Sup norm bounds for some theta lifts to $\mathrm{O}(1, 8n+1)$

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Abstract

This article is a write-up of the talk presented by the second named author at RIMS workshop "Analytic Number Theory and Related Topics". Our main result is upper and lower bounds in terms of the Laplace eigenvalues for the sup-norms of the cusp forms on the orthogonal groups $\mathrm{O}(1, 8n+1)$ given by theta lifts from Maass cusp forms of level one. Our result is based on an explicit formula for the Petersson norms of the theta lifts with an explicit archimedean local factor and a pre-trace formula for arithmetic quotients of real hyperbolic spaces, the latter of which is a general formula.

1 Main result and background

The problem of estimating the sup-norm of Laplace eigenfunctions on a compact Riemannian manifold is a fundamental one in harmonic analysis and mathematical physics. The most basic result on this problem is due to Avacumović [1] and Levitan [21]. If X is a compact Riemannian manifold without boundary and ϕ is an eigenfunction of the Laplacian on X with eigenvalue $-\Lambda \leq 0$, they show that

$$\frac{||\phi||_\infty}{||\phi||_2} \ll \Lambda^{\frac{\dim X - 1}{4}}. \tag{1}$$

In general, this is the best possible estimate and is indeed achieved when X is a sphere. On the other hand, one expects stronger bounds to hold on a generic manifold. For instance, when X is negatively curved, Bérard [2] showed that the bound (1) can be improved by a factor of $\sqrt{\log \Lambda}$. Moreover, it is often possible to improve (1) by a power of Λ under arithmetic assumptions on X and ϕ. The first such improvement is due to Iwaniec and Sarnak [17]. They considered X a congruence arithmetic hyperbolic surface arising from a quaternion algebra over $\mathbb{Q}$ (possibly split), and ϕ a Hecke–Maass cusp form. In this case, they improved on (1) to obtain

$$\frac{||\phi||_\infty}{||\phi||_2} \ll_\epsilon \Lambda^{\frac{5}{24}+\epsilon}. \tag{2}$$

They also obtained a lower bound $\sqrt{\log\log \Lambda}$ for an infinite sequence of ϕ, which was later improved to $\exp((1+o(1))\sqrt{\log \Lambda / \log\log \Lambda})$ by Milićević [24]. Let us also mention several other works on the sup-norm problem [8], [9], [11], [13], [31], [32]. Among other things we also cite [27], which includes an excellent review of the sup-norm problems in terms of elliptic differential operators.

We note that in these arithmetic settings, where X is taken to be a locally symmetric space, one generally assumes that ϕ is an eigenfunction not just of the Laplacian, but of the full ring of invariant differential operators. Under this assumption on ϕ, it was shown by Sarnak [33] that the bound (1) can be strengthened to

$$\frac{||\phi||_\infty}{||\phi||_2} \ll \Lambda^{\frac{\dim X - \mathrm{Rank} X}{4}}. \tag{3}$$

This is sharp on spaces of compact type, and is the natural analog of (1) for these eigenfunctions.

It is also interesting to investigate the sup-norm of eigenfunctions in the case when X is not compact. In this case, we now assume that ϕ is square-integrable. In the noncompact setting, the bound (1) (and (3), under the appropriate assumptions) continues to hold on fixed compact sets, and it is natural to ask whether it in fact holds globally (i.e. over the entire manifold). For this to happen one certainly needs $\|\phi\|_\infty$ to be finite, and this is not generally true, even when X is a finite volume locally symmetric space. However, it is true if one assumes that X is a finite volume locally symmetric space and ϕ is cuspidal (cf. [16]), and so we make these assumptions from now on.

In some cases, it is known that the bounds (1) and (3) hold globally. See for instance [7] for the case of Hecke–Maass forms on GL_2 over a number field, and [6] for Hecke–Maass forms on the space $\mathrm{PGL}_3(\mathbb{Z})\backslash\mathrm{PGL}_3(\mathbb{R})/\mathrm{PO}(3)$. (Note that [6] establishes a bound stronger than (1), but not as strong as (3).) On the other hand, it was shown in [13] that (1) fails on $\mathrm{PGL}_n(\mathbb{Z})\backslash\mathrm{PGL}_n(\mathbb{R})/\mathrm{PO(n)}$ for $n \geq 6$. The reason for this failure is the large peaks of the GL_n Whittaker function in higher rank, which lead to large values of cusp forms via the Fourier expansion. Moreover, it is generally expected that the large values produced in this way occur high in the cusp, at height roughly $\sqrt{\Lambda}$; [13] establishes the weaker result that the suprema of these cusp forms occur at points tending to infinity. See [5] for an upper bound that complements the lower bound of [13].

