Algebraic independence of the values of a certain function invariant under the action of the dihedral group

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1 Introduction

Liouville series $L(z) = \sum_{k=0}^{\infty} z^{k!}$ is known to have the following property:

Theorem 1.1 (Nishioka [3]). *Let $q_1, \ldots, q_r$ be algebraic numbers with $0 < |q_i| < 1$ $(1 \leq i \leq r)$. Then the values $L(q_1), \ldots, L(q_r)$ are algebraically dependent if and only if there exist i, j with $1 \leq i < j \leq r$ such that q_i/q_j is a root of unity.*

In what follows, $\overline{\mathbb{Q}}^{\times}$ denotes the set of nonzero algebraic numbers. Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ and let μ_∞ be the multiplicative group consisting of all the roots of unity. Theorem 1.1 means that the points of $D \cap \overline{\mathbb{Q}}^{\times}$ at which $L(z)$ takes algebraically dependent values lie on the same orbit under the action of the group μ_∞ defined by $\mu_\infty \times D \ni (\zeta, z) \to \zeta z \in D$.

Although μ_∞ is an infinite group, in this paper we consider the functions of several variables with the following property: The points at which such a function takes the same value always lie on the identical orbit under the action of a finite group including non-commutative one, except trivial cases such as $f(X, Y) = g(X^N, Y^N)$ with the multiplicative action of the cyclic group $G = \langle(\zeta_N, \zeta_N)\rangle$, where ζ_N is a primitive N-th root of unity. Let $\{R_k\}_{k\geq 1}$ be a linear recurrence of positive integers satisfying

$$R_{k+n} = c_1 R_{k+n-1} + \cdots + c_n R_k \quad (k \geq 1), \tag{1}$$

where $n \geq 2$ and $c_1, \ldots, c_n$ are nonnegative integers with $c_n \neq 0$. In this paper we assume that g.c.d.$(R_k, R_{k+1}, \ldots, R_{k+n-1}) = 1$ for all $k \geq 1$ to exclude the trivial case above. The author [8] studied the two-variable function $E(x, q)$ defined by

$$E(x, q) = \sum_{k=1}^{\infty} \prod_{l=1}^{k} \frac{xq^{R_l}}{1 - q^{R_l}} = \sum_{k=1}^{\infty} \frac{x^k q^{R_1 + R_2 + \cdots + R_k}}{(1 - q^{R_1})(1 - q^{R_2}) \cdots (1 - q^{R_k})},$$

which may be regarded as an analogue of q-exponential function

$$E_q(x) = 1 + \sum_{k=1}^{\infty} \frac{x^k q^{1+2+\cdots+k}}{(1 - q)(1 - q^2) \cdots (1 - q^k)}$$

(cf. Gasper and Rahman [1]), if we replace k in the exponent of q in $E_q(x)$ with $\{R_k\}_{k\geq 1}$ defined by (1). Let

$$\Phi(X) = X^n - c_1 X^{n-1} - \cdots - c_n.$$

The author proved the following:

Theorem 1.2 (Corollary 4 of [8]). *Let $\{R_k\}_{k\ge1}$ be a linear recurrence satisfying* (1)*. Suppose that $\Phi(\pm1)\neq 0$ and that the ratio of any pair of distinct roots of $\Phi(X)$ is not a root of unity. Assume that* g.c.d.$(R_k, R_{k+1},\dots,R_{k+n-1}) = 1$ *for all $k\ge 1$. Then the infinite set of the values*

$$\left\{ E(x,q) \;\middle|\; x,q\in\overline{\mathbb{Q}}^{\times},\ |q|<1 \right\}$$

is algebraically independent.

Remark 1.3. It is shown in Remark 2 of [5] that, if $\Phi(\pm1)\neq0$ and if the ratio of any pair of distinct roots of $\Phi(X)$ is not a root of unity, then the $\{R_k\}_{k\ge1}$ defined by (1) is expressed as $R_k = c\rho^k + o(\rho^k)$ with $c>0$ and $\rho>1$, which guarantees the convergence of the series $E(x,q)$.

Remark 1.4. The condition that g.c.d.$(R_k, R_{k+1},\dots,R_{k+n-1}) = 1$ for all $k\ge1$ implies that the sequence $\{R_k\}_{k\ge1}$ is not a geometric progression, which is a technically inevitable condition for the values $E(x,q)$ to be treated as special values of Mahler functions of several variables similarly to our poof in Section 3.

Example 1.5. Let $\{F_k\}_{k\ge1}$ be the sequence of Fibonacci numbers defined by $F_1=F_2=1$ and $F_{k+2}=F_{k+1}+F_k$ $(k\ge1)$. Since $\{F_k\}_{k\ge1}$ satisfies all the conditions of Theorem 1.2, the infinite set of the values

$$\left\{ \sum_{k=1}^{\infty}\frac{x^k q^{F_1+F_2+\cdots+F_k}}{(1-q^{F_1})(1-q^{F_2})\cdots(1-q^{F_k})} \;\middle|\; x,q\in\overline{\mathbb{Q}}^{\times},\ |q|<1 \right\}$$

is algebraically independent.

