ZEROS OF POLYNOMIALS ASSOCIATED TO SPECIAL VALUES OF L-FUNCTIONS

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Abstract. In this article, we give a brief overview of the problem of zero distribution of families of self-reciprocal polynomials whose coefficients arise from special values of L-functions.

A non-zero polynomial $P(x) \in \mathbb{C}[x]$ of degree d is said to be *self-inversive* if there exists $\varepsilon \in \mathbb{C}$, with $|\varepsilon| = 1$ such that

$$\varepsilon\, z^d\, \overline{P}\left(\frac{1}{z}\right) = P(z).$$

We call a self-inversive polynomial as *self-reciprocal* if $P(x) \in \mathbb{R}[x]$ and $\varepsilon = \pm 1$. It is clear that if $0 \neq \alpha$ is a root of a self-inversive polynomial $P(x)$, then so is $\overline{1/\alpha}$. Thus, in the same spirit as the Riemann hypothesis, it is natural to investigate whether all zeros of a self-inversive polynomial $P(x)$ lie on its 'curve of symmetry', namely the unit circle S^1. Clearly, one must impose additional conditions on $P(x)$ to ensure unimodularity of its zeros.

In this note, we focus on families of self-inversive polynomials whose coefficients are composed of special values of certain L-functions. We will give an outline of the history and context of the problem, highlight the important methods used and mention relevant conjectures. This article is based on the talk delivered by the author at the RIMS Analytic Number Theory and Related Topics Workshop in October 2024, and two forthcoming papers, [3] and [4].

1. Special Values of $\zeta(s)$

The study of L-functions is an important theme in number theory and their special values at integer points are expected to capture significant arithmetic information. Moreover, their values at positive integers are believed to be a rich source of (new) transcendental numbers. The origin of this line of inquiry can be traced back to Euler's resolution of the Basel problem.

For $\mathrm{Re}(s) > 1$, define the Riemann zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\text{ - prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Then Euler demonstrated that for a positive integer $k \geq 1$,

$$\zeta(2k) = -\frac{(2\pi i)^{2k}\, B_{2k}}{2(2k)!},$$

where B_n denotes the n-th Bernoulli number given by the generating function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!}\, t^n,$$

so that $B_0 = 1$, $B_1 = -1/2$, $B_{2m+1} = 0$ and $B_{2m} \in \mathbb{Q}$ for $m \geq 1$. Thus, $\zeta(2k)$ is a transcendental number. A natural question is whether the same holds for $\zeta(2k+1)$, i.e., are the odd zeta-values also transcendental? This is one of the major open problems in transcendental number theory. It is widely believed that

Conjecture. *The numbers*
$$\{\pi\} \cup \{\zeta(2k+1) \,:\, k \in \mathbb{Z},\, k \geq 1\}$$
are algebraically independent over $\mathbb{Q}$.

To date, there have been several attempts to gain deeper understanding about the odd-zeta values. One such attempt can be found in Ramanujan's notebooks [16] as the following formula: for positive real numbers α and β with $\alpha\beta = \pi^2$ and any positive integer k,

$$\alpha^{-k}\left\{\frac{1}{2}\zeta(2k+1) + \sum_{n=1}^{\infty}\frac{n^{-2k-1}}{e^{2\alpha n}-1}\right\} - (-\beta)^{-k}\left\{\frac{1}{2}\zeta(2k+1) + \sum_{n=1}^{\infty}\frac{n^{-2k-1}}{e^{2\beta n}-1}\right\}$$
$$= -2^{2k}\alpha^{k+1}\sum_{j=0}^{k+1}(-1)^j\frac{B_{2j}\,B_{2k+2-2j}}{(2j)!\,(2k+2-2j)!}\left(\frac{\beta}{\alpha}\right)^j.$$

The above formula was independently discovered by E. Grosswald [11] in 1970 in the context of transformation formulae of Eichler integrals. Let $\mathbb{H}$ denote the upper half plane, that is the set of complex numbers with strictly positive imaginary part. For any positive integer k, set

$$R_{2k+1}(x) := \sum_{j=0}^{k+1}\frac{B_{2j}\,B_{2k+2-2j}}{(2j)!\,(2k+2-2j)!}\,x^{2j} \tag{1}$$

Then we have the following.