This phenomenon of cusp forms having a peak high in the cusp is a general one, which is already present for GL_2 where it is caused by the transition range of the Bessel function $K_{\sqrt{-1}r}(y)$ at $y \sim r$ [33]. However, in this case the peak produced is only of size $\Lambda^{1/12}$, which is smaller than (1). These results lead one to ask the general question of whether a sequence of cusp forms on X realize their suprema in a fixed compact set, or at a sequence of points tending to infinity.

We remark that if, instead of Maass forms, one considers holomorphic modular forms of growing weight for the group $\mathrm{SL}_2(\mathbb{Z})$, then the growth rate of the sup norm was determined by Xia [34] using the Fourier expansion. Similar results in the case of Siegel modular forms were obtained by Blomer [4].

To state he main result of this article we introduce an even unimodular lattice (L, S) of rank N with the quadratic form defined by the positive definite even integral matrix S. We note that N is divisible by 8. Let $L \oplus \mathbb{Z}^2$ with the quadratic form defined by $Q = \begin{pmatrix} & & 1 \\ & -S & \\ 1 & & \end{pmatrix}$, where $\mathbb{Z}^2$ is viewed as a lattice of the hyperbolic plane. The symmetric matrix Q defines the orthogonal group $\mathrm{O}(1, N+1)$ over $\mathbb{Q}$. The main result is a formula for sup norm bounds of cusp forms F_f on orthogonal groups $\mathrm{O}(1, N+1)$ obtained by theta lifts from f belonging to the space $S(\mathrm{SL}_2(\mathbb{Z}); -\frac{r^2+1}{4})$ of Maass cusp forms of level one with the spectral parameter r. The automorphy of cusp forms F_f respects the arithmetic subgroup of $\mathrm{O}(1, N+1)$ attached to the lattice L.

Theorem. 1.1. *Let $f \in$ be as above and assume that f is a non-zero Hecke eigenform and the Fourier coefficients $\{c(m) : m \in \mathbb{Z}\}$ of f satisfy $c(m) = \pm c(-m)$ for all $m \in \mathbb{Z}$. Let F_f be the lift of f to a cusp form on $\mathrm{O}(1, N+1)$ introduced above. Let $-\Lambda$ be the Laplace eigenvalue of F_f. Then for any $\epsilon > 0$ and any r, we have*

$$\frac{||F_f||_\infty}{||F_f||_2} \ll_{N,\epsilon,\Gamma} \Lambda^{\frac{N}{4}+\frac{N(1+2\theta)}{8(N+1+2\theta)}+\epsilon} \leq \Lambda^{\frac{N}{4}+\frac{\theta}{4}+\frac{1}{8}+\epsilon},$$

where $\theta = 7/64$ is the current best estimate towards the Ramanujan conjecture for Maass forms (cf. [18]). We also have the lower bound

$$\Lambda^{\frac{N}{8}+\frac{1}{12}-\epsilon} \ll_{N,\epsilon,\Gamma} \frac{||F_f||_\infty}{||F_f||_2}. \tag{4}$$

We note that the bound (1) in this case reads $||F_f||_\infty/||F_f||_2 \ll \Lambda^{N/4}$, so that we are not quite able to obtain this globally. We remark that we are free to assume that $r \gg 1$ when proving Theorem 1.1. On the other hand, this condition holds automatically, as it is known that any cuspidal Laplace eigenfunction on the modular surface has eigenvalue $-\Lambda < -1/4$, see for instance [10, 29].