Since $R_k = c\rho^k + o(\rho^k)$ with $c>0$ and $\rho>1$ by Remak 1.3, the two-variable function $E(x,q)$ converges on the domain

$$(\mathbb{C}\times\{|q|<1\})\cup(\{|x|<1\}\times\{|q|>1\}) := \{(x,q)\in\mathbb{C}^2 \mid |q|<1\vee(|x|<1\wedge|q|>1)\},$$

whereas a ‘balanced’ analogue

$$\Theta(x,q) = \sum_{k=1}^{\infty}\prod_{l=1}^{k}\frac{xq^{R_l}}{1-q^{2R_l}} = \sum_{k=1}^{\infty}\frac{x^k q^{R_1+R_2+\cdots+R_k}}{(1-q^{2R_1})(1-q^{2R_2})\cdots(1-q^{2R_k})}$$

converges on the wider domain

$$\mathbb{C}\times\{|q|\neq1\} := \{(x,q)\in\mathbb{C}^2 \mid |q|\neq1\}.$$

Indeed, if $q\neq0$, then $\Theta(x,q)$ is invariant under the map

$$\sigma_1 \;:\; (x,q)\longmapsto(-x,q^{-1}),$$

namely

$$\Theta(\sigma_1(x,q)) = \sum_{k=1}^{\infty}\frac{(-x)^k q^{-R_1-R_2-\cdots-R_k}}{(1-q^{-2R_1})(1-q^{-2R_2})\cdots(1-q^{-2R_k})} = \Theta(x,q)$$

and so $\Theta(x,q)$ converges on $\mathbb{C}\times\{|q|\neq 1\}$ by the similar reason to the convergence of $E(x,q)$ on $\mathbb{C}\times\{|q|<1\}$.

Moreover, if $\{R_k\}_{k\geq 1}$ is a sequence of odd integers, then $\Theta(x,q)$ is invariant also under the maps

$$\sigma_2 \;:\; (x,q)\longmapsto(-x,-q) \qquad\text{and}\qquad \sigma_3 \;:\; (x,q)\longmapsto(x,-q^{-1}).$$

Since $\sigma_1\circ\sigma_1=\sigma_2\circ\sigma_2=\mathrm{id}$ and $\sigma_1\circ\sigma_2=\sigma_2\circ\sigma_1=\sigma_3$, we see that $G_4=\{\mathrm{id},\,\sigma_1,\,\sigma_2,\,\sigma_3\}$ is Klein four-group. Therefore, $\Theta(x,q)$ can be regarded as a map defined on the set of orbits $(\mathbb{C}\times\{|q|\neq 0,1\})/G_4$, where $\mathbb{C}\times\{|q|\neq 0,1\}=\{(x,q)\in\mathbb{C}^2 \mid |q|\neq 0,1\}$, namely the map

$$\widetilde{\Theta} \;:\; (\mathbb{C}\times\{|q|\neq 0,1\})/G_4 \longrightarrow \Theta(\mathbb{C}\times\{|q|\neq 0,1\})$$

given by

$$\overline{(x,q)}\longmapsto\Theta(x,q),$$

where $\overline{(x,q)}$ denotes the orbit represented by (x,q), is a well-defined surjection. Hence the restriction to algebraic points

$$\widetilde{\Theta}:\left((\mathbb{C}\times\{|q|\neq 0,1\})\cap\left(\overline{\mathbb{Q}}^{\times}\right)^2\right)\Big/G_4\longrightarrow\Theta\left((\mathbb{C}\times\{|q|\neq 0,1\})\cap\left(\overline{\mathbb{Q}}^{\times}\right)^2\right),$$

or equivalently

$$\widetilde{\Theta}:\left(\overline{\mathbb{Q}}^{\times}\times(\overline{\mathbb{Q}}^{\times}\setminus\{|q|=1\})\right)\Big/\,G_4\longrightarrow\Theta\left(\overline{\mathbb{Q}}^{\times}\times(\overline{\mathbb{Q}}^{\times}\setminus\{|q|=1\})\right)$$

is also a well-defined surjection. The author proved the following:

Theorem 1.6 (Theorem 1 of [9])**.** *Let $\{R_k\}_{k\geq 1}$ be a linear recurrence satisfying* (1)*. Suppose that $\Phi(-1)\neq 0$, $\Phi(2)<0$, and the ratio of any pair of distinct roots of $\Phi(X)$ is not a root of unity. Assume that* g.c.d.$(R_k,R_{k+1},\ldots,R_{k+n-1})=1$ *for all $k\geq 1$. Assume further that $\{R_k\}_{k\geq 1}$ is a sequence of odd integers. Then the infinite set of the values*

$$\widetilde{\Theta}\left(\left(\overline{\mathbb{Q}}^{\times}\times(\overline{\mathbb{Q}}^{\times}\setminus\{|q|=1\})\right)\Big/\,G_4\right)$$

is algebraically independent.