Theorem (Grosswald)**.** *Let* $k \geq 1$ *be an integer and* $\sigma_k(n) := \sum_{d|n} d^k$. *For* $z \in \mathbb{H}$, *define*
$$\mathcal{F}_k(z) := \sum_{n=1}^{\infty}\frac{\sigma_k(n)}{n^k}\,e^{2\pi i n z}.$$
Then
$$\mathcal{F}_{2k+1}(z) - z^{2k}\,\mathcal{F}_{2k+1}\left(-\frac{1}{z}\right) = \frac{1}{2}\zeta(2k+1)\left(z^{2k}-1\right) + \frac{(2\pi i)^{2k+1}}{2z}R_{2k+1}(z). \tag{2}$$

One can recover Ramanujan's formula from the above identity by specializing at $z = i\alpha/\pi$, $-1/z = i\beta/\pi$ and noting that
$$\mathcal{F}_k(z) = -\zeta(k) - \sum_{n=1}^{\infty}\frac{n^{-k}}{e^{-2\pi i n z}-1}.$$

2. Ramanujan polynomials

From equation (2), it is clear that if z_0 is a zero of $R_{2k+1}(x)$, and z_0 is not a $2k$-th root of unity, then
$$\zeta(2k+1) = \frac{2}{\left(z_0^{2k}-1\right)}\left(\mathcal{F}_{2k+1}(z_0) - z_0^{2k}\mathcal{F}_{2k+1}\left(\frac{-1}{z_0}\right)\right).$$
Therefore, zeros of the polynomial $R_{2k+1}(x)$ have the potential to shed light on the nature of odd zeta-values.

With this motivation, in 2011, M. R. Murty, C. Smyth and R. Wang [15] isolated the polynomial $R_{2k+1}(x)$ for independent study and called them *'Ramanujan polynomials'*. Observe that $R_{2k+1}(x)$ is an even and self-reciprocal polynomial, that is,
$$R_{2k+1}(-x) = R_{2k+1}(x) \qquad \text{and} \qquad x^{2k+2}\,R_{2k+1}(1/x) = R_{2k+1}(x)$$

respectively. Hence, if z_0 is a zero of $R_{2k+1}(x)$, then so are $-z_0$, z_0^{-1} and $-z_0^{-1}$. In the same paper, Murty-Smyth-Wang proved the following.

Theorem 2.1 (Murty-Smyth-Wang)**.** *For an integer $k \geq 2$, the following holds.*

(a) The polynomial $R_{2k+1}(x)$ has 4 distinct real zeros of the form $\alpha_k > 1$, $-\alpha_k$, α_k^{-1} and $-\alpha_k^{-1}$. All the remaining zeros of $R_{2k+1}(x)$ lie on the unit circle and are uniformly distributed as $k \to \infty$.

(b) Apart from $\pm i$ when k is even and $\pm e^{2\pi i/3}$, $\pm e^{-2\pi i/3}$ when $3 \mid k$, none of the other roots of unity occur as zeros of $R_{2k+1}(x)$.

3. 'Riemann hypothesis' for period polynomials

A flurry of activity followed Theorem 2.1 because one can recognize the Ramanujan polynomial $R_{2k+1}(z)$ as the odd part of the period polynomial associated to the normalized Eisenstein series $E_{2k+2}(z)$ of weight $2k+2$ with respect to $\mathrm{SL}_2(\mathbb{Z})$. More specifically, for $k \in \mathbb{Z}_{\geq 1}$ and $f \in M_{2k+2}(\mathrm{SL}_2(\mathbb{Z}))$, let

$$f(z) = a_f(0) + \sum_{n=1}^{\infty} a_f(n)\, e^{2\pi i n z}.$$

Then, one can attach an L-function to f as

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s},$$

which is absolutely convergent in $\mathrm{Re}(s) > 2k+2$. Set $\Lambda_f(s) := (2\pi)^{-s}\,\Gamma(s)\,L(s,f)$ to be the completed L-function. Then $\Lambda_f(s)$ extends to a holomorphic function on $\mathbb{C}$, with only possible simple poles at $s = 0$ and $s = 2k+2$ when $a_f(0) \neq 0$. Define

$$\widetilde{f}^*(z) := \sum_{n=1}^{\infty} \frac{a_f(n)}{n^{2k+1}}\, e^{2\pi i n z}.$$