Remark 1. As alluded to above, the lower bound we prove for F_f is obtained high in the cusp. In contrast to this, there are several papers that establish power growth of eigenfunctions on hyperbolic manifolds, and more generally on locally symmetric spaces of noncompact type, in *fixed* compact sets [11, 12, 14, 20, 23, 30]. We mention in particular [12, 14], which apply to the higher-dimensional hyperbolic setting considered here. (The results of [11] also include this, but the growth exponents produced are ineffective.) In [14], Donnelly constructs compact hyperbolic $(N+1)$-manifolds for $N \geq 4$, and sequences of Laplace eigenfunctions $\{\phi_i\}_i$ that satisfy $\|\phi_i\|_\infty/\|\phi_i\|_2 \gg \Lambda_i^{(N-3)/4}$. In [12], this lower bound was improved to $\Lambda_i^{(N-1)/4-\epsilon}$ for N even, and it is expected that this parity condition can be removed. Moreover, similarly to F_f, the forms constructed in these papers are theta lifts from SL_2. These results lead one to hope that F_f might satisfy the same lower bound, which should be realized in a fixed compact subset of the manifold. As this is larger than the lower bound of $\Lambda^{\frac{N}{8}+\frac{1}{12}-\epsilon}$ obtained from the peak of the Whittaker function, one might therefore expect F_f to realize their sup norms in the bulk, rather than the cusp.

Notation

If $A \in \mathbb{C}$ and $B \in \mathbb{R}$, we use the notation $A \ll B$ to mean that there is a constant $C > 0$ such that $|A| \leq CB$, and $A \sim B$ means that $A \ll B \ll A$.

2 Theta lifts F_f and its Petersson norms

The real Lie group for $\mathrm{O}(1, N+1)$, say G, admits an Iwasawa decomposition $G = NAK$, where

$$N := \left\{ n(x) = \begin{pmatrix} 1 & {}^t xS & \frac{1}{2}{}^t xSx \\ & 1_N & x \\ & & 1 \end{pmatrix} \;\middle|\; x \in \mathbb{R}^N \right\}, \quad A := \left\{ a_y = \begin{bmatrix} y & & \\ & 1_N & \\ & & y^{-1} \end{bmatrix} \;\middle|\; y \in \mathbb{R}_+^\times \right\}, \tag{5}$$

and K denotes a maximal compact subgroup. From the Iwasawa decomposition we can identify the homogeneous space G/K with the $(N+1)$-dimensional real hyperbolic space $H_N := \{(x, y) \mid x \in \mathbb{R}^N,\ y \in \mathbb{R}_{>0}\}$ by the natural map

$$n(x)a_y \mapsto (x, y).$$

The cusp forms we are going to study are regarded as cusp forms on the real hyperbolic space H_N.

For $\lambda \in \mathbb{C}$ and a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ we denote by $S(\Gamma, \lambda)$ the space of Maass cusp forms of weight 0 on the complex upper half plane $\mathfrak{h} := \{u + \sqrt{-1}v \in \mathbb{C} \mid v > 0\}$ whose eigenvalue with respect to the hyperbolic Laplacian is λ.

For $r \in \mathbb{C}$ we denote by $\mathcal{M}(\Gamma_S, r)$ the space of smooth functions F on G satisfying the following conditions:

i) $\Omega \cdot F = \dfrac{1}{2N}\left(r^2 - \dfrac{N^2}{4}\right)F$, where Ω is the Casimir operator defined in [22, (2.3)],

ii) for any $(\gamma, g, k) \in \Gamma \times G \times K$, we have $F(\gamma g k) = F(g)$,

iii) F is of moderate growth.

Let $r \in \mathbb{R}$. We say that $F \in \mathcal{M}(\Gamma_S, \sqrt{-1}r)$ is a cusp form if it vanishes at the cusps of Γ_S. By [22, Section 2.3], any cusp form $F \in \mathcal{M}(\Gamma_S, \sqrt{-1}r)$ has a Fourier expansion of the form

$$F(n(x)a_y) = \sum_{\lambda \in L\setminus\{0\}} A(\lambda) y^{\frac{N}{2}} K_{\sqrt{-1}r}(4\pi|\lambda|_S y) \exp(2\pi\sqrt{-1}\,{}^t\lambda Sx). \tag{6}$$

Here, $|\lambda|_S := \sqrt{q_S(\lambda)}$ and $K_{\sqrt{-1}r}$ is the K-Bessel function.