Remark 1.7. It is easily seen that $\Phi(2)<0$ implies that $\Phi(1)\neq 0$.

Corollary 1.8. *Let $\{R_k\}_{k\geq 1}$ be as in Theorem* 1.6*. Then the infinite set of the values*

$$\left\{\sum_{k=1}^{\infty}\frac{x^k q^{R_1+R_2+\cdots+R_k}}{(1-q^{2R_1})(1-q^{2R_2})\cdots(1-q^{2R_k})}\;\middle|\;x,q\in\overline{\mathbb{Q}}^{\times},\;|q|\neq 1\right\}$$

is a '4-fold' algebraically independent set, namely in this set each value appears 4 *times and the infinite set consisting of all the distinct values of this set is algebraically independent.*

Example 1.9. Let $\{P_k\}_{k\geq 1}$ be the sequence defined by $P_1=P_2=1$ and $P_{k+2}=2P_{k+1}+P_k$ $(k\geq 1)$. Since $\{P_k\}_{k\geq 1}$ satisfies all the conditions of Theorem 1.6, the infinite set of the values

$$\left\{\sum_{k=1}^{\infty}\frac{x^k q^{P_1+P_2+\cdots+P_k}}{(1-q^{2P_1})(1-q^{2P_2})\cdots(1-q^{2P_k})}\;\middle|\;x,q\in\overline{\mathbb{Q}}^{\times},\;|q|\neq 1\right\}$$

is a '4-fold' algebraically independent set.

2 Main results

The three-variable function

$$\mathcal{F}(x,a,q)=\sum_{k=1}^{\infty}\prod_{l=1}^{k}\frac{xq^{R_l}}{1-aq^{2R_l}}=\sum_{k=1}^{\infty}\frac{x^kq^{R_1+R_2+\cdots+R_k}}{(1-aq^{2R_1})(1-aq^{2R_2})\cdots(1-aq^{2R_k})} \tag{2}$$

converges on the domain

$$\begin{aligned} U:=&\{(x,a,q)\in\mathbb{C}^3 \mid |q|<1,\ aq^{2R_k}\neq 1 \text{ for any } k\geq 1\}\\ &\cup\{(x,a,q)\in\mathbb{C}^3 \mid a\neq 0,\ |q|>1,\ aq^{2R_k}\neq 1 \text{ for any } k\geq 1\}.\end{aligned}$$

If $a,q\neq 0$, then $\mathcal{F}(x,a,q)$ is invariant under the map

$$\tau\ :\ (x,a,q)\longmapsto(-a^{-1}x,a^{-1},q^{-1}), \tag{3}$$

namely

$$\mathcal{F}(\tau(x,a,q))=\sum_{k=1}^{\infty}\frac{(-1)^ka^{-k}x^kq^{-R_1-R_2-\cdots-R_k}}{(1-a^{-1}q^{-2R_1})(1-a^{-1}q^{-2R_2})\cdots(1-a^{-1}q^{-2R_k})}=\mathcal{F}(x,a,q).$$

Here and hereafter, N denotes an integer greater than 1. Let ζ_N be a primitive N-th root of unity and let $\delta\in\{1,\dots,N-1\}$ be such that g.c.d.$(\delta,N)=1$. If $R_k\equiv\delta \pmod N$ for all $k\geq 1$, then $\mathcal{F}(x,a,q)$ is invariant also under the map

$$\sigma\ :\ (x,a,q)\longmapsto(\zeta_N^{-\delta}x,\zeta_N^{-2\delta}a,\zeta_N\,q), \tag{4}$$

namely $\mathcal{F}(\sigma(x,a,q))=\mathcal{F}(x,a,q)$. In what follows, we denote by σ^i and $\sigma\tau$ the compositions of maps $\underbrace{\sigma\circ\cdots\circ\sigma}_{i}$ and $\sigma\circ\tau$, respectively. By (3) and (4) we have $\sigma^N=\tau^2=\mathrm{id}$,

$$\begin{aligned}\sigma^2\ :\ (x,a,q)\longmapsto(\zeta_N^{-2\delta}x,\zeta_N^{-4\delta}a,\zeta_N^{2}\,q),\ \dots,\ \sigma^{N-1}\ :\ (x,a,q)\longmapsto(\zeta_N^{\delta}x,\zeta_N^{2\delta}a,\zeta_N^{-1}\,q),\\ \sigma\tau\ :\ (x,a,q)\longmapsto(-\zeta_N^{-\delta}a^{-1}x,\zeta_N^{-2\delta}a^{-1},\zeta_N\,q^{-1}),\\ \sigma^2\tau\ :\ (x,a,q)\longmapsto(-\zeta_N^{-2\delta}a^{-1}x,\zeta_N^{-4\delta}a^{-1},\zeta_N^{2}\,q^{-1}),\\ \vdots\end{aligned}$$

and

$$\sigma^{N-1}\tau=\tau\sigma\ \ :\ \ (x,a,q)\longmapsto(-\zeta_N^{\delta}a^{-1}x,\zeta_N^{2\delta}a^{-1},\zeta_N^{-1}\,q^{-1}).$$