The function $\widetilde{f}^*(z)$ is an Eichler integral of the second kind. One can view it (essentially) as a $(2k+1)$-th anti-derivative of $f(z)$. The following result, proved in various contexts by M. Razar [17], A. Weil [18] and L. Goldstein-M. Razar [10], measures the 'deviation from modularity' of the function $\widetilde{f}^*(z)$.

Theorem 3.1 (Razar, Weil, Goldstein-Razar)**.**

$$\widetilde{f}^*(z) - z^{2k}\widetilde{f}^*(-1/z) = -\frac{(2\pi i)^{2k+1}\, a_f(0)}{(2k+1)!}\left(z^{2k+1} - \frac{1}{z}\right) + \sum_{j=0}^{2k} \frac{(2\pi i)^j\, L(2k+1-j,f)}{j!}\, z^j \;=: r_f(z).$$

When f is a cusp form, $r_f(z)$ is a self-inversive polynomial due to the functional equation of $\Lambda_f(s)$. This polynomial is referred to as the 'period polynomial' of $f(z)$. These play a central role in the Eichler-Shimura cohomology. If $a_f(0) \neq 0$, then $z\, r_f(z)$ is a self-inversive polynomial. In particular, the Ramanujan polynomial $R_{2k+1}(z)$ is the odd part of $z\, r_{E_{2k+2}}(z)$. Thus, it is natural to investigate whether the zeros of period polynomials associated to general modular forms also lie on the unit circle. This theme is known as the Riemann hypothesis for period polynomials. The unimodularity of zeros of $z\, r_{E_{2k+2}}(z)$ was shown by M. Lalín and C. Smyth in 2012. For weight one Hecke eigenforms, this was proved by A. El-Guindy and W. Raji in 2014 and for Hecke eigencuspforms of higher level by S. Jin, W. Ma, K. Ono, K. Soundararajan in 2016. We refer the reader to the excellent survey [7] for further details.

4. **Zeros of self-inversive polynomials**

As Ramanujan polynomials and period polynomials are special cases of self-inversive polynomials, we recount some techniques involved in the study of their zeros.

In 1922, A. Cohn showed that all zeros of a polynomial $P(x)$ lie on S^1 if and only if $P(x)$ is self-inversive and all zeros of its derivative $P'(x)$ lie inside the closed unit disk, $D := \{z \in \mathbb{C} : |z| \leq 1\}$. In specific situations, verification of the condition on the zeros of the derivative may not be easy. Thus, other sufficient conditions that force all zeros of a self-inversive polynomial $P(x)$ to lie on S^1 were obtained by several authors (see [13] for details). In particular, M. Lalín and C. Smyth [13, Theorem 1] proved the following.

Theorem 4.1 (Lalín-Smyth)**.** *Let $0 \neq h(x) \in \mathbb{C}[x]$ with $\deg h = n$ be such that all zeros of $h(x)$ lie in D. Set $h^*(x) := z^n \overline{h}(1/z)$. Then for any $d > n$ and $\lambda \in S^1$, the self-inversive polynomial*

$$P^{\{\lambda\}}(x) = x^{d-n}\, h(x) + \lambda\, h^*(x)$$

has all its zeros on S^1.

In this context, we have below the well-known result of Eneström[9], independently proved by Kakeya [12].