Let

$$f(\tau) = \sum_{n\neq 0} c(n) W_{0,\frac{\sqrt{-1}r}{2}}(4\pi|n|v)\exp(2\pi\sqrt{-1}nu) \in S(\mathrm{SL}_2(\mathbb{Z}); -\frac{r^2+1}{4}) \tag{7}$$

be a Maass cusp form on $\mathfrak{h}$, where we use the Whittaker function $W_{0,\frac{\sqrt{-1}r}{2}}$ to describe the Fourier expansion of f. Recall that we have supposed that L is an even unimodular lattice of rank N divisible by 8. For $0 \neq \lambda \in L$, define

$$A(\lambda) := |\lambda|_S \sum_{d|d_\lambda} c\left(-\frac{|\lambda|_S^2}{d^2}\right) d^{\frac{N}{2}-2}, \tag{8}$$

where d_λ denotes the greatest integer such that $\frac{1}{d_\lambda}\lambda \in L$. By Theorems 3.1, 3.3 and 4.11 from [22] we have

Theorem. 2.1. *Let L be an even unimodular lattice. Let $f \in S(\mathrm{SL}_2(\mathbb{Z}); -\frac{r^2+1}{4})$ with Fourier expansion (7). Let $F_f : H_N \to \mathbb{C}$ be given by the Fourier expansion (6) with Fourier coefficients $A(\lambda)$ defined in (8). Then*

i) The map $f \to F_f$ is a map from $S(\mathrm{SL}_2(\mathbb{Z}); -\frac{r^2+1}{4})$ to $\mathcal{M}(\Gamma_S, \sqrt{-1}r)$ preserving cuspidality.

ii) If f is a Hecke eigenform, then so is F_f.

The second assertion needs the adelic reformulation of F_f. Such formulation is useful to obtain the explicit formula for the Petersson norm of F_f as follows:

Theorem. 2.2. *Let $f \in S(\mathrm{SL}_2(\mathbb{Z}); -\frac{r^2+1}{4})$ be a Hecke Maass eigenform with respect to $\mathrm{SL}_2(\mathbb{Z})$. Then the Petersson norm of F_f is given by the formula*

$$||F_f||^2 = \frac{L(\frac{N}{2}, \pi, \mathrm{Ad})}{\zeta(\frac{N}{2}+1)\zeta(N)} \Big(2^{1-\frac{N}{2}}\pi^2 \frac{\Gamma(\frac{N}{4}+\frac{\sqrt{-1}r}{2})\Gamma(\frac{N}{4}-\frac{\sqrt{-1}r}{2})}{\Gamma(\frac{N}{4}+\frac{1}{2})^2}\Big)||f||^2.$$

Here, $L(s, \pi, \mathrm{Ad})$ is the finite part of the degree 3 adjoint L-function of π.

To get this, we use the well known method of [26]. We first obtain an adelization of the theta lift construction of Borcherds for our case, which is nothing but the adelization mentioned above. For this, we follow the work of Kudla [19] where the adelization of the Borcherds construction is obtained for orthogonal groups of signature $(p, 2)$. The L^2-norm of the adelization Φ of F_f can then be rewritten as

$$||\Phi||^2 = \int_{\mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A})} \Big(\int_{\mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A})} f_0(g_1) E((g_1^*, g_2), s_0; \Xi_0) dg_1 \Big) \overline{f_0(g_2)} dg_2.$$

Here, f_0 is the adelization of f, Ξ_0 is a section in an induced representation of $\mathrm{Sp}_4(\mathbb{A})$ obtained from the Weil representation corresponding to the theta lifting, and E is an Eisenstein series on Sp_4 obtained from the Siegel Weil formula. As in [25], the inner integral is Eulerian and can be written as a product of local integrals (see also [15, Section 11]). Since we are restricted to Maass forms f of level 1, in the non-archimedean case, all the data is unramified and the integral is obtained in [25]. The main contribution here is the archimedean integral computation, which is never an immediate consequence from general formulas e.g. [26], [15] etc. The key ingredient of the archimedean computation is to realize that the archimedean section Ξ_∞ is in the trivial K-type of the $\mathrm{Sp}_4(\mathbb{R})$ representation, i.e. it is invariant under the maximal compact subgroup of $\mathrm{Sp}_4(\mathbb{R})$. Though the inner product formula is a well-known method to show the injectivity of theta lifts (cf. [26]), let us remark here that, in [22], the injectivity of the map $f \to F_f$ was proven for general even unimodular lattices by using the Fourier expansion at a cusp corresponding to a sum of copies of the E_8-lattice, which has the property that it contains vectors of squared length M for *all* $M \in \mathbb{Z}_{>0}$. This yields another proof for the injectivity of the map $f \to F_f$, which also works in full generality.