Hence $\{\mathrm{id},\,\sigma,\,\sigma^2,\dots,\sigma^{N-1},\,\tau,\,\sigma\tau,\,\sigma^2\tau,\dots,\sigma^{N-1}\tau\}$ is isomorphic to the dihedral group D_{2N} of order $2N$. Since the equation $aq^{2R_k}=1$ is invariant under σ and τ, the dihedral group D_{2N} acts on the set

$$U^*:=\{(x,a,q)\in\mathbb{C}^3 \mid aq^{2R_k}\neq 1 \text{ for any } k\geq 1,\ a\neq 0,\ |q|\neq 0,1\}.$$

Therefore, $\mathcal{F}(x,a,q)$ can be regarded as a map defined on the set of orbits U^*/D_{2N}, namely the map

$$\widetilde{\mathcal{F}}\ :\ U^*/D_{2N}\longrightarrow\mathcal{F}(U^*)$$

given by

$$\overline{(x,a,q)} \longmapsto \mathcal{F}(x,a,q),$$

where $\overline{(x,a,q)}$ denotes the orbit represented by (x,a,q), is a well-defined surjection. In other words, the image of the function $\mathcal{F}(x,a,q)$ is a $2N$-fold cover. Hence the restriction to algebraic points

$$\widetilde{\mathcal{F}} \ : \ U_{\mathcal{A}}/D_{2N} \longrightarrow \mathcal{F}(U_{\mathcal{A}}),$$

is also a well-defined surjection, where

$$U_{\mathcal{A}} = U^* \cap \left(\overline{\mathbb{Q}}^{\times}\right)^3 = \{(x,a,q) \mid x,a,q \in \overline{\mathbb{Q}}^{\times},\ aq^{2R_k} \neq 1 \text{ for any } k \geq 1,\ |q| \neq 1\}.$$

In this paper we prove the following:

Main Theorem 2.1. *Let $\{R_k\}_{k\geq 1}$ be a linear recurrence satisfying* (1) *and let $\delta \in \{1,\ldots,N-1\}$ be such that* g.c.d.$(\delta,N) = 1$. *Suppose that $\Phi(-1) \neq 0$, $\Phi(2) < 0$, and the ratio of any pair of distinct roots of $\Phi(X)$ is not a root of unity. Assume that* g.c.d.$(R_k, R_{k+1},\ldots,R_{k+n-1}) = 1$ *and $R_k \equiv \delta \pmod N$ for all $k \geq 1$. Then the infinite set of the values $\widetilde{\mathcal{F}}(U_{\mathcal{A}}/D_{2N})$ is algebraically dependent if and only if there exist distinct orbits $\overline{(x_1,a_1,q_1)},\ldots,\overline{(x_s,a_s,q_s)} \in U_{\mathcal{A}}/D_{2N}$, algebraic numbers $c_1,\ldots,c_s$, not all zero, and a sufficiently large positive integer u such that, for any $k \geq u$,*

$$a_1 q_1^{2R_k} = a_i q_i^{2R_k} \quad (2 \leq i \leq s) \qquad \textit{and} \qquad \sum_{i=1}^{s} c_i x_i^{k-u+1} q_i^{R_u+\cdots+R_k} = 0. \tag{5}$$

Remark 2.2. The equations (5) depend only on the orbits $\overline{(x_i,a_i,q_i)}$ but not on the choice of the representatives (x_i,a_i,q_i), namely, under the action of the dihedral group D_{2N}, the equations (5) are invariant.

Main Theorem 2.3. *Let $\{R_k\}_{k\geq 1}$ be a linear recurrence satisfying* (1), *let N be an odd integer, and let $\delta \in \{1,\ldots,N-1\}$ be such that* g.c.d.$(\delta,N) = 1$. *Suppose that $R_k \equiv \delta \pmod N$ and* g.c.d.$(R_{k+1}-R_k,\ R_{k+2}-R_{k+1},\ldots,R_{k+n}-R_{k+n-1}) = N$ *for all $k \geq 1$. Assume that $\Phi(-1) \neq 0$, $\Phi(2) < 0$, and the ratio of any pair of distinct roots of $\Phi(X)$ is not a root of unity. Then the infinite set of the values $\widetilde{\mathcal{F}}(U_{\mathcal{A}}/D_{2N})$ is algebraically dependent if and only if there exist distinct orbits $\overline{(x_1,a,q)},\ldots,\overline{(x_s,a,q)},\overline{(x_{s+1},a,-q)},\ldots,\overline{(x_{s+t},a,-q)} \in U_{\mathcal{A}}/D_{2N}$, algebraic numbers $c_1,\ldots,c_{s+t}$, not all zero, and a sufficiently large positive integer u such that, for any $k \geq u$,*

$$\sum_{i=1}^{s} c_i x_i^{k-u+1} + \sum_{i=s+1}^{s+t} c_i (-1)^{R_u+\cdots+R_k} x_i^{k-u+1} = 0.$$

Remark 2.4. The condition of Main Theorem 2.3 that $R_k \equiv \delta \pmod N$ and g.c.d.$(R_{k+1}-R_k,\ R_{k+2}-R_{k+1},\ldots,R_{k+n}-R_{k+n-1}) = N$ for all $k \geq 1$ implies that g.c.d.$(R_k,R_{k+1},\ldots,R_{k+n-1}) = 1$ for all $k \geq 1$.

