satisfy the inequalities $\| \cdot \|_n \leq \| \cdot \|_{n+1}$, $n = 1, 2, \cdots$, and each $(\Phi, \langle \cdot, \cdot \rangle_n)$ is a pre-Hilbert space.
- **(ii)** Induce a locally convex space (Φ, τ_Φ) from the family of the obtained norms $\{\| \cdot \|_n\}$.
- **(iii)** The completion of (Φ, τ_Φ) is called a CHS.

2-B. A compact operator T on a Hilbert space $\mathcal{H}_1$ into a Hilbert space $\mathcal{H}_2$ is said to be *nuclear* provided that the relation $\sum_{n=1}^{\infty} \lambda_n < +\infty$ holds, where each λ_n is the eigenvalue of the operator T obtained in the polar decomposition of T.

2-C. In the construction of CHS (2-A.), let Φ_n be the completion of the norm space $(\Phi, \|\cdot\|_n)$ $(n = 1, 2, \cdots)$. Whenever $m \leq n$, the identity map $\phi = \phi^n \mapsto \phi = \phi^m$ on Φ can be extended to the continuous linear map $T^n_m : \Phi_n \to \Phi_m$. We call T^n_m $(m \leq n)$ a *canonical mapping.*

2-D. A CHS (Φ, τ_Φ) is called a *nuclear space* provided that for any m there exists an n such that the canonical mapping $T^n_m : \Phi_n \to \Phi_m$ is a nuclear operator. In other words, T^n_m has the form

$$T^n_m \phi = \sum_{k=1}^{\infty} \lambda_n \langle \phi, \varphi_k \rangle_n \psi_k \tag{3}$$

satisfying $\sum_{n=1}^{\infty} \lambda_n < +\infty$, where $\{\varphi_k\}$ and $\{\psi_k\}$ are orthonormal systems of Φ_n and Φ_m, respectively.

(For a more general definition of nuclear space, see [32]).

2-E. Let Λ be a locally compact separable space and μ a positive measure on Λ. For each $\lambda \in \Lambda$, set a (separable) Hilbert space $(H(\lambda), \langle \cdot, \cdot \rangle_\lambda)$. A *vector field* x is an element of $\Pi_{\lambda \in \Lambda} H(\lambda)$. A countable family (x_i) of vector fields is called a *fundamental family* if (i) all functions, $\Lambda \ni \lambda \mapsto \langle x_i(\lambda), x_j(\lambda) \rangle_\lambda \in \mathbb{C}$, are measurable for all i, j and (ii) $(x_i(\lambda))$ spans $H(\lambda)$ for each λ. A vector field x is *measurable* if all the functions, $\lambda \mapsto \langle x(\lambda), x_i(\lambda) \rangle_\lambda$ $(i = 1, 2, \cdots)$, are measurable, where (x_i) is a fundamental family. Let $\mathcal{M} = \{x \in \Pi_{\lambda \in \Lambda} H(\lambda), x \text{ is measurable} \}$. The *direct integral*

$$\int_\Lambda H(\lambda) d\mu \tag{4}$$

of Hilbert spaces $H(\lambda)$ is the space of equivalence classes of measurable vector fields $x \in \mathcal{M}$ satisfying

$$\int_\Lambda \|x\|^2 d\mu < \infty. \tag{5}$$

Here, two fields are equivalent if they are equal μ-almost everywhere on Λ. If the direct integral equips the inner product,

$$\widetilde{\langle x, y \rangle} = \int_\Lambda \langle x, y \rangle d\mu, \tag{6}$$

where $\langle x, y \rangle : \Lambda \to \mathbb{C}, \lambda \mapsto \langle x(\lambda), y(\lambda) \rangle_\lambda$ ($\|x\|^2 = \langle x, x \rangle$) is measurable if $x \in \mathcal{M}$, then it is a Hilbert space.

2-G. Statement.(Maurins' Nuclear Spectral Theorem)[2] *Let* $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ *be a Hilbert space,* (Φ, τ_Φ) *a nuclear space such that (i)* Φ *is a dense subspace of* $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ *and (ii) the identity* $i : (\Phi, \tau_\Phi) \to (\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}}), \phi \mapsto \phi$ *is the continuous embedding. Then, Fourier transformation* $F : \mathcal{H} \to \hat{\mathcal{H}}$ *is given by the formulae;*

$$\hat{\phi}_k(\lambda) := \langle \hat{\phi}(\lambda), f_k(\lambda) \rangle_\lambda = (F\phi)_k(\lambda) = e_k(\lambda)(\phi) \quad (k = 1, 2, \ldots, dim \hat{\mathcal{H}}(\lambda)) \tag{7}$$

where $e_k(\lambda) \in \Phi'$, $\{f_k(\lambda)\}_{k=1,\ldots,dim\hat{\mathcal{H}}(\lambda)}$ *is an orthonormal base of* $\hat{\mathcal{H}}(\lambda)$*, and* $\hat{\mathcal{H}} = \int_\Lambda \hat{\mathcal{H}}(\lambda) d\mu$ *is the direct integral of the family of (sebarable) Hilbert spaces* $\{\hat{\mathcal{H}}(\lambda)\}_{\lambda\in\Lambda}$ *indexed by the locally compact space* Λ *with the Borel measure* μ*. Moreover, suppose that a normal operator* A *is continuous and* $A\Phi \subset \Phi$*. Then, the spaces* $\hat{\mathcal{H}}(\lambda)$ *are common eigenspace of* A*, which is diagonolized by the transformation* F*.*

