On implementation of discrete and continuous shift translations on quantum lattice systems

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2023.12.6

Abstract

We discuss the continuous extension of the discrete shift translations on the one-dimensional fermion lattice system to C^*-flows.

1 Introduction

To investigate a continuous dynamical system, it is useful to consider its discretized dynamics by using certain reduction methods. Conversely, we sometimes guess the whole continuous dynamics from its partial discrete data. We are interested in the latter for quantum dynamics on infinitely extended large systems. Concretely, we propose the following problems:

Problem I: For a given discrete quantum dynamics on a large quantum system, is it possible to extend it to some continuous quantum dynamics?

Problem II: What dynamical properties of a given discrete dynamics persist or fail for its continuous extension(s)?

In this note, we specialize in the shift-translation automorphism group of the one-dimensional fermion lattice system [5]. We shall now refer to

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some relevant works. The pursuit of continuous extensions of a discrete automorphism group (or a single automorphism) finds its origins in Kuipier's seminal work [4], which explores the topological structure of unitaries on infinite-dimensional Hilbert spaces. It has been shown that continuous extension of the shift-translation automorphism group on the one-dimensional quantum spin lattice system can not be constructed by local Hamiltonians satisfying certain locality condition in [6]. On the other hand, we consider a much wider class of quantum dynamics. We search for realization of continuous shift-translations by C^*-flows (strongly continuous one-parameter automorphism groups on the C^*-algebra) with no explicit restriction upon local Hamiltonians.

2 Mathematical formulation

2.1 Fermion lattice systems

We introduce the fermion lattice system on $\mathbb{Z}$. For a subset $\mathrm{I} \subset \mathbb{Z}$, $|\mathrm{I}|$ denotes the number of sites in I. If $\mathrm{I} \subset \mathbb{Z}$ has a finite $|\mathrm{I}| < \infty$, we shall denote $\mathrm{I} \Subset \mathbb{Z}$. Let c_i and c_i^* denote the annihilation operator and the creation operator of a fermion at $i \in \mathbb{Z}$, respectively. These obey the canonical anti-commutation relations (CARs):

$$\{c_i^*, c_j^*\} = \{c_i, c_j\} = 0, \quad \{c_i^*, c_j\} = \delta_{i,j}\, 1, \quad i, j \in \mathbb{Z}. \tag{1}$$

For each $\mathrm{I} \Subset \mathbb{Z}$, define the finite system $\mathfrak{A}^F(\mathrm{I})$ by the $*$-algebra generated by $\{c_i^*,\, c_i\, ;\ i \in \mathrm{I}\}$. It is isomorphic to $\mathrm{M}_{2^{|\mathrm{I}|}}(\mathbb{C})$, the algebra of all $2^{|\mathrm{I}|} \times 2^{|\mathrm{I}|}$ complex matrices. For $\mathrm{I} \subset \mathrm{J} \Subset \mathbb{Z}$, $\mathfrak{A}^F(\mathrm{I})$ is naturally embedded into $\mathfrak{A}^F(\mathrm{J})$ as a subalgebra. Let us take

$$\mathfrak{A}^F_\circ := \bigcup_{\mathrm{I} \Subset \mathbb{Z}} \mathfrak{A}^F(\mathrm{I}). \tag{2}$$

By taking the norm completion of $\mathfrak{A}^F_\circ$ we obtain a C^*-algebra $\mathfrak{A}^F$ called the CAR algebra. The dense $*$-subalgebra $\mathfrak{A}^F_\circ$ in $\mathfrak{A}^F$ is called the local algebra of $\mathfrak{A}^F$.

2.2 Discrete and continuous shift translations

We define the discrete shift translations on $\mathfrak{A}^F$. For each $i \in \mathbb{Z}$, let

$$\tau_i^F(c_j) = c_{j+i},\ \tau_i^F(c_j^*) = c_{j+i}^*, \quad j \in \mathbb{Z}. \tag{3}$$

The above formulas determine the shift-translation automorphism group $\{\tau_i^F \in \mathrm{Aut}(\mathfrak{A}^F),\ i \in \mathbb{Z}\}$ on $\mathfrak{A}^F$.

In the following, we construct a continuous shift-translation group on $\mathfrak{A}^F$. Our fermion lattice system $\mathfrak{A}^F$ is identical to the CAR algebra over the complex Hilbert space $l^2(\mathbb{Z})$. This identification is explicitly given by setting $c_j = c(\chi_j)$ for $j \in \mathbb{Z}$, where $\chi_j \in l^2(\mathbb{Z})$ denotes the indicator determined by

$$\chi_j(k) := \delta_{jk},\ (k \in \mathbb{Z}). \tag{4}$$

For each $f = (f_j)_{j\in\mathbb{Z}} \in l^2(\mathbb{Z})$, define

$$c(f) := \sum_{j\in\mathbb{Z}} f_j c_j \in \mathfrak{A}^F, \quad c^*(f) := \sum_{j\in\mathbb{Z}} f_j c_j^* \in \mathfrak{A}^F. \tag{5}$$