Example 2.5. Let $\{R_k\}_{k\geq 1}$ be the sequence defined by $R_1 = R_2 = 1$ and $R_{k+2} = NR_{k+1} + R_k$ $(k \geq 1)$, where $N \geq 3$ is an odd integer. Then $\{R_k\}_{k\geq 1}$ satisfies all the conditions of Main Theorem 2.3 and the infinite set consisting of the distinct values of

$$\left\{ \sum_{k=1}^{\infty} \frac{x^k q^{R_1+R_2+\cdots+R_k}}{(1-aq^{2R_1})(1-aq^{2R_2})\cdots(1-aq^{2R_k})} \;\middle|\; x,a,q \in \overline{\mathbb{Q}}^{\times},\ aq^{2R_k} \neq 1\ (k \geq 1),\ |q| \neq 1 \right\}$$

is algebraically dependent. In fact, there exists the linear relation

$$\mathcal{F}(x,a,q) - 2\zeta_3^2 \mathcal{F}(\zeta_3 x,a,q) - 2\zeta_3 \mathcal{F}(\zeta_3^2 x,a,q) - 3\mathcal{F}(x,a,-q) = 0$$

holds for $\mathcal{F}(x,a,q)$ of the form (2). Note that $\overline{(x,a,q)}, \overline{(\zeta_3 x,a,q)}, \overline{(\zeta_3^2 x,a,q)}$, and $\overline{(x,a,-q)}$ are the distinct orbits of $U_{\mathcal{A}}/D_{2N}$, since $\mathcal{F}(x,a,q), \mathcal{F}(\zeta_3 x,a,q), \mathcal{F}(\zeta_3^2 x,a,q)$, and $\mathcal{F}(x,a,-q)$ are distinct as series in x, a, and q. Moreover, by Main Theorem 2.3 we have

$$\text{trans.deg}_{\mathbb{Q}}\, \mathbb{Q}\Big(\mathcal{F}(x,a,q), \mathcal{F}(\zeta_3 x,a,q), \mathcal{F}(\zeta_3^2 x,a,q), \mathcal{F}(x,a,-q)\Big) = 3,$$

or more precisely, any three of the numbers $\mathcal{F}(x,a,q), \mathcal{F}(\zeta_3 x,a,q), \mathcal{F}(\zeta_3^2 x,a,q)$, and $\mathcal{F}(x,a,-q)$ are algebraically independent.

3 Proof of the main theorems

Let $\Omega = (\omega_{ij})$ be an $n \times n$ matrix with nonnegative integer entries, where n is as in (1). We define a multiplicative transformation $\Omega\colon \mathbb{C}^n \to \mathbb{C}^n$ by

$$\Omega \boldsymbol{z} = \left(\prod_{j=1}^{n} z_j^{\omega_{1j}}, \prod_{j=1}^{n} z_j^{\omega_{2j}}, \ldots, \prod_{j=1}^{n} z_j^{\omega_{nj}} \right) \tag{6}$$

for any $\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$. Then $\Omega(\Omega^k \boldsymbol{z}) = \Omega^{k+1}\boldsymbol{z}$ $(k = 0,1,2,\ldots)$. Let $\{R_k\}_{k\geq 1}$ be a linear recurrence satisfying (1) and define a monomial

$$M(\boldsymbol{z}) = z_1^{R_n} \cdots z_n^{R_1}. \tag{7}$$

Let

$$\Omega_1 = \begin{pmatrix} c_1 & 1 & 0 & \ldots & 0 \\ c_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ c_n & 0 & \ldots & \ldots & 0 \end{pmatrix}. \tag{8}$$

It follows from (1), (6), and (7) that

$$M(\Omega_1^k \boldsymbol{z}) = z_1^{R_{k+n}} \cdots z_n^{R_{k+1}} \quad (k \geq 0). \tag{9}$$

Sketch of the proof of Main Theorem 2.1. First we prove that, if there exist distinct orbits $\overline{(x_1,a_1,q_1)}, \ldots, \overline{(x_s,a_s,q_s)} \in U_{\mathcal{A}}/D_{2N}$, algebraic numbers $c_1, \ldots, c_s$, not all zero, and a sufficiently large positive integer u such that the equations (5) hold for any $k \geq u$, then the values $\mathcal{F}(x_1,a_1,q_1), \ldots, \mathcal{F}(x_s,a_s,q_s)$ are algebraically dependent, which means that the infinite set of the values $\widetilde{\mathcal{F}}(U_{\mathcal{A}}/D_{2N})$ is algebraically dependent. Indeed, using (5) and letting $a_i q_i^{2R_k} = \eta_k$ $(1 \leq i \leq s,\ k \geq u)$, we have