3 Upper bounds using the pre-trace formula

In this section, we use a pre-trace inequality to obtain upper bounds on the lifted form F_f. The bound we prove holds for any square-integrable Laplace eigenfunction on any hyperbolic orbifold X of finite volume with the Laplacian Δ. To state it, we will need the notion of the height of a point $x \in X$ in the cusp, denoted by $\mathrm{ht}(x)$, which we recall in this section. We note that only for this section, $\sqrt{-1}r$ will denote the spectral parameter of an eigenfunction on X, instead of the parameter of the Maass form f. We note that these two spectral parameters are the same size, so this should not lead to any confusion.

Theorem. 3.1. *Let X be a finite-volume hyperbolic orbifold of dimension $N+1$. Let $\psi \in L^2(X)$ be an L^2-normalized Laplace eigenfunction with spectral parameter $\sqrt{-1}r$, so that $(\Delta + r^2 + N^2/4)\psi = 0$. We have*

$$\psi(x) \ll (1+|r|)^{N/2} + \mathrm{ht}(x)^{N/2}(1+|r|)^{N/4}, \quad x \in X.$$

We shall assume in the proof that $r > 1$, as the other case is similar.

3.1 Background on hyperbolic geometry

We let G^0 be the group of isometries of H_N, which can be identified with the index two subgroup of $\mathrm{O}(1,N)$ preserving the upper sheet of the two-sheeted hyperboloid. Let d be the standard distance function on H_N, and let $\partial H_N \simeq S^N$ denote the boundary sphere.

Let $X = \Gamma \backslash H_N$, where $\Gamma < G^0$ is a lattice. Let $O(\Gamma) \subset \partial H_N$ denote the set of fixed points of parabolic elements of Γ. We fix a set of representatives Ξ for the Γ-orbits in $O(\Gamma)$, which can be identified with the set of cusps of X. We let Γ_ξ be the stabilizer of $\xi \in \Xi$ in Γ. For each $\xi \in \Xi$ we choose a horoball B_ξ tangent to the boundary at ξ, and denote $V_\xi = \Gamma_\xi \backslash B_\xi$. By [28, Thm 12.7.4], we may choose each B_ξ so that the following hold:

i) V_ξ is mapped isometrically to its image in X.

ii) We have $X = X_0 \coprod V_\xi$, where X_0 is compact.

iii) If $\gamma \in \Gamma - \Gamma_\xi$, then $B_\xi \cap \gamma B_\xi = \varnothing$.

We note that [28] states this theorem for manifolds, rather than orbifolds, but one may easily obtain the result for an orbifold by using Selberg's lemma to pass to a finite-index torsion-free subgroup of Γ. We now define the height function $\mathrm{ht} : X \to \mathbb{R}_{\geq 1}$. If $x \in X_0$, we set $\mathrm{ht}(x) = 1$. Next, suppose that $x \in V_\xi$ for some ξ. Let B_∞ denote the standard horoball $\{(x,y) : x \in \mathbb{R}^N, y > 1\}$, and choose $g_\xi \in G^0$ such that $g_\xi B_\xi = B_\infty$. We define $\mathrm{ht}(x) = y(g_\xi x)$ (the y-coordinate of $g_\xi x$), which is independent of the choice of g_ξ. This has the key property that for any $C > 0$, the set of points $x \in X$ with $\mathrm{ht}(x) \leq C$ is compact.