Theorem 4.2 (Eneström-Kakeya)**.** *Let $F(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0 \in \mathbb{R}[x]$ be such that $0 \leq b_0 \leq b_1 \leq \cdots \leq b_n$. Then all the zeros of $F(x)$ lie in D.*

Therefore, if $0 \neq f(x) \in \mathbb{R}[x]$ is a self-reciprocal polynomial given by

$$f(x) = a_{2n}x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_{n+1}x^{n+1} + 2a_n x^n + \varepsilon\left(a_{n+1}x^{n-1} + \cdots + a_{2n-1}x + a_{2n}\right)$$

$$(\text{ or } \quad f(x) = a_{2n-1}x^{2n-1} + a_{2n-2}x^{2n-2} + \cdots + a_n x^n + \varepsilon\left(a_n x^{n-1} + \cdots + a_{2n-2}x + a_{2n-1}\right) \quad \text{resp.}),$$

for $\varepsilon = \pm 1$ such that $a_{2n} \geq a_{2n-1} \geq \cdots \geq a_{n+1} \geq a_n$ (or $a_{2n-1} \geq a_{2n-2} \geq \cdots \geq a_{n+1} \geq a_n$ resp.), then all zeros of $f(x)$ lie on the unit circle. This monotonicity property of coefficients is at the heart of many of the results mentioned earlier.

The above observation implies the following general proposition [4].

Proposition 4.3 (Charan-Pathak)**.** *Let $L(s) = \sum_{n\geq 1} a_n\, \lambda_n^{-s}$ be a general Dirichlet series, convergent in Re$(s) > 1$ with*

- *$a_n \geq 0$ for $n \geq 1$, and*
- *$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ and $\lambda_n \to \infty$ as $n \to \infty$.*

Then, for $k \geq 4$, all zeros of $\mathcal{A}_{k,L}(x) = \sum_{j=0}^{k-2} L(j+2)\, L(k-2-j)\, x^j$ lie on the unit circle.

As immediate consequences, we derive the unimodularity of zeros of $\mathcal{A}_{k,L}(x)$ for $L(s) = L(s, \chi_{0,n})$ where $\chi_{0,n}$ is the principal Dirichlet character modulo n (a result of Chourasiya-Jamal-Maji [6]), $L(s) = \zeta(s, \alpha)$ (the Hurwitz zeta-function when $0 < \alpha \leq 1$), $L(s) = \zeta_K(s)$ (the Dedekind zeta-function associated to a number field $K/\mathbb{Q}$), $L(s) = L(s, f \otimes f)$ (the Rankin-Selberg convolution of a holomorphic newform f with itself), $L(s) = \zeta_{\mathbb{F}_q[t]}(s)$ (the zeta-function associated to polynomials over the finite field $\mathbb{F}_q$) and $L(s) = \zeta_Q(s)$ (the Epstein zeta-function attached to a positive definite quadratic form).

5. **Variants of Ramanujan Polynomials**

Recently, several generalizations of Ramanujan's formula have appeared in the literature. Each of these formulae have corresponding polynomials with coefficients arising from special values of Dirichlet series. Analogous to Murty-Smyth-Wang, one can investigate the location of zeros of these self-reciprocal polynomials. Since they also involve values of the Dirichlet series at $s = 0$, that is, outside the region of absolute convergence, one cannot apply Proposition 4.3. Therefore, one must treat these instances individually, relying on specific estimates on zeta-values that allow the application of the intermediate value theorem. We treat two such scenarios below.

5.1. **Powers of zeta-values.** In 2019, A. A. Dixit and R. Gupta [8, Theorem 2.1] obtained an analog of Ramanujan's formula for the square of odd zeta values.

Theorem (A. A. Dixit, R. Gupta)**.** *Let $\tau(r) = \#$divisors of r, $\epsilon = e^{i\pi/4}$, $\widetilde{\sigma}_s(n) := \sum_{d|n} \tau(d)\,\tau(n/d)\,d^s$, and*

$$\mathcal{G}_{2k+1}(x) := \zeta^2(2k+1)\left(\gamma + \log\left(\frac{x}{\pi}\right) - \frac{\zeta'(2k+1)}{\zeta(2k+1)}\right) + \sum_{n=1}^{\infty} \frac{\widetilde{\sigma}_{2k+1}(n)}{n^{2k+1}}\left(K_0(4\epsilon\sqrt{nx}) + K_0(4\bar{\epsilon}\sqrt{nx})\right), \qquad x > 0,$$