$$\sum_{i=1}^{s} c_i \prod_{k=1}^{u-1} \frac{1 - a_i q_i^{2R_k}}{x_i q_i^{R_k}} \left(\mathcal{F}(x_i,a_i,q_i) - \sum_{k=1}^{u-1} \prod_{l=1}^{k} \frac{x_i q_i^{R_l}}{1 - a_i q_i^{2R_l}} \right)$$

$$= \sum_{i=1}^{s} c_i \sum_{k=u}^{\infty} \prod_{l=u}^{k} \frac{x_i q_i^{R_l}}{1 - a_i q_i^{2R_l}}$$

$$= \sum_{k=u}^{\infty} \sum_{i=1}^{s} c_i \frac{\prod_{l=u}^{k} x_i q_i^{R_l}}{\prod_{l=u}^{k}(1-\eta_l)} = \sum_{k=u}^{\infty} \frac{\sum_{i=1}^{s} c_i x_i^{k-u+1} q_i^{R_u+\cdots+R_k}}{\prod_{l=u}^{k}(1-\eta_l)} = 0.$$

Next assume that the infinite set of the values $\widetilde{\mathcal{F}}(U_{\mathcal{A}}/D_{2N})$ is algebraically dependent. Then there exist distinct orbits $\overline{(x_1, a_1, q_1)}, \overline{(x_2, a_2, q_2)}, \ldots, \overline{(x_r, a_r, q_r)} \in U_{\mathcal{A}}/D_{2N}$ such that the values

$$\theta_i := \mathcal{F}(x_i, a_i, q_i) = \sum_{k=1}^{\infty} \prod_{l=1}^{k} \frac{x_i q_i^{R_l}}{1 - a_i q_i^{2R_l}} \quad (i = 1, \ldots, r)$$

are algebraically dependent. Then there exist multiplicatively independent algebraic numbers $\beta_1, \ldots, \beta_t$ with $0 < |\beta_j| < 1$ $(1 \le j \le t)$ and a primitive N_1-th root of unity ζ, which may be different form the ζ_N defined in Section 2, such that

$$q_i = \zeta^{m_i} \prod_{j=1}^{t} \beta_j^{e_{ij}} \quad (1 \le i \le r), \tag{10}$$

where $m_1, \ldots, m_r$ are integers with $0 \le m_i \le N_1 - 1$ and e_{ij} $(1 \le i \le r,\ 1 \le j \le t)$ are nonnegative integers not all zero (cf. Loxton and van der Poorten [2], Nishioka [4]). We can choose a positive integer κ and a sufficiently large integer u such that $R_{k+\kappa} \equiv R_k \pmod{N_1}$ for any $k \ge u+1$, where u will be determined later. Let $p = \kappa N_1$. If necessary, we replace p with its multiple such that all the entries of Ω_1^p are positive, where Ω_1 is the matrix defined by (8). In fact, we can choose such a p. (For the proof see [5].) Let $y_{j\lambda}$ $(1 \le j \le t,\ 1 \le \lambda \le n)$ be variables and let $\boldsymbol{y}_j = (y_{j1}, \ldots, y_{jn})$ $(1 \le j \le t)$, $\boldsymbol{y} = (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_t)$. Define

$$f_i(\boldsymbol{y}) = \sum_{k=u}^{\infty} \prod_{l=u}^{k} \frac{x_i \zeta^{m_i R_{l+1}} \prod_{j=1}^{t} M(\Omega_1^l \boldsymbol{y}_j)^{e_{ij}}}{1 - a_i \left(\zeta^{m_i R_{l+1}} \prod_{j=1}^{t} M(\Omega_1^l \boldsymbol{y}_j)^{e_{ij}}\right)^2} \quad (1 \le i \le r),$$

where $M(\boldsymbol{z})$ is defined by (7). Letting $\boldsymbol{\beta} = (\underbrace{1, \ldots, 1}_{n-1}, \beta_1, \ldots\ldots, \underbrace{1, \ldots, 1}_{n-1}, \beta_t)$, we see by (9) and (10) that

$$f_i(\boldsymbol{\beta}) = \sum_{k=u}^{\infty} \prod_{l=u}^{k} \frac{x_i q_i^{R_{l+1}}}{1 - a_i q_i^{2R_{l+1}}} = \sum_{k=u+1}^{\infty} \prod_{l=u+1}^{k} \frac{x_i q_i^{R_l}}{1 - a_i q_i^{2R_l}}$$

and so

$$\theta_i = \left(\prod_{k=1}^{u} \frac{x_i q_i^{R_k}}{1 - a_i q_i^{2R_k}}\right) f_i(\boldsymbol{\beta}) + \sum_{k=1}^{u} \prod_{l=1}^{k} \frac{x_i q_i^{R_l}}{1 - a_i q_i^{2R_l}}.$$