$\mathfrak{A}^F$ is also isomorphic to the CAR algebra over $L^2[-\pi,\pi]$ as C^*-algebras by the isometry of $l^2(\mathbb{Z})$ and $L^2[-\pi,\pi]$ as Hilbert spaces. Here, the closed interval $[-\pi,\pi]$ can be interpreted as the range of the possible momentum of each fermion particle. The connection between $f = (f_j)_{j\in\mathbb{Z}} \in l^2(\mathbb{Z})$ and

$\tilde{f}(k) \in L^2[-\pi, \pi]$ is given by the Fourier transformation as follows:

$$\tilde{f}(k) := \sum_{j \in \mathbb{Z}} f_j e^{ijk}, \quad k \in [-\pi, \pi],$$
$$f_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{f}(k) e^{-ijk}\, dk \quad j \in \mathbb{Z}. \tag{6}$$

Hence for each $j \in \mathbb{Z}$

$$\widetilde{\chi_j}(k) = e^{ijk}, \quad k \in [-\pi, \pi]. \tag{7}$$

The one-step (right) shift translation U_1 on the Hilbert space $l^2(\mathbb{Z})$ is given by

$$(U_1 f)_j := f_{j-1}, \quad j \in \mathbb{Z}, \quad \text{for } f = (f_j)_{j \in \mathbb{Z}} \in l^2(\mathbb{Z}). \tag{8}$$

The $\mathbb{Z}$-group of unitaries $\{U_i \in \mathcal{U}(L^2[-\pi, \pi]);\ i \in \mathbb{Z}\}$ is given by

$$U_0 = I,\ U_k := \underbrace{U_1 \cdots U_1}_{k-\text{times}},\ U_{-k} := {U_k}^{-1}\ (k \in \mathbb{N}).$$

By the Fourier transformation, $U_1 \in \mathcal{U}(l^2(\mathbb{Z}))$ has the following expression on $L^2[-\pi, \pi]$

$$\widetilde{U_1 f}(k) = e^{ik} \tilde{f}(k), \quad k \in [-\pi, \pi]. \tag{9}$$

By the second quantization procedure, the $\mathbb{Z}$-group of unitaries $\{U_i \in \mathcal{U}(L^2[-\pi, \pi]);\ i \in \mathbb{Z}\}$ generates the discrete-shift translation automorphism group $\{\tau_i^F \in \mathrm{Aut}(\mathfrak{A}^F),\ i \in \mathbb{Z}\}$ given in (3). To interpolate $\{U_i \in \mathcal{U}(L^2[-\pi, \pi]);\ i \in \mathbb{Z}\}$, we define unitaries U_t $(t \in \mathbb{R})$ on $L^2[-\pi, \pi]$ as

$$\widetilde{U_t f}(k) := e^{itk} \tilde{f}(k), \quad k \in [-\pi, \pi]. \tag{10}$$

Now we obtain a strongly continuous one-parameter group of unitaries:

$$\{U_t \in \mathcal{U}(L^2[-\pi,\pi]);\ t \in \mathbb{R}\}, \tag{11}$$

equivalently,

$$\{U_t \in \mathcal{U}(l^2(\mathbb{Z});\ t \in \mathbb{R}\}. \tag{12}$$

By Stone's theorem, there exists a unique self-adjoint operator $\bar{h}$ on $L^2[-\pi,\pi]$ such that

$$U_t = e^{it\bar{h}}, \quad t \in \mathbb{R}. \tag{13}$$

This one-parameter group of unitaries generates a strongly continuous one parameter group of quasi-free automorphisms on $\mathfrak{A}^F$:

$$\{\tau_t^F \in \mathrm{Aut}(\mathfrak{A}^F),\ t \in \mathbb{R}\}. \tag{14}$$

By construction, it extends the discrete shift-automorphism group $\{\tau_i^F \in \mathrm{Aut}(\mathfrak{A}^F),\ i \in \mathbb{Z}\}$ from the discrete parameter $\mathbb{Z}$ to the continuous parameter $\mathbb{R}$.

The time evolution $\{\tau_t^F \in \mathrm{Aut}(\mathfrak{A}^F),\ t \in \mathbb{R}\}$ (14) on $\mathfrak{A}^F$ is associated to the second quantized Hamiltonian $\tilde{H}_\infty = \Gamma(\bar{h})$ of the one-particle Hamiltonian $\bar{h}$. In the following, we shall derive a concrete expression of $\tilde{H}_\infty$ by a formal (non rigorous) method. Let

$$\hat{c}_k := \sum_{j\in\mathbb{Z}} c_j e^{ijk}, \quad \hat{c}_k^* := \sum_{j\in\mathbb{Z}} c_j^* e^{ijk},\ k \in [-\pi,\pi] \tag{15}$$