where $K_z(x) := \frac{\pi}{2}\frac{I_{-z}(x) - I_z(x)}{\sin \pi z}$ is the modified Bessel function of the second kind of order z, and I_z is that of the first kind. Then, for positive real number α and β with $\alpha\beta = \pi^2$,

$$\alpha^{-2k}\,\mathcal{G}_{2k+1}(\alpha) - (-1)^k \beta^{-2k}\,\mathcal{G}_{2k+1}(\beta) = -\pi\, 2^{4k}\,\beta^{2k+2}\sum_{j=0}^{k+1}(-1)^j\left(\frac{B_{2j}}{(2j)!}\,\frac{B_{2k+2-2j}}{(2k+2-2j)!}\right)^2\left(\frac{\alpha}{\beta}\right)^{2j}.$$

This was further generalized to any positive integer power of odd zeta-values by S. Banerjee and V. Sahani [1, Theorem 1.1] in 2023.

Motivated by the analogy of the sums in terms of Bernoulli numbers appearing above with Ramanujan polynomials, and supported by numerical evidence, B. Maji and T. Sarkar in [14] proposed the following conjecture.

Conjecture 1 (Maji-Sarkar)**.** *For positive integers k and ℓ, set*

$$R_{k,\ell}(x) := \sum_{j=0}^{k+1}(-1)^{(\ell+1)j}\left(\frac{B_{2j}}{(2j)!}\,\frac{B_{2k+2-2j}}{(2k+2-2j)!}\right)^{\ell} x^j. \tag{3}$$

Then all the non-real zeros of $R_{k,\ell}(x)$ lie on the unit circle.

Note that $R_{k,1}(x^2) = R_{2k+1}(x)$ is the Ramanujan polynomial. Thus, for $\ell = 1$, the conjecture follows from the theorem of M. R. Murty, C. Smyth, and R. Wang [15].

In a recent joint work with M. Charan and J. Meher [3], we settle the above conjecture and prove

Theorem 5.1 (Charan-Meher-Pathak)**.** *For positive integers k and ℓ, Conjecture 1 is true. More specifically, except for two real zeros of the form $\alpha_{k,\ell}$ and ${\alpha_{k,\ell}}^{-1}$ with $\alpha_{k,\ell} > 1$, the remaining zeros of $R_{k,\ell}(x)$ lie on the unit circle and are simple. Furthermore, for a fixed ℓ, as $k \to \infty$, the non-real zeros of $R_{k,\ell}(x)$ are equidistributed on the unit circle.*

The method of proof of Theorem 5.1 is inspired by that in [15]. However, the corresponding estimates in our case require one to determine the explicit dependence on ℓ, which adds to the difficulty. A brief outline of the proof is as follows.

Sketch of Proof of Theorem 5.1. We work with the monic counterpart of $R_{k,\ell}$, namely

$$M_{k,\ell}(x) = \frac{(-1)^{(\ell+1)(k+1)}}{(B_{2k+2}/(2k+2)!)^{\ell}} R_{k,\ell}(x^2).$$

In terms of zeta-values, Euler's formula implies that

$$M_{k,\ell}(x) = x^{2k+2} + (-1)^{(\ell+1)(k+1)} - 2^{\ell} \sum_{j=1}^{k} (-1)^{(\ell+1)(k+j)} q_j^{\ell} x^{2j},$$

with

$$q_j := \frac{\zeta(2j)\,\zeta(2k+2-2j)}{\zeta(2k+2)} \qquad j = 1, 2, \cdots, k.$$

The case when ℓ is odd and when ℓ is even are treated separately. First, we deduce the following proposition by comparing values of $M_{k,\ell}(z)$ at $z = 1$ and as $z \to \infty$ on the real line, and applying the intermediate value theorem. When k and ℓ are both even, $M_{k,\ell}(1) = 0$, and one needs to study $M'_{k,\ell}(1)$.