Since $\theta_1, \ldots, \theta_r$ are algebraically dependent, so are $f_i(\boldsymbol{\beta})$ $(1 \le i \le r)$. Let

$$\Omega_2 = \mathrm{diag}(\underbrace{\Omega_1^p, \ldots, \Omega_1^p}_{t}).$$

Then, using the fact that $R_{k+p} \equiv R_k \pmod{N_1}$ for any $k \geq u+1$, we see that each $f_i(\boldsymbol{y})$ satisfies the functional equation

$$\begin{aligned} f_i(\boldsymbol{y}) &= \left(\prod_{k=u}^{p+u-1} \frac{x_i \zeta^{m_i R_{k+1}} \prod_{j=1}^t M(\Omega_1^k \boldsymbol{y}_j)^{e_{ij}}}{1 - a_i \left(\zeta^{m_i R_{k+1}} \prod_{j=1}^t M(\Omega_1^k \boldsymbol{y}_j)^{e_{ij}} \right)^2} \right) f_i(\Omega_2 \boldsymbol{y}) \\ &\quad + \sum_{k=u}^{p+u-1} \prod_{l=u}^{k} \frac{x_i \zeta^{m_i R_{l+1}} \prod_{j=1}^t M(\Omega_1^l \boldsymbol{y}_j)^{e_{ij}}}{1 - a_i \left(\zeta^{m_i R_{l+1}} \prod_{j=1}^t M(\Omega_1^l \boldsymbol{y}_j)^{e_{ij}} \right)^2}, \end{aligned}$$

where $\Omega_2 \boldsymbol{y} = (\Omega_1^p \boldsymbol{y}_1, \ldots, \Omega_1^p \boldsymbol{y}_t)$. Let $D = |\det(\Omega_1 - E)|$, where E is the identity matrix. Put $M(D(\Omega_1 - E)^{-1} \Omega_1^k \boldsymbol{z}) = \boldsymbol{z}^{\boldsymbol{S}(k)}$. By the condition that $\Phi(-1) \neq 0$, $\Phi(2) < 0$, and the ratio of any pair of distinct roots of $\Phi(X)$ is not a root of unity and by Remark 1.7, D is a positive integer and there exists a sufficiently large integer u such that we have the componentwise inequality among the vectors with integral components

$$D(R_{u+n}, \ldots, R_{u+1}) > \boldsymbol{s}(u) > \mathbf{0}. \tag{11}$$

Changing the indices i if necessary, we may assume by Nishioka [4, p. 110, Theorem 3.5 and p. 115, Theorem 3.6.4], by [5, Lemma 4 and Proof of Theorem 2], by [6, Theorem 2], and by Remark 1.7 that there exist a positive integer $s \leq r$, nonzero algebraic numbers $\eta_u, \ldots, \eta_{p+u-1}, \xi$, and nonnegative integers $e_1, \ldots, e_t$ such that for any $i \in \{1, \ldots, s\}$ we have $a_i \zeta^{2m_i R_{k+1}} = \eta_k$ $(u \leq k \leq p+u-1)$, $(e_{i1}, \ldots, e_{it}) = (e_1, \ldots, e_t) \neq \mathbf{0}$, and $x_i^p \zeta^{m_i(R_{u+1} + \cdots + R_{p+u})} = \xi$, and there exits a $G(\boldsymbol{z}) \in \overline{\mathbb{Q}}(z_1, \ldots, z_n)$ satisfying the functional equation

$$\begin{aligned} G(\boldsymbol{z}) &= \xi \left(\prod_{k=u}^{p+u-1} \frac{M(\Omega_1^k \boldsymbol{z})^{2D\ell}}{1 - \eta_k M(\Omega_1^k \boldsymbol{z})^{2D\ell}} \right) G(\Omega_1^p \boldsymbol{z}) \\ &\quad + \boldsymbol{z}^{-\ell \boldsymbol{S}(u)} \sum_{k=u}^{p+u-1} \left(\sum_{i=1}^{s} c_i x_i^{k-u+1} \zeta^{m_i(R_{u+1} + \cdots + R_{k+1})} \right) \prod_{l=u}^{k} \frac{M(\Omega_1^l \boldsymbol{z})^{D\ell}}{1 - \eta_l M(\Omega_1^l \boldsymbol{z})^{2D\ell}}, \end{aligned}$$

where $\ell = \sum_{j=1}^t e_j \nu^j$ with ν a sufficiently large positive integer, $\boldsymbol{s}(u)$ is as in (11), and $c_1, \ldots, c_s$ are algebraic numbers not all zero. Letting $G(\boldsymbol{z}) = A(\boldsymbol{z})/B(\boldsymbol{z})$, where $A(\boldsymbol{z})$ and $B(\boldsymbol{z})$ are coprime polynomials in $\overline{\mathbb{Q}}[z_1, \ldots, z_n]$ with $B \not\equiv 0$, we have