2-H. By using the Fourier transformation $F : \mathcal{H} \to \hat{\mathcal{H}}$ that is isometric isomorphic obtained the above statement, for $\phi, \psi \in \Phi$, we have

$$\langle \phi, \psi \rangle_{\mathcal{H}} = \int_\Lambda \langle \hat{\phi}(\lambda), \hat{\psi}(\lambda) \rangle_\lambda d\mu. \tag{8}$$

By $\hat{\mathcal{H}}(\lambda) \ni \hat{\phi}(\lambda) = \sum_{k=1}^{dim\hat{\mathcal{H}}(\lambda)} \langle \hat{\phi}(\lambda), f_k(\lambda) \rangle_\lambda f_k(\lambda)$,

$$\langle \hat{\phi}(\lambda), \hat{\psi}(\lambda) \rangle_\lambda = \sum_{k=1}^{dim\hat{\mathcal{H}}(\lambda)} e_k^*(\lambda)(\phi) e_k(\lambda)(\psi) \tag{9}$$

is satisfied. Assuming $dim\hat{\mathcal{H}}(\lambda) \equiv 1$ for any λ, we have

$$\langle \phi, \psi \rangle_{\mathcal{H}} = \int_{Sp(A)} e(\lambda)^*(\phi) e(\lambda)(\psi) d\mu.$$

Therefore, Maurins' Nuclear Spectral Theorem can be represented by the following simple statement.

Statement. *Let* $\Phi \subset \mathcal{H} \subset \Phi'$ *be a RHS,* $A : \mathcal{D}(A) \to \mathcal{H}$*, self-adjoint with a dense domain* $\mathcal{D}(A)$*. In addition, assume that* A *is continuous on* Φ *and* $A\Phi \subset \Phi$*. Then,* A *has the set* $\{e(\xi)\}$ *of the generalized eigenvectors in* Φ' *corresponding to generalized eigenvalues* ξ*, where* ξ *goes through the spectra* $Sp(A)$ *of* A*. Furthermore, for any* $\varphi, \psi \in \Phi$*, we obtain the relations*

$$\begin{aligned} \langle \varphi, \psi \rangle_{\mathcal{H}} &= \int_{Sp(A)} e(\xi)^*(\varphi) e(\xi)(\psi) d\nu(\xi), \\ \langle \varphi, A\psi \rangle_{\mathcal{H}} &= \int_{Sp(A)} \xi \ e(\xi)^*(\varphi) e(\xi)(\psi) d\nu(\xi), \end{aligned} \tag{10}$$

where $\nu(\xi)$ *is a Borel measure on* $Sp(A)$*.*

Here, $\Phi' \ni F : \Phi \to \mathbb{C}$ is called a generalized eigenvector of A corresponding to the eigenvalue ξ, if F satisfies $F(A\phi) = \xi F(\phi)$ for every $\phi \in \Phi$.

2.2 Bra-ket formalism

The main features of Dirac's bra-ket formalism that are handled by the RHS are listed as follows.

- To each point of the spectrum of an observable (self-adjoint operator) A, there correspond left and right eigenvectors, denoted by $\langle a|$ and $|a\rangle$, satisfying

$$\langle a| A = a \langle a| \quad \text{and} \quad A |a\rangle = a |a\rangle , \tag{11}$$

respectively.

- These (eigen) bra and kets form complete bases; any bra $\langle\varphi|$ and ket $|\varphi\rangle$ can be expanded as

$$\langle\varphi| = \sum_n \langle\varphi|a_n\rangle\langle a_n| + \int\langle\varphi|a\rangle\langle a|\,d\mu(a),$$
$$\langle A\varphi| = \sum_n a_n\langle\varphi|a_n\rangle\langle a_n| + \int a\,\langle\varphi|a\rangle\langle a|\,d\mu(a), \tag{12}$$

and

$$|\varphi\rangle = \sum_n \langle a_n|\varphi\rangle|a_n\rangle + \int\langle a|\varphi\rangle|a\rangle\,d\mu(a),$$
$$|A\varphi\rangle = \sum_n a_n\langle a_n|\varphi\rangle|a_n\rangle + \int a\,\langle a|\varphi\rangle|a\rangle\,d\mu(a). \tag{13}$$

(a_n and a correspond to the discrete and continuous eigenvalues).

- They are normalized with respect to the following relations:

$$\langle a_n|a_m\rangle = \delta_{nm}, \quad \langle a|a'\rangle = \delta(a-a'), \tag{14}$$

where δ_{nm} is the Kronecker delta and $\delta(a-a')$ is the Dirac delta function.