Proposition 5.2. *Let k and ℓ be positive integers.*

(a) If ℓ is odd, then $M_{k,\ell}(z)$ has exactly four real zeros, of the form $\beta_{k,\ell}$, $-\beta_{k,\ell}$, $1/\beta_{k,\ell}$ and $-1/\beta_{k,\ell}$ with $\beta_{k,\ell} > 1$.

(b) Suppose that ℓ is even.

(i) If k is odd, then $M_{k,\ell}(z)$ has at least four real zeros of the form $\beta_{k,\ell}$, $-\beta_{k,\ell}$, $1/\beta_{k,\ell}$ and $-1/\beta_{k,\ell}$ with $\beta_{k,\ell} > 1$.

(ii) If k is even, then $M_{k,\ell}(z)$ has at least six real zeros of the form -1, 1, $\beta_{k,\ell}$, $-\beta_{k,\ell}$, $1/\beta_{k,\ell}$ and $-1/\beta_{k,\ell}$ with $\beta_{k,\ell} > 1$.

To complete the proof of Theorem 5.1, we study the number of zeros of $M_{k,\ell}(z)$ on the unit circle. The idea is to approximate $M_{k,\ell}(z)$ by a suitable trigonometric polynomial $A_{k,\ell}(z)$ and count the number of sign changes of the associated real-valued function on the unit circle.

For positive integers k and ℓ with $k \geq 3$, let

$$A_{k,\ell}(z) := z^{2k+2} + (-1)^{(\ell+1)(k+1)} - (2q_1)^{\ell}\left(z^{2k} + (-1)^{(\ell+1)(k+1)} z^2\right) - 2^{\ell} \sum_{j=2}^{k-1} (-1)^{(\ell+1)(k+j)} z^{2j},$$

and set

$$\begin{aligned} \Delta_{k,\ell}(z) &:= M_{k,\ell}(z) - A_{k,\ell}(z) \\ &= -2^{\ell} \sum_{j=2}^{k-1} (-1)^{(\ell+1)(k+j)} (q_j^{\ell} - 1)\, z^{2j}. \end{aligned} \tag{4}$$

We first establish that $A_{k,\ell}(z)$ is a 'good' approximation for $M_{k,\ell}(z)$ on the unit circle.

Proposition 5.3. *For all $z \in \mathbb{C}$ such that $|z| = 1$,*

$$\left|\Delta_{k,\ell}(z)\right| < 2^{\ell} c_{\ell} \times 0.2762, \qquad \text{with} \qquad c_{\ell} = \frac{(1.306)^{\ell} - 1}{0.306}, \tag{5}$$

for $k \geq 3$ and $\ell \geq 1$.

Then, we investigate the behaviour of $M_{k,\ell}(z)$ and $A_{k,\ell}(z)$ on the unit circle. For $\theta \in [0, 2\pi]$, let

$$F_{k,\ell}(\theta) = \begin{cases} -i\, e^{-i(k+1)\theta} M_{k,\ell}(e^{i\theta}) & \text{if } k \text{ and } \ell \text{ are both even,} \\ e^{-i(k+1)\theta} M_{k,\ell}(e^{i\theta}) & \text{otherwise.} \end{cases}$$

The reciprocal nature of $M_{k,\ell}(z)$ implies that the function $F_{k,\ell}(\theta)$ is real-valued. Similarly, let

$$f_{k,\ell}(\theta) = \begin{cases} -i\, e^{-i(k+1)\theta} A_{k,\ell}(e^{i\theta}) & \text{if } k \text{ and } \ell \text{ are both even,} \\ e^{-i(k+1)\theta} A_{k,\ell}(e^{i\theta}) & \text{otherwise.} \end{cases}$$

By the symmetry of coefficients, it is evident that $f_{k,\ell}(\theta)$ is also real-valued. Proposition 5.3 is equivalent to the statement that

$$|F_{k,\ell}(\theta) - f_{k,\ell}(\theta)| < 2^{\ell} c_{\ell} \times 0.2762, \quad \text{with} \quad c_{\ell} = \frac{(1.306)^{\ell} - 1}{0.306}.$$

We choose $\theta_j \in (0, 2\pi)$ for $1 \le j \le 2k-3$ such that $f_{k,\ell}(\theta_j)$ has sign $(-1)^{j+1}$ and

$$\left| f_{k,\ell}(\theta_j) \right| > 2^{\ell} c_{\ell} \times 0.2762.$$

Thus, we establish that $F_{k,\ell}(\theta)$ changes sign at least $2k-1$ times in $[0, 2\pi]$. This proves the following.