3.2 Test functions

In this section, we construct a test function for use in the pre-trace inequality. We shall do this using the Harish-Chandra transform, which we now recall. Define φ_s to be the standard spherical function on H_N or G^0 with spectral parameter s. We continue to normalize s so that $\sqrt{-1}\mathbb{R}$ is the tempered axis. Let $K \simeq \mathrm{O}(N)$ be the standard maximal compact subgroup of G^0. For a K-biinvariant function $k \in C_c^\infty(G^0)$, we define its Harish-Chandra transform by

$$\widehat{k}(s) = \int_{G^0} k(g)\varphi_s(g)dg.$$

This is inverted by

$$k(g) = C_N \int_{\sqrt{-1}\mathbb{R}} \widehat{k}(s)\varphi_s(g)|c(s)|^{-2}ds$$

for a constant C_N, where $c(s)$ is Harish-Chandra's c-function. We may now define the test function we shall use.

Lemma. 3.2. *There exists a K-biinvariant function $k_r \in C_c^\infty(G^0)$ with the following properties:*

(i) *k_r is supported in a fixed compact set that is independent of r.*

(ii) *$\widehat{k}_r(s) \geq 0$ for $s \in \sqrt{-1}\mathbb{R} \cup (0, N/2]$.*

(iii) *$\widehat{k}_r(\sqrt{-1}r) = 1$.*

(iv) *$k_r(g) \ll r^N(1 + rd(g, e))^{-N/2}$. In particular, $\|k_r\|_\infty \ll r^N$.*

3.3 The pre-trace inequality

Let k_r be as in Lemma 3.2. The fundamental inequality we shall use to bound ψ is

$$|\psi(x)|^2 \leq \sum_{\gamma \in \Gamma} k_r(x^{-1}\gamma x). \tag{9}$$

This may be derived from the pre-trace formula, by using the positivity property (ii) of $\widehat{k}_r$ to drop all terms on the spectral side other than $|\psi(x)|^2$ (including the continuous spectrum). For this we remark that the parameters of the discrete spectrum are contained in $\sqrt{-1}\mathbb{R} \cup (0, N/2]$, which parametrize equivalence classes of irreducible unitary spherical principal series representations of $O(1, N)$ together with spherical complimentary series representations.

It may also be proved in an elementary way by an application of Cauchy–Schwartz and unfolding, see for instance [12, Lemma 6.5], with the test function ω there taken to be the function k_r^0 satisfying $\widehat{k}_r^0 = h_r$ so that $k_r^0 * (k_r^0)^* = k_r$.

Let $R > 0$ be a constant, independent of r, such that the support of k_r is contained in the open ball of radius R about the origin $(0, 1)$ in H_N. For x in any fixed compact subset of X, (9) and (iv) give $|\psi(x)| \ll r^{N/2}$, so we may assume that $\mathrm{ht}(x) > e^R > 1$. In particular, x is contained in a cusp neighborhood V_ξ, which we shall assume to be fixed for the rest of the proof. Moreover, until further notice we identify x with a choice of lift $x \in B_\xi$. Under this assumption, we shall show that only $\gamma \in \Gamma_\xi$ contribute to (9). Indeed, suppose that $\gamma \in \Gamma$ satisfies $k_r(x^{-1}\gamma x) \neq 0$. This implies that $d(\gamma x, x) < R$. This, together with $\mathrm{ht}(x) > e^R$, implies that $\gamma x \in B_\xi$, so that $\gamma B_\xi \cap B_\xi \neq \varnothing$, and hence that $\gamma \in \Gamma_\xi$ as required.

We now assume that our cusp ξ is the standard point at infinity, and denote Γ_ξ and B_ξ by Γ_∞ and B_∞, as the proof for the other cusps is similar. We therefore have

$$|\psi(x)|^2 \leq \sum_{\gamma \in \Gamma_\infty} k_r(x^{-1}\gamma x).$$

We next apply our upper bound (iv) for k_r, which gives

$$|\psi(x)|^2 \ll r^N \sum_{\substack{\gamma \in \Gamma_\infty \\ d(\gamma x, x) < R}} (1 + rd(\gamma x, x))^{-N/2}.$$