The RHS formulation of the above features is summarized as follows. First we construct bra-ket vectors. Set a RHS of the type (1),

$$\Phi \subset \mathcal{H} \subset \Phi', \Phi^\times. \tag{15}$$

For $\varphi \in \Phi$, define a map $|\varphi\rangle_\mathcal{H} : \Phi \to \mathbb{C}$ by $|\varphi\rangle_\mathcal{H}(\psi) \equiv \langle\psi,\,\varphi\rangle_\mathcal{H}$ where $\psi \in \Phi$. This map $|\varphi\rangle_\mathcal{H}$ is called a *ket vector* of φ. Similarly, a *bra vector* of φ is the map $\langle\varphi|_\mathcal{H} : \Phi \to \mathbb{C}$ of the complex conjugate of $|\varphi\rangle_\mathcal{H}$: $\langle\varphi|_\mathcal{H}(\psi) = |\varphi\rangle^*_\mathcal{H}(\psi) = (|\varphi\rangle_\mathcal{H}(\psi))^* = \langle\varphi,\,\psi\rangle_\mathcal{H}$. Clearly, $\langle\varphi|_\mathcal{H} \in \Phi'$ and $|\varphi\rangle_\mathcal{H} \in \Phi^\times$ hold.

For the representations in (10), as $\{e(\xi)\}(\subset \Phi')$ are the generalized eigenvectors of A, they satisfy

$$e(\xi)(A\phi) = \xi e(\xi)(\phi) \quad (\phi \in \Phi). \tag{16}$$

Here, we adopt the following notations:

$$e(\xi) \to \langle\xi|_\mathcal{H}, \quad e(\xi)^* \to |\xi\rangle_\mathcal{H}, \quad e(\xi)(\varphi) \to \langle\xi|\varphi\rangle_\mathcal{H}, \quad e(\xi)^*(\varphi) \to \langle\varphi|\xi\rangle_\mathcal{H} \quad (\varphi \in \Phi),$$

and set the extension $\hat{A}$ on $\Phi' \cup \Phi^\times$ where $\hat{A}(f)(\phi) = f(A\phi)$ $(f \in \Phi' \cup \Phi^\times, \phi \in \Phi)$ (see also Sec. 3). Then, from (16) we derive

$$\hat{A}\,\langle\xi|_\mathcal{H} = \xi\,\langle\xi|_\mathcal{H}, \quad \hat{A}\,|\xi\rangle_\mathcal{H} = \xi\,|\xi\rangle_\mathcal{H}. \tag{17}$$

They show the eigen equations with respect to the bra and kets, individually. Moreover, the relations in (10) convert to

$$\langle\psi,\varphi\rangle_\mathcal{H} = \int_{Sp(A)} \langle\psi|\xi\rangle_\mathcal{H}\,\langle\xi|\varphi\rangle_\mathcal{H}\,d\nu(\xi), \quad \langle\psi,A\varphi\rangle_\mathcal{H} = \int_{Sp(A)} \xi\,\langle\psi|\xi\rangle_\mathcal{H}\,\langle\xi|\varphi\rangle_\mathcal{H}\,d\nu(\xi). \tag{18}$$

Therefore, the spectral expansions can be obtained as follows.

Statement. *The spectral expansions for the bra $\langle\varphi|_{\mathcal{H}}$ and ket $|\varphi\rangle_{\mathcal{H}}$ in Φ' and $\Phi^{\times}$ by the generalized eigenvectors $\{\langle\xi|_{\mathcal{H}}\}$ and $\{|\xi\rangle_{\mathcal{H}}\}$ of A represent, for $\varphi \in \Phi$,*

$$
\begin{aligned}
\langle\varphi|_{\mathcal{H}} &= \int_{Sp(A)} \langle\varphi|\xi\rangle_{\mathcal{H}} \langle\xi|_{\mathcal{H}}\, d\nu(\xi), \quad \langle A\varphi|_{\mathcal{H}} = \int_{Sp(A)} \xi \langle\varphi|\xi\rangle_{\mathcal{H}} \langle\xi|_{\mathcal{H}}\, d\nu(\xi), \\
|\varphi\rangle_{\mathcal{H}} &= \int_{Sp(A)} \langle\xi|\varphi\rangle_{\mathcal{H}} |\xi\rangle_{\mathcal{H}}\, d\nu(\xi), \quad |A\varphi\rangle_{\mathcal{H}} = \int_{Sp(A)} \xi \langle\xi|\varphi\rangle_{\mathcal{H}} |\xi\rangle_{\mathcal{H}}\, d\nu(\xi).
\end{aligned} \tag{19}
$$

Note that in physical literature, (19) are represented in the following forms:

$$
|\varphi\rangle_{\mathcal{H}} = \sum_{\xi_n \in Sp(A)} \langle\xi_n|\varphi\rangle_{\mathcal{H}} |\xi_n\rangle_{\mathcal{H}} + \int_{\xi \in Sp(A)} \langle\xi|\varphi\rangle_{\mathcal{H}} |\xi\rangle_{\mathcal{H}}\, d\nu(\xi), \tag{20}
$$

and

$$
\langle\varphi|_{\mathcal{H}} = \sum_{\xi_n \in Sp(A)} \langle\varphi|\xi_n\rangle_{\mathcal{H}} \langle\xi_n|_{\mathcal{H}} + \int_{\xi \in Sp(A)} \langle\varphi|\xi\rangle_{\mathcal{H}} \langle\xi|_{\mathcal{H}}\, d\nu(\xi), \tag{21}
$$

where the sum is taken over the discrete spectrum and integral over the continuous spectrum.