Let $\hat{N}_k := \hat{c}_k^*\hat{c}_k$, the number operator for the momentum $k \in [-\pi,\pi]$. In terms of $\hat{N}_k$, $\tilde{H}_\infty$ has the following expression in the momentum space:

$$\tilde{H}_\infty = \int_{-\pi}^{\pi} k\hat{N}_k\,dk. \tag{16}$$

Using

$$[\hat{N}_k, \hat{c}_k] = -i\hat{c}_k,\ \ [\hat{N}_k, \hat{c}_k^*] = i\hat{c}_k^*$$

we rewrite $\tilde{H}_\infty$ in the lattice space as

$$\tilde{H}_\infty = \int_{-\pi}^{\pi} \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} k e^{ik(n-m)} c_n^* c_m \, dk = \sum_{n,m\in\mathbb{Z}} h_{n,m} c_n^* c_m,$$

where for $n, m \in \mathbb{Z}$

$$h_{n,m} = G(n-m) \in \mathbb{C},\ \ G(x) := \int_{-\pi}^{\pi} k e^{ikx} \, dk.$$

By integration by parts

$$\begin{aligned} G(x) &= \left[k \frac{e^{ikx}}{ix} \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{e^{ikx}}{ix} \, dk = \pi \frac{e^{i\pi x} + e^{-i\pi x}}{ix} + \frac{e^{i\pi x} - e^{-i\pi x}}{x^2} \\ &= -2\pi i \frac{\cos(\pi x)}{x} + 2i \frac{\sin(\pi x)}{x^2} \end{aligned}$$

By plugging $n-m$ to x, we obtain

$$\tilde{H}_\infty = \sum_{n,m\in\mathbb{Z}} h_{n,m} c_n^* c_m,\ \ h_{n,m} = \frac{-2\pi i (-1)^{n-m}}{n-m}. \tag{17}$$

3 Results

We shall respond to **Problem I** and **Problem II** for the quasi-free C^*-flow that extends the discrete-shift translations on the fermion lattice system given in the preceding section. The following statement is given in [5]. Here, "the locality of dynamics" refers to Definition 2.2.1 of [2]: There exists a strict local operator whose time-translation becomes non-local.

Theorem 1. The discrete shift-translation automorphism group on the one-dimensional fermion lattice system $\mathfrak{A}^F$ can be extended to a quasi-free C^*-flow $\{\tau_t^F \in \mathrm{Aut}(\mathfrak{A}^F),\ t \in \mathbb{R}\}$ on $\mathfrak{A}^F$. It has the following dynamical formulas in terms of the sinc function $\mathrm{sinc}x := \frac{\sin x}{x}$ as follows. For $j \in \mathbb{Z}$ and $t \in \mathbb{R}\setminus\mathbb{Z}$,

$$\tau_t^F(c_j) = \sum_{l\in\mathbb{Z}} \mathrm{sinc}(\pi(j+t-l))c_l = \sum_{l\in\mathbb{Z}} \frac{(-1)^{j-l}\sin(\pi t)}{\pi(j+t-l)} c_l \in \mathfrak{A}^F,$$
$$\tau_t^F(c_j^*) = \sum_{l\in\mathbb{Z}} \mathrm{sinc}(\pi(j+t-l))c_l^* = \sum_{l\in\mathbb{Z}} \frac{(-1)^{j-l}\sin(\pi t)}{\pi(j+t-l)} c_l^* \in \mathfrak{A}^F. \tag{18}$$

For $j \in \mathbb{Z}$ and $k \in \mathbb{Z}$,

$$\tau_k^F(c_j) = c_{j+k} \in \mathfrak{A}^F_\circ, \quad \tau_k^F(c_j^*) = c_{j+k}^* \in \mathfrak{A}^F_\circ. \tag{19}$$

The time-shift translation $\tau_t^F \in \mathrm{Aut}(\mathfrak{A}^F)$ violates the locality of dynamics for $\forall t \in \mathbb{R} \setminus \mathbb{Z}$.

Some comments are in order. The formal Hamiltonian $\tilde{H}_\infty$ of the continuous shift translations on the one-dimensional fermion lattice system consists of the two-body translation invariant interactions with $\frac{1}{r}$-decay rate (17). We note that its derivation is not rigorous; the meaning of $\tilde{H}_\infty$ in C^*-theory remains unclear. More precisely, we do not know whether and how it is associated to a pre-generator of the C^*-flow $\{\tau_t^F \in \mathrm{Aut}(\mathfrak{A}^F),\ t \in \mathbb{R}\}$. In general, such long-range interactions do not generate a C^*-flow. For **Problem II**, we address the Rohlin property as a dynamical property which is not kept by the continuous extension as follows.

Corollary 1. The discrete shift-translation automorphism group $\{\tau_i^F \in \mathrm{Aut}(\mathfrak{A}^F),\ i \in \mathbb{Z}\}$ satisfies the Rohlin property [1], whereas its continuous extension $\{\tau_t^F \in \mathrm{Aut}(\mathfrak{A}^F),\ t \in \mathbb{R}\}$ does not.

Proof. It has been shown in [1] that the shift-translation automorphism group $\{\tau_i^F \in \mathrm{Aut}(\mathfrak{A}^F),\ i \in \mathbb{Z}\}$ satisfies the Rohlin property [1]. On the other hand, since any quasi-free C^*-flow on the CAR algebra has a (unique) KMS state, it does not satisfy the Rohlin property [3]. □

Acknowledgments

This research is supported by Kakenhi (grant no. 21K03290).

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