ON THE CONSTANTS IN SOME CLASSICAL RESULTS ON DIOPHANTINE APPROXIMATION

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Abstract. We use continued fractions to refine information about the constants in classical theorems of Borel, Hurwitz, Nathanson, and Vahlen on diophantine approximation.

1. Introduction

Continued fractions are used analyse how good rational approximations are under certain specific circumstances. For instance, we can consider two results due to A. Hurwitz.

Theorem 1.1 (Hurwitz, 1891, [12]). *For every irrational number α there are infinitely many pairs $(p, q) \in (\mathbb{Z} \times \mathbb{Z})$ such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2\sqrt{5}}.$$

The second result shows that the golden ratio has the worst approximation.

Theorem 1.2 (Hurwitz, 1891, [12]). *For every irrational number α which is not equivalent to the golden ratio there are infinitely many pairs $(p, q) \in (\mathbb{Z} \times \mathbb{Z})$ such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2\sqrt{8}}.$$

Equivalence is defined in the next section. Approximation by two successive convergents was considered by K. Th. Vahlen and by three consecutive convergents by É.Borel.

Theorem 1.3 (Vahlen, 1895, [19]). *For a real number α, suppose $\frac{p}{q}$ represents one of any pair of consecutive convergents to α. Then at least one of the pair satisfies the inequality*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}\,.$$

Theorem 1.4 (Borel, 1903, [2]). *Suppose $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$. Also suppose $\frac{p_{n-1}}{q_{n-1}}$, $\frac{p_n}{q_n}$ and $\frac{p_{n+1}}{q_{n+1}}$ are three consecutive convergents of the number α. Then at least one of them satisfies the inequality*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}\,.$$

In 1974, inspired by a simple proof due to Cohn [3] of Borel's theorem, the American mathematician Melvyn B. Nathanson introduced a special set of real numbers that are still better approximated.

Definition 1. For $k \geq 1$, let $F(k)$ denote the set of all real numbers α such that $0 \leq \alpha \leq 1$ and the simple continued fraction for α has no partial quotient greater than k.

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Theorem 1.5 (Nathanson, 1974, [16])**.** *Suppose $k \geq 1$ and let α be a real irrational number not equivalent to an element of $F(k-1)$. Then there are infinitely many rational numbers $\frac{p}{q}$ such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2\sqrt{k^2+4}}.$$

The constant $\frac{1}{\sqrt{k^2+4}}$ is best possible.

In this paper, we discuss improvments in approximation by successively eliminating poorer approximation results using a sequence of numbers called Lagrange numbers which converge to 3. Specifically, the n-th Lagrange number has the formula

$$L_n = \sqrt{9 - \frac{4}{M_n^2}},$$

where M_n is the n-th Markoff number which is the n-th smallest interger m satisfing the equation

$$m^2 + m_1^2 + m_2^2 = 3mm_1m_2$$

for some positive intergers m_1 and m_2.

We will improve the above results with the idea of replacing the existing constants by functions that will converge to these constants from below as the denominator of the rational number $\frac{p}{q}$ tends to infinity.

2. Notation

Throughout the paper, $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{R}$ will denote the sets of integers, positive integers, non-negative integers and real numbers respectively. Let α be a real number and suppose $n \in \mathbb{N}_0$. For $\alpha \in [0,1)$ write $T(\alpha) = 0$ if $\alpha = 0$ and $T(\alpha) = \{\frac{1}{\alpha}\}$ otherwise, where for a real number x we have used $\{x\}$ to donote its fractional part. Also set $[x] := x - \{x\}$. We now set $a_0(\alpha) = [\alpha]$ and set $a_n(\alpha) = a_{n-1}(T(\alpha))$ $(n = 1, 2, \ldots)$, which we call the partial quotients of the simple continued fraction of α. Two real numbers. α and β are called equivalent if $T^m(\alpha) = T^n(\beta)$ for some $m, n \in \mathbb{N}_0$. Let

$$\alpha = [a_0; a_1, a_2, \ldots] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}},$$

be the simple continued fraction expansion of α and let

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, \ldots, an] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots a_n}}}}, \qquad (n = 0, 1, \ldots)$$

be its n-th convergent. The following recurrence relations for convergents are known

$$p_0 = a_0\,, \qquad p_1 = a_1a_0 + 1\,, \qquad p_{n+2} = a_{n+2}p_{n+1} + p_n\,,$$
$$q_0 = 1\,, \qquad q_1 = a_1\,, \qquad q_{n+2} = a_{n+2}q_{n+1} + q_n\,.$$

For a simple continued fraction expansion we have that

$$\alpha = [a_0; a_1, a_2, \dots] = [a_0; a_1, a_2, \dots, a_n, [a_{n+1}; a_{n+2}, a_{n+3}, \dots]].$$

Taking a difference of two consecutive convergents we obtain that

$$q_{n+1}p_n - p_{n+1}q_n = (-1)^{n+1} .$$

Finally we have the identity most frequently used in this article.

$$\left|\alpha - \frac{p_n}{q_n}\right| = \frac{1}{q_n^2([a_{n+1}; a_{n+2}, \dots] + [0; a_n, a_{n-1}, \dots, a_1])} ,$$

where if $n = 0$, we set $[0; a_n, a_{n-1}, \dots, a_1] = 0$.

For simple continued fraction expansions, if we have $\alpha = [a_0; a_1, a_2, \dots, a_k]$ for finite $k \geq 1$, then, of the two possibilities available, we choose $a_k \neq 1$. More details on the discussion in this section can be found in [10], [11], [13], [14], [15], [17], [18], pages 7 to 10 and [20].