3 Construction of η-RHS

Let η be a positive operator on the Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$. Given RHS (1), we assume that η is continuous on the nuclear space (Φ, τ_{Φ}) and satisfies $\eta\Phi \subset \Phi$. Set an hermitian form $\langle\cdot,\cdot\rangle_{\eta}$ over $\mathcal{H}$ by

$$
\langle\phi,\psi\rangle_{\eta} = \langle\phi,\eta\psi\rangle_{\mathcal{H}} \quad (\phi,\psi \in \mathcal{H}). \tag{22}
$$

Then, the hermitian form endows the inner product where the vector space $\mathcal{H}$ becomes a pre-Hilbert space; such inner product is denoted by $\langle\cdot,\cdot\rangle_{\eta}$ again and the pre-Hilbert space is denoted by $\mathcal{H}_{\eta} = (\mathcal{H}, \langle\cdot,\cdot\rangle_{\eta})$. Also, we set the completion $\widetilde{\mathcal{H}_{\eta}} = (\widetilde{\mathcal{H}_{\eta}}, \widetilde{\langle\cdot,\cdot\rangle_{\eta}})$ of $\mathcal{H}_{\eta}$. This procedure shows that η defines a new metric, called the positive-definite metric, in $\mathcal{H}$ such that $\mathcal{H}$ becomes a Hilbert space with respect to $\widetilde{\langle\cdot,\cdot\rangle_{\eta}}$. It is shown that Φ is a dense linear subspace of $\widetilde{\mathcal{H}_{\eta}}$ and the inner product $\widetilde{\langle\cdot,\cdot\rangle_{\eta\Phi}}$, which is the restriction of $\widetilde{\langle\cdot,\cdot\rangle_{\eta}}$ to Φ, is separately continuous on the nuclear space (Φ, τ_{Φ}). Therefore, the family

$$
\Phi \subset \widetilde{\mathcal{H}_{\eta}} \subset \Phi', \Phi^{\times} \tag{23}
$$

becomes an RHS; hereafter, we call (23) an η-RHS.

An extension of η to the dual and anti-dual spaces Φ' and $\Phi^{\times}$ are easily established as follows. The assumption wherein η is continuous on (Φ, τ_{Φ}) and has $\eta\Phi \subset \Phi$ provides the mappings $\hat{\eta}' : \Phi' \to \Phi'$ and $\hat{\eta}^{\times} : \Phi^{\times} \to \Phi^{\times}$ by $(\hat{\eta}^{j}(f))(\phi) := f(\eta(\phi))$ for $\phi \in \Phi,\ f \in \Phi^{j}$ where $j = \prime, \times$. From these mappings we can derive the operator $\hat{\eta} : \Phi' \cup \Phi^{\times} \to \Phi' \cup \Phi^{\times}$, where

$$
(\hat{\eta}(f))(\phi) = f(\eta(\phi)) \tag{24}
$$

for $f \in \Phi' \cup \Phi^{\times}$ and $\phi \in \Phi$. It is easy to check the relations, $\hat{\eta}(f) \in \Phi^j$ for $f \in \Phi^j$ $(j = \prime, \times)$ and $\hat{\eta}(f^*) = \hat{\eta}(f)^*$. Note that $\Phi' \cap \Phi^{\times} = \{\hat{0}\}$ ($\hat{0}$ stands for the zero-valued functional on Φ). Furthermore, we can show

$$\langle\varphi|_{\mathcal{H}}\, \hat{\eta} = \langle\varphi|_{\eta}\,, \quad \hat{\eta}\, |\varphi\rangle_{\mathcal{H}} = |\varphi\rangle_{\eta}\,, \tag{25}$$

where we denote $\hat{\eta}(\langle\varphi|_{\mathcal{H}})$ by $\langle\varphi|_{\mathcal{H}}\, \hat{\eta}$. The relations of (25) indicate that the operator $\hat{\eta}$, extended from η, transforms the bra and ket vectors of $\mathcal{H}$-system into those of the $\widetilde{\mathcal{H}_{\eta}}$-system.