We may identify Γ_∞ with a lattice in $\mathrm{Isom}(\mathbb{R}^N)$. By [28, Thm 7.5.2], Γ_∞ has a finite index subgroup of translations of rank N, which we identify with a lattice $L \subset \mathbb{R}^N$. The action of L on H_N will be written additively. We let $\gamma_1, \ldots, \gamma_k$ be coset representatives for $L \backslash \Gamma_\infty$, and define $x_i = \gamma_i x$. We therefore have

$$|\psi(x)|^2 \ll r^N \sum_{i=1}^{k} \sum_{\substack{\ell \in L \\ d(\ell + x_i, x) < R}} (1 + rd(\ell + x_i, x))^{-N/2}.$$

Theorem 3.1 now follows from the following lemma.

Lemma. 3.3. *Let $v_1, v_2 \in \mathbb{R}^N$, and let $x_i = (v_i, y) \in H_N$ with $y > 1$. We have*

$$\sum_{\substack{\ell \in L \\ d(\ell + x_1, x_2) < R}} (1 + rd(\ell + x_1, x_2))^{-N/2} \ll 1 + y^N r^{-N/2}.$$

4 Outline of the proof of Theorem 1.1

In this section we will give an outline for the proof of Theorem 1.1. The lower bound comes from the first nonzero Fourier coefficients of F_f. Regarding the upper bound, first note that Theorem 3.1 gives

$$|F(n_x a_y h)|/\|F_f\|_2 \ll r^{N/2} + y^{N/2} r^{N/4}$$

for all $h \in \mathcal{H}(\mathbb{A}_f)$ and all $y \gg 1$, where we are using the Fourier expansion to convert between the classical and adelic pictures. By the Fourier expansion combined with some transition behavior of the Bessel function, we have

$$\frac{1}{\|F_f\|_2} |F_f(n(x)a_y)| \ll_{\epsilon,N,L} \begin{cases} y^{-N/2-1-2\theta} r^{3N/4+1+2\theta+\epsilon} & 1 \ll y \leq r^{11/12}; \\ y^{-N/2+1-2\theta} r^{3N/4-5/6+2\theta+\epsilon} & r^{11/12} < y \leq r/2\pi; \\ e^{-Cy} & r/2\pi < y. \end{cases} \tag{10}$$

Here, $\theta = 7/64$ is the current best estimate towards the Ramanujan conjecture for Maass forms, and we note that $\Lambda \sim r^2$. Thus it suffices to consider the range where $1 \ll y < r/2\pi$. We consider the point $y_0 = r^{(N/2+1+2\theta)/(N+1+2\theta)}$, which is chosen so that the expressions $y^{N/2} r^{N/4}$ and $y^{-N/2-1-2\theta} r^{3N/4+1+2\theta}$ obtained by the pre-trace formula approach (cf. Theorem 3.1) and the Fourier expansion approach (cf. (10)) are both equal to $r^{N/2+N(1+2\theta)/4(N+1+2\theta)}$ (our desired upper bound) when evaluated at y_0. An elementary computation gives us that $y_0 < r^{11/12}$.

The expression $r^{N/2} + y^{N/2} r^{N/4}$ is increasing in y, so for $1 \ll y \leq y_0$ we have

$$\begin{aligned} |F(n_x a_y h)|/\|F_f\|_2 &\ll r^{N/2} + y^{N/2} r^{N/4} \leq r^{N/2} + y_0^{N/2} r^{N/4} \\ &\ll r^{N/2+N(1+2\theta)/4(N+1+2\theta)}. \end{aligned}$$

We next suppose that $y_0 \leq y < r/2\pi$. Because the upper bound given by (10) is decreasing in y (including across the transition point at $y = r^{11/12}$), we may obtain an upper bound for $|F(n_x a_y h)|/\|F_f\|_2$ by evaluating the upper bound at y_0. As $y_0 < r^{11/12}$, this gives

$$|F(n_x a_y h)|/\|F_f\|_2 \ll y_0^{-N/2-1-2\theta} r^{3N/4+1+2\theta+\epsilon} = r^{N/2+N(1+2\theta)/4(N+1+2\theta)+\epsilon}.$$

Note that we have $\Lambda \sim r^2$ since $r \gg 1$. Combining the above computations completes the proof of the upper bound in Theorem 1.1.

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