Theorem 5.4. *For positive integers k and ℓ with $k \ge 3$, the polynomial $M_{k,\ell}(z)$ has at least $2k-2$ zeros on the unit circle.*

Combining the above result with Proposition 5.2 concludes the proof.

□

5.2. **Values of the Hurwitz zeta-function.** For $0 < \alpha \le 1$ and $\mathrm{Re}(s) > 1$, the Hurwitz zeta function is given by

$$\zeta(s, \alpha) := \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s}.$$

It is known that $\zeta(s, \alpha)$ can be analytically continued to $\mathbb{C}$ except for a simple pole at $s = 1$ with residue 1. Recently, P. Chavan [5] obtained analogs of Ramanujan's formulae for values of the Hurwitz zeta-function. One can consider the polynomials occurring in these identities as analogs of Ramanujan polynomials for the Hurwitz zeta-function. With this analogy, jointly with M. Charan [4] we investigate the unimodularity of their zeros and prove the following.

Theorem 5.5 (M. Charan, S. Pathak)**.** *Let $k \ge 4$ be a positive integer and $0.1 < \alpha \le 1$, $\alpha \ne 1/2$. Define*

$$\mathcal{P}_{k,\alpha}(z) := \sum_{j=0}^{k+1} \zeta(2j, \alpha)\, \zeta(2k + 2 - 2j, \alpha)\, z^j.$$

Then except for 2 real zeros of the form $\gamma_{k,\alpha} > 1$ and $\gamma_{k,\alpha}^{-1}$, all other zeros of $\mathcal{P}_{k,\alpha}(x)$ lie on the unit circle.

The proof is along similar lines as that of Theorem 5.1. However, one needs estimates on values of the Hurwitz zeta-function which make the dependence on α explicit. Although the theorem is proved when $0.1 < \alpha$, computational evidence suggests that the result also holds for $0 < \alpha \le 0.1$. To prove it when α is close to zero, more precise estimates are required than those that are currently known. This is because as $\alpha \to 0$, $\gamma_{k,\ell} \to 1$.

6. **Concluding remarks**

Ramanujan polynomials and their variants, as defined in (1) and (3) are polynomials with rational coefficients. Thus, their zeros are algebraic numbers. An immediate question that arises is whether any of these zeros that are on the unit circle are roots of unity. For Ramanujan polynomials, this was answered in [15]. However, this is not so easy for the variants appearing in the Maji-Sarkar conjecture. Investigating other algebraic properties of these polynomials, such as their irreducibility or Galois groups will be of interest as well.

In the context of polynomials associated to values of Dirichlet L-functions, there are several conjectures proposed by Berndt and Straub [2] which remain open. For instance, they conjecture the following.

Conjecture (Berndt-Straub [2], Conjecture 7.9)**.** *For positive integers L and M, let*

$$S_k(x;\chi,\psi) := \sum_{s=0}^{k} \frac{B_{s,\chi}\, B_{k-s,\psi}}{s!\,(k-s)!} \left(\frac{Lx}{M}\right)^{k-s-1},$$

where $B_{n,\chi}$ are generalized Bernoulli numbers defined by

$$\sum_{a=1}^{L} \chi(a)\, \frac{t\, e^{at}}{e^{Lt}-1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!}$$

for a character χ modulo L. Then for all non-principal real Dirichlet characters χ modulo L, all the non-zero roots of $S_k(x;\chi,\chi)$ lie on the unit circle.

Due to oscillations in character values, the methods discussed in this article are not amenable to approach the above problem and new techniques are necessary for its resolution. We relegate these as avenues for future research.

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