If we set a positive invertible operator η on $\mathcal{H}$, then $\widetilde{\mathcal{H}_{\eta}} = \mathcal{H}_{\eta} = (\mathcal{H}, \langle\cdot,\cdot\rangle_{\eta})$, since $\langle\cdot,\cdot\rangle_{\eta}$ produces the equivalent norm of $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. The extension of the inverse η^{-1} of η, $\widehat{\eta^{-1}} : \Phi' \cup \Phi^{\times} \to \Phi' \cup \Phi^{\times}$, is defined by

$$(\widehat{\eta^{-1}}(f))(\phi) = f(\eta^{-1}(\phi)) \tag{26}$$

for $f \in \Phi' \cup \Phi^{\times}$ and $\phi \in \Phi$. It follows that $\widehat{\eta^{-1}}$ is the inverse of $\hat{\eta}$, namely,

$$\hat{\eta}^{-1} = \widehat{\eta^{-1}}. \tag{27}$$

Also, the following inverse relations to (25) are obtained:

$$\langle\varphi|_{\eta}\, \hat{\eta}^{-1} = \langle\varphi|_{\mathcal{H}}\,, \quad \hat{\eta}^{-1}\, |\varphi\rangle_{\eta} = |\varphi\rangle_{\mathcal{H}}\,. \tag{28}$$

4 Bra-ket Formalism for quasi-Hermitian system

4.1 Spectral expansions

Let an η-quasi Hermitian operator $A : \mathcal{D}(A) \to \mathcal{H}$ in a Hilbert space $\mathcal{H} = (\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$, where a positive operator η is assumed to be invertible[29, 30, 31]. Then, by its definition, A satisfies the following symmetric structure :

$$A^{\dagger} = \eta A \eta^{-1}. \tag{29}$$

Here η is intertwining between A and $A^{\dagger}$. Regarding the positive operator η we set the η-RHS by (23), and we assume that $A\Phi \subset \Phi$, $\eta\Phi = \Phi$, and A is continuous on (Φ, τ_{Φ}). Using this assumption and the fact that A is a self-adjoint operator in the Hilbert space $\mathcal{H}_{\eta}$, the nuclear spectral theorem based on η-RHS can be applied to A and hence we have the following representations : for any $\varphi, \phi \in \Phi$,

$$\langle\phi, \varphi\rangle_{\eta} \;=\; \int_{Sp(A)} \langle\phi|\lambda\rangle_{\eta}\, \langle\lambda|\varphi\rangle_{\eta}\, d\mu(\lambda), \tag{30}$$

and

$$\langle\phi, A\varphi\rangle_{\eta} \;=\; \int_{Sp(A)} \lambda_A\, \langle\phi|\lambda\rangle_{\eta}\, \langle\lambda|\varphi\rangle_{\eta}\, d\mu(\lambda), \tag{31}$$

where $\mu(\lambda)$ is a Borel measure on the spectrum of A. $\langle\lambda|_{\eta}$ and $|\lambda\rangle_{\eta}$ $(\equiv \langle\lambda|_{\eta}^{*})$ represent the generalized eigenvectors for A; they satisfy the following (generalized) eigen equations,

$$\langle\lambda|_{\eta}\, (A\phi) = \lambda_A\, \langle\lambda|_{\eta}\, (\phi), \;\; |\lambda\rangle_{\eta}\, (A\phi) = \lambda_A\, |\lambda\rangle_{\eta}\, (\phi) \tag{32}$$

for any $\phi \in \Phi$. From (30)-(31), we obtain the spectral expansions of the bra and ket in the $\mathcal{H}_\eta$-system by the generalized eigenvectors of A : for $\varphi \in \Phi$,

$$\begin{aligned} \langle\varphi|_\eta &= \int_{Sp(A)} \langle\varphi|\lambda\rangle_\eta \langle\lambda|_\eta \, d\mu(\lambda), \\ \langle A\varphi|_\eta &= \int_{Sp(A)} \lambda_A \langle\varphi|\lambda\rangle_\eta \langle\lambda|_\eta \, d\mu(\lambda), \end{aligned} \tag{33}$$

and

$$\begin{aligned} |\varphi\rangle_\eta &= \int_{Sp(A)} \langle\lambda|\varphi\rangle_\eta |\lambda\rangle_\eta \, d\mu(\lambda), \\ |A\varphi\rangle_\eta &= \int_{Sp(A)} \lambda_A \langle\lambda|\varphi\rangle_\eta |\lambda\rangle_\eta \, d\mu(\lambda). \end{aligned} \tag{34}$$

Using the transformations (25) and (28), the following relations with respect to the bra-ket of $\mathcal{H}$-system can be derived from (33) and (34) :

$$\begin{aligned} \langle\varphi|_{\mathcal{H}} &= \int_{Sp(A)} \left\langle \eta^{-1}\varphi \middle| \lambda \right\rangle_\eta \langle\lambda|_\eta \, d\mu(\lambda) \\ &= \int_{Sp(A)} \langle\varphi|_{\mathcal{H}} \, \hat{\eta}^{-1} |\lambda\rangle_\eta \langle\lambda|_\eta \, d\mu(\lambda), \\ |\varphi\rangle_{\mathcal{H}} &= \int_{Sp(A)} \left\langle \lambda \middle| \eta^{-1}\varphi \right\rangle_\eta |\lambda\rangle_\eta \, d\mu(\lambda) \\ &= \int_{Sp(A)} \langle\lambda|_\eta \, \hat{\eta}^{-1} |\varphi\rangle_{\mathcal{H}} |\lambda\rangle_\eta \, d\mu(\lambda). \end{aligned} \tag{35}$$