$$\begin{aligned} & A(\boldsymbol{z}) B(\Omega_1^p \boldsymbol{z}) \boldsymbol{z}^{\ell \boldsymbol{S}(u)} \prod_{k=u}^{p+u-1} \left(1 - \eta_k M(\Omega_1^k \boldsymbol{z})^{2D\ell}\right) \\ = \;& \xi A(\Omega_1^p \boldsymbol{z}) B(\boldsymbol{z}) \boldsymbol{z}^{\ell \boldsymbol{S}(u)} \prod_{k=u}^{p+u-1} M(\Omega_1^k \boldsymbol{z})^{2D\ell} \\ & + \sum_{k=u}^{p+u-1} \left(\sum_{i=1}^{s} c_i x_i^{k-u+1} \zeta^{m_i(R_{u+1} + \cdots + R_{k+1})} \right) B(\boldsymbol{z}) B(\Omega_1^p \boldsymbol{z}) \prod_{l=u}^{k} M(\Omega_1^l \boldsymbol{z})^{D\ell} \\ & \times \prod_{l'=k+1}^{p+u-1} \left(1 - \eta_{l'} M(\Omega_1^{l'} \boldsymbol{z})^{2D\ell}\right). \end{aligned} \tag{12}$$

We can put the greatest common divisor of $A(\Omega_1^p \boldsymbol{z})$ and $B(\Omega_1^p \boldsymbol{z})$ as $\boldsymbol{z}^{I(p)}$, where $I(p)$ is an n-dimensional vector with nonnegative integer components, by [4, Lemma 3.2.3]. Then $B(\Omega_1^p \boldsymbol{z})$ divides $B(\boldsymbol{z})\boldsymbol{z}^{\ell \boldsymbol{S}(u)+I(p)} \prod_{k=u}^{p+u-1} M(\Omega_1^k \boldsymbol{z})^{2D\ell}$ by (12). Therefore $B(\boldsymbol{z})$ is a monomial in $z_1, \ldots, z_n$ by [6, Lemma 12] with Remark 1.7.

By (9) and (11) we see that $z_1 \cdots z_n \boldsymbol{z}^{\ell \boldsymbol{S}(u)}$ divides $M(\Omega_1^u \boldsymbol{z})^{D\ell}$. Since p and u are independent, the right-hand side of (12) is divisible by $z_1 \cdots z_n \boldsymbol{z}^{\ell \boldsymbol{S}(u)} B(\Omega_1^p \boldsymbol{z})$ for sufficiently large u and thus $A(\boldsymbol{z})$ is divisible by $z_1 \cdots z_n$. Since $A(\boldsymbol{z})$ and $B(\boldsymbol{z})$ are coprime, $B(\boldsymbol{z}) \in \overline{\mathbb{Q}}^{\times}$. If $A(\boldsymbol{z}) \notin \overline{\mathbb{Q}}$, then by [7, Lemma 6] and the fact that all the entries of Ω_1^p are positive, $\deg_{\boldsymbol{z}} A(\Omega_1^p \boldsymbol{z}) > \deg_{\boldsymbol{z}} A(\boldsymbol{z})$, which is a contradiction by comparing the total degrees of both sides of (12). Hence $A(\boldsymbol{z}) \in \overline{\mathbb{Q}}$ and so $A(\boldsymbol{z}) = 0$ by comparing the coefficients of $\boldsymbol{z}^{\ell \boldsymbol{S}(u)}$, the term of the smallest degree among the terms appearing in (12). Then again by (12), we see that $\sum_{i=1}^s c_i x_i^{k-u+1} \zeta^{m_i(R_{u+1}+\cdots+R_{k+1})} = 0$ $(u \le k \le p+u-1)$. Since $R_{k+p} \equiv R_k \pmod{N_1}$ for any $k \ge u+1$, we have $a_i \zeta^{2m_i R_{k+1}} = a_{i'} \zeta^{2m_{i'} R_{k+1}}$ $(k \ge u)$ for any $i, i' \in \{1, \ldots, s\}$. Noting that $p = \kappa N_1$ with $R_{k+\kappa} \equiv R_k \pmod{N_1}$ for any $k \ge u+1$, we see that $\sum_{l=k}^{k+p-1} R_{l+1} \equiv 0 \pmod{N_1}$ for any $k \ge u$ and so $\sum_{l=u}^{k} R_{l+1} \equiv \sum_{l=u}^{k+pk'} R_{l+1} \pmod{N_1}$ for any k with $u \le k \le p+u-1$ and for all $k' \ge 0$. Hence $\sum_{i=1}^s c_i x_i^{k-u+1} \zeta^{m_i(R_{u+1}+\cdots+R_{k+1})} = 0$ $(k \ge u)$. By (10) with $(e_{i1}, \ldots, e_{it}) = (e_{i'1}, \ldots, e_{i't})$ we get $a_i q_i^{2R_{k+1}} = a_{i'} q_{i'}^{2R_{k+1}}$ $(k \ge u)$ for any $i, i' \in \{1, \ldots, s\}$ and $\sum_{i=1}^s c_i x_i^{k-u+1} q_i^{R_{u+1}+\cdots+R_{k+1}} = 0$ $(k \ge u)$. This completes the proof of the theorem. □

Sketch of the proof of Main Theorem 2.3. Choosing a suitable complete set of representatives of the orbits $U_{\mathcal{A}}/D_{2N}$, we can prove the theorem. □

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