These relations formulate the spectral expansions in terms of the bra $\langle\varphi|_{\mathcal{H}}$ and ket $|\varphi\rangle_{\mathcal{H}}$. Similarly, we have the spectral expansions,

$$\begin{aligned} \langle A\varphi|_{\mathcal{H}} &= \int_{Sp(A)} \lambda_A \langle\varphi|_{\mathcal{H}} |\lambda\rangle_\eta \langle\lambda|_\eta \, \hat{\eta}^{-1} d\mu(\lambda), \\ |A\varphi\rangle_{\mathcal{H}} &= \int_{Sp(A)} \lambda_A \langle\lambda|_\eta |\varphi\rangle_{\mathcal{H}} \, \hat{\eta}^{-1} |\lambda\rangle_\eta \, d\mu(\lambda), \end{aligned} \tag{36}$$

and

$$\begin{aligned} \left\langle A^\dagger\varphi \right|_{\mathcal{H}} &= \int_{Sp(A)} \lambda_A \langle\varphi|_{\mathcal{H}} \, \hat{\eta}^{-1} |\lambda\rangle_\eta \langle\lambda|_\eta \, d\mu(\lambda), \\ \left| A^\dagger\varphi \right\rangle_{\mathcal{H}} &= \int_{Sp(A)} \lambda_A \langle\lambda|_\eta \, \hat{\eta}^{-1} |\varphi\rangle_{\mathcal{H}} |\lambda\rangle_\eta \, d\mu(\lambda). \end{aligned} \tag{37}$$

Note that the region of integral can convert from $Sp(A)$ into the real line $\mathbb{R}$ and hereafter we take $\mathbb{R}$.

4.2 Complete bi-orthogonal system

From the spectral expansions obtained in the previous section, the complete bi-orthogonal system is composed of the generalized eigenvectors of A. The completeness is easily found from the relations in (35), which forms

$$I = \int_{\mathbb{R}} |\lambda\rangle_\eta \langle\lambda|_\eta \, \hat{\eta}^{-1} d\mu(\lambda) = \int_{\mathbb{R}} \hat{\eta}^{-1} |\lambda\rangle_\eta \langle\lambda|_\eta \, d\mu(\lambda), \tag{38}$$

where I is the identity for the bra $\langle\cdot|_{\mathcal{H}}$ and ket $|\cdot\rangle_{\mathcal{H}}$. Here, we define the operators $\langle\lambda|_{\mathcal{H}}$ and $|\lambda\rangle_{\mathcal{H}}$ in Φ' and $\Phi^{\times}$ concerning $\mathcal{H}$-system by

$$\langle\lambda|_{\mathcal{H}} = \langle\lambda|_{\eta}\,\hat{\eta}^{-1} \text{ and } |\lambda\rangle_{\mathcal{H}} = \hat{\eta}^{-1}\,|\lambda\rangle_{\eta}\,. \tag{39}$$

Then, (38) is reformed to

$$I = \int_{\mathbb{R}} |\lambda\rangle_{\eta}\,\langle\lambda|_{\mathcal{H}}\,d\mu(\lambda) = \int_{\mathbb{R}} |\lambda\rangle_{\mathcal{H}}\,\langle\lambda|_{\eta}\,d\mu(\lambda). \tag{40}$$

Regarding the bi-orthogonality, we set $\varphi^{\eta}(\lambda) = \langle\lambda|_{\eta}\,(\varphi)$ for $\langle\lambda|_{\eta}$ and $\varphi\in\Phi$. Using (38), we have

$$\begin{aligned}
&\int_{\mathbb{R}} \langle\lambda|\lambda'\rangle_{\eta}\,\varphi^{\eta}(\lambda')d\mu(\lambda') \\
=\;& \int_{\mathbb{R}} \langle\lambda|_{\eta}\cdot|\lambda'\rangle_{\mathcal{H}}\,\langle\lambda'|_{\eta}\,(\varphi)d\mu(\lambda') \\
=\;& \int_{\mathbb{R}} \langle\lambda|_{\eta}\cdot\hat{\eta}^{-1}\,|\lambda'\rangle_{\eta}\,\langle\lambda'|_{\eta}\cdot|\varphi\rangle_{\mathcal{H}}\,d\mu(\lambda') \\
=\;& \langle\lambda|_{\eta}\cdot|\varphi\rangle_{\mathcal{H}} \\
=\;& \varphi^{\eta}(\lambda).
\end{aligned} \tag{41}$$

Thus, the orthonormal relation is obtained as

$$\langle\lambda|_{\mathcal{H}}\cdot|\lambda'\rangle_{\eta} = \langle\lambda|_{\eta}\cdot|\lambda'\rangle_{\mathcal{H}} = \delta(\lambda-\lambda'). \tag{42}$$

where δ is Diracs' δ-function as the normalization factor of the eigenvectors of A. Therefore, the relations (40) and (42) show that the pair $\{|\lambda\rangle_{\eta}\,, |\lambda\rangle_{\mathcal{H}}\}$ is the complete bi-orthogonal system.

4.3 Transformation Theory

Let a self-adjoint operator $B:\mathcal{D}(B)\to\mathcal{H}$ in $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ satisfying continuity on (Φ,τ_{Φ}) and the relation $B\Phi\subset\Phi$. ($B=\eta$ is possible.) From the nuclear spectral theorem based on RHS (1), each of the bra $\langle\varphi|_{\mathcal{H}}$ and ket $|\varphi\rangle_{\mathcal{H}}$ can be expanded in the $\mathcal{H}$-system by using the generalized eigenvectors, denoted by $\{\langle\omega|_{\mathcal{H}}\}$, of B. Then, we have

$$\begin{aligned}
\langle\varphi|_{\mathcal{H}} &= \int_{\mathbb{R}} \langle\varphi|\omega\rangle_{\mathcal{H}}\,\langle\omega|_{\mathcal{H}}\,d\mu(\omega), \\
|\varphi\rangle_{\mathcal{H}} &= \int_{\mathbb{R}} \langle\omega|\varphi\rangle_{\mathcal{H}}\,|\omega\rangle_{\mathcal{H}}\,d\mu(\omega),
\end{aligned} \tag{43}$$

where $\mu(\omega)$ is a Borel measure on the spectrum of B and $\langle\omega|\varphi\rangle_{\mathcal{H}} = \langle\omega|_{\mathcal{H}}\,(\varphi)$ and $\langle\varphi|\omega\rangle_{\mathcal{H}} = |\omega\rangle_{\mathcal{H}}\,(\varphi)$. The generalized eigenvectors $\{\langle\omega|_{\mathcal{H}}\}$ composes of the complete orthonormal system for the $\mathcal{H}$-system[17], i.e.,

$$\langle\omega|\omega'\rangle_{\mathcal{H}} = \delta(\omega-\omega') \tag{44}$$

and

$$\int_{\mathbb{R}} |\omega\rangle_{\mathcal{H}}\,\langle\omega|_{\mathcal{H}}\,d\mu(\omega) = I \tag{45}$$

are satisfied. Now we consider the transformations of the bra-ket vectors of the representations between the η-quasi Hermitian A and the above self-adjoint B. First, using the complete relations (45) for $\{|\omega\rangle_{\mathcal{H}}\}$ and (40) for $\{|\lambda\rangle_{\mathcal{H}}, |\lambda\rangle_{\eta}\}$, we obtain

$$\begin{aligned}\langle\omega|\varphi\rangle_{\mathcal{H}} &= \langle\omega|_{\mathcal{H}}\cdot|\varphi\rangle_{\mathcal{H}} \\ &= \int_{\mathbb{R}} \langle\omega|_{\mathcal{H}}\cdot|\lambda\rangle_{\eta}\,\langle\lambda|_{\mathcal{H}}\cdot|\varphi\rangle_{\mathcal{H}}\,d\mu(\lambda) \\ &= \int_{\mathbb{R}} \langle\omega|\lambda\rangle_{\eta}\,\langle\lambda|\varphi\rangle_{\mathcal{H}}\,d\mu(\lambda), \end{aligned} \tag{46}$$

for any $\varphi\in\Phi$. The relation (46) shows the transformation of the representations from $\langle\lambda|_{\mathcal{H}}$ to $\langle\omega|_{\mathcal{H}}$ by means of the transformation factor $\langle\omega|_{\mathcal{H}}\cdot|\lambda\rangle_{\eta} = \langle\omega|\lambda\rangle_{\eta}$. Additionally, the transformation from $\langle\lambda|_{\eta}$ to $\langle\omega|_{\mathcal{H}}$ is derived from

$$\begin{aligned}\langle\omega|\varphi\rangle_{\mathcal{H}} &= \int_{\mathbb{R}} \langle\omega|_{\mathcal{H}}\cdot|\lambda\rangle_{\mathcal{H}}\,\langle\lambda|_{\eta}\cdot|\varphi\rangle_{\mathcal{H}}\,d\mu(\lambda) \\ &= \int_{\mathbb{R}} \langle\omega|\lambda\rangle_{\mathcal{H}}\,\langle\lambda|\varphi\rangle_{\eta}\,d\mu(\lambda), \end{aligned} \tag{47}$$

where $\langle\omega|_{\mathcal{H}}\cdot|\lambda\rangle_{\mathcal{H}} = \langle\omega|\lambda\rangle_{\mathcal{H}}$ represents the transformation factor. The other factors $\langle\lambda|\omega\rangle_{\mathcal{H}}$ and $\langle\lambda|\omega\rangle_{\eta}$ that characterize the transformation equations from $\langle\omega|_{\mathcal{H}}$ to $\langle\lambda|_{\mathcal{H}}$ and $\langle\lambda|_{\eta}$ are obtained by

$$\langle\lambda|\varphi\rangle_{\mathcal{H}} = \int_{\mathbb{R}} \langle\lambda|\omega\rangle_{\mathcal{H}}\,\langle\omega|\varphi\rangle_{\mathcal{H}}\,d\mu(\omega) \tag{48}$$

and

$$\langle\lambda|\varphi\rangle_{\eta} = \int_{\mathbb{R}} \langle\lambda|\omega\rangle_{\eta}\,\langle\omega|\varphi\rangle_{\mathcal{H}}\,d\mu(\omega). \tag{49}$$

Note that the following relations of the complex conjugates are satisfied:

$$\langle\lambda|\omega\rangle_{\eta}^{*} = \langle\omega|\lambda\rangle_{\eta}, \quad \langle\lambda|\omega\rangle_{\mathcal{H}}^{*} = \langle\omega|\lambda\rangle_{\mathcal{H}}. \tag{50}$$

We note that the transformation of the representations between the $\mathcal{H}$-system and $\mathcal{H}_{\eta}$-system is distinct from the typical transformation for Hermitian systems, such as the Fourier transformation between the real space and the momentum space. To see it, we consider the transformation factors as the Fourier transformation, namely, we set $\langle\omega|\lambda\rangle_{\eta} = \frac{1}{\sqrt{2\pi\hbar}}e^{-i\omega\lambda/\hbar}$ in (46) for which λ and ω are like the position and momentum operators in Hermitian quantum mechanics. Then, from (50), we have $\langle\lambda|\omega\rangle_{\eta} = \langle\omega|\lambda\rangle_{\eta}^{*} = \frac{1}{\sqrt{2\pi\hbar}}e^{i\omega\lambda/\hbar}$. Therefore, we obtain by the transformations in (49)

$$\langle\lambda|\varphi\rangle_{\eta} = \frac{1}{\sqrt{2\pi\hbar}}\int_{\mathbb{R}} e^{i\omega\lambda/\hbar}\,\langle\omega|\varphi\rangle_{\mathcal{H}}\,d\mu(\omega) = \langle\lambda|\varphi\rangle_{\mathcal{H}}$$

for any $\varphi\in\Phi$. The relation induces $\langle\lambda|_{\eta} = \langle\lambda|_{\mathcal{H}}$, which shows that $\hat{\eta}$ is the identity.

5 Extensions

Since A is continuous on (Φ, τ_Φ) with $A\Phi \subset \Phi$, the operator $\hat{A}$ on $\Phi' \cup \Phi^\times$ can be defined by

$$(\hat{A}(f))(\phi) := f(A(\phi)), \tag{51}$$

for any $\phi \in \Phi$ and $f \in \Phi' \cup \Phi^\times$. The extension $\hat{A}$ satisfies the eigenequations represented by (32); for the generalized eigenvectors $\{\langle\lambda|_\eta\}$ and $\{|\lambda\rangle_\eta\}$ of A we have

$$\langle\lambda|_\eta \hat{A} = \lambda_A \langle\lambda|_\eta\,, \ \hat{A}\,|\lambda\rangle_\eta = \lambda_A\,|\lambda\rangle_\eta\,, \tag{52}$$

where we denote $\hat{A}(\langle\lambda|_\eta)$ by $\langle\lambda|_\eta \hat{A}$. Similarly, the adjoint operator $A^\dagger$ can be extended to the operator $\hat{A}^\dagger$ on $\Phi' \cup \Phi^\times$, where

$$(\hat{A}^\dagger(f))(\phi) := f(A^\dagger(\phi)) \tag{53}$$

for $\phi \in \Phi,\ f \in \Phi' \cup \Phi^\times$. As well as (52), the eigenequations

$$\hat{A}^\dagger \langle\lambda|_{\mathcal{H}} = \lambda_A \langle\lambda|_{\mathcal{H}}\,, \ \hat{A}^\dagger\,|\lambda\rangle_{\mathcal{H}} = \lambda_A\,|\lambda\rangle_{\mathcal{H}}\,, \tag{54}$$

are satisfied. Between these extensions $\hat{A}$ and $\hat{A}^\dagger$, the extension $\hat{\eta}$ of η is intertwining. Actually, the following symmetric structure is shown:

$$\langle\phi|\,\hat{A}^\dagger\,|\varphi\rangle_{\mathcal{H}} = \langle\phi|\,\hat{\eta}\hat{A}\hat{\eta}^{-1}\,|\varphi\rangle_{\mathcal{H}}\,, \tag{55}$$

where $\phi, \varphi \in \Phi$. Considering that the relation (55) is satisfied, $\hat{A}$ is identified with the $\hat{\eta}$-quasi Hermitian operator for the bra-ket space.

The extensions constructed in the section can be applied to the practical physical non-Hermitian system, such as Swanson model[33]. Regarding the details of the topics, see [26].

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