3. New results

We first state an improvement of Hurwitz's first result in which we replace the constant $\sqrt{5}$ by the function $f(q) = \frac{\sqrt{5}}{2}\left(1 + \sqrt{1 + \frac{4}{5q^2}}\right)$.

Theorem 3.1 (Jaroslav Hančl, 2015, [6]). *For every irrational number α there are infinitely many pairs $(p, q) \in (\mathbb{Z} \times \mathbb{Z})$ such that*

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{\frac{q^2\sqrt{5}}{2}\left(1 + \sqrt{1 + \frac{4}{5q^2}}\right)}.$$

Hurwitz's second result can also be improved using the same idea.

Theorem 3.2 (Jaroslav Hančl, 2016, [7]). *To every irrational number α which is not equivalent to the golden ratio we can find infinitely many pairs $(p, q) \in (\mathbb{Z} \times \mathbb{Z})$ such that*

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{\frac{q^2\sqrt{8}}{2}\left(1 + \sqrt{1 + \frac{4}{8q^2}}\right)}.$$

Nathanson's result, suggests a generalization to Hurwitz's theorem. Recalling the definition of the set $F(k)$ we have the following result.

Theorem 3.3 (Jaroslav Hančl and Tho Phuoc Nguyen, 2024, [9]). *Let $\alpha = [a_0; a_1, \dots]$ be an irrational number not equivalent to an element of $F(k-1)$. Then there are infinitely many integers p and q such that*

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{\frac{q^2\sqrt{k^2+4}}{2}\left(1 + \sqrt{1 + \frac{4}{(k^2+4)q^2}}\right)} = \frac{1}{f(q)}.$$

The function $f(q)$ is best possible in the sense that there isn't any function $g(q)$ such that $g(q) > f(q)$ for all positive integers q and $g(q)$ satisfies $\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{g(q)}$ for infinitely many integers p and q. The equality is obtained only for the number $\alpha = \frac{\sqrt{k^2+4} - k}{2} = [0; \overline{k}]$ or the number $\alpha = \frac{k + 2 - \sqrt{k^2+4}}{2} = [0; 1, k-1, \overline{k}]$ plus an integer.

From this result we see that the cases when $k=1$ and $k=2$ give the functions that appear in the two improved results for Hurwitz's theorem, respectively.

The ability to approximate an irrational number from its two successive convergences in Vahlen's theorem is improved by replacing 2 by a function $f(q)=2+\frac{4(q-1)}{q(q^2+q+6)}$, which in fact converges to 2 as q tends to infinity.

Theorem 3.4 (Hančl, Bahnerová 2021, [1]). *Suppose $\alpha\in\mathbb{R}$ and $n\in\mathbb{N}$. Also suppose $\frac{p_{n-1}}{q_{n-1}}$, $\frac{p_n}{q_n}$ are two consecutive convergents of the number α. Then at least one of them satisfies the inequalty*

$$\left|\alpha-\frac{p}{q}\right|<\frac{1}{\left(2+\frac{4(q-1)}{q(q^2+q+6)}\right)q^2}.$$

Another improved result is stated using the ratio of the denominators of two consecutive convergents.

Theorem 3.5 (Hančl, Bahnerová 2021, [1]). *Suppose $\alpha\in\mathbb{R}$ and suppose $n\in\mathbb{N}$. Also suppose $\frac{p_{n-1}}{q_{n-1}}$, $\frac{p_n}{q_n}$ are two consecutive convergents of the number α. Suppose $q_{-1}=q_{-2}=0$. Then there exists $i\in\{n-1,n\}$ such that*

$$\left|\alpha-\frac{p_i}{q_i}\right|<\frac{1}{\left(2+\frac{2}{3}\frac{q_{i-2}^2}{q_i^2}\right)q_i^2}.$$

The constant $\frac{2}{3}$ in this result can also be replaced by $\frac{\sqrt{1017}-29}{4}$ to get a better result as follows.

Theorem 3.6 (Hančl, Bahnerová 2021, [1]). *Suppose $\alpha\in\mathbb{R}$ and suppose n is a sufficiently large integer. Suppose $\frac{p_{n-1}}{q_{n-1}}$, $\frac{p_n}{q_n}$ are two consecutive convergents of the number α. Then there exists $i\in\{n-1,n\}$ such that*

$$\left|\alpha-\frac{p_i}{q_i}\right|<\frac{1}{\left(2+\frac{\sqrt{1017}-29}{4}\frac{q_{i-2}^2}{q_i^2}\right)q_i^2}.$$

On the other hand, we can also give an upper bound for this sequence of results.

Theorem 3.7 (Hančl, Bahnerová 2021, [1]). *Let ϵ be a positive real number. Then there exists a real number $\alpha=[0;a_1,a_2,a_3,\dots]$ with infinitely many pairs of consecutive convergents $\frac{p_{n-1}}{q_{n-1}}$, $\frac{p_n}{q_n}$ such that for every $i\in\{n-1,n\}$ we have*

$$\left|\alpha-\frac{p_i}{q_i}\right|>\frac{1}{\left(2+\left(\sqrt{61}-7+\epsilon\right)\frac{q_{i-2}^2}{q_i^2}\right)q_i^2}.$$

This idea was further developed to improve Borel's result for approximating real numbers by its three successive convergences. In 2024, we had the following results.

Theorem 3.8 (Hančl, Nair 2024, [8]). *Suppose $\alpha\in\mathbb{R}$ and $n\in\mathbb{Z}^+$. Also suppose $\frac{p_{n-1}}{q_{n-1}}$, $\frac{p_n}{q_n}$ and $\frac{p_{n+1}}{q_{n+1}}$ are three consecutive convergents of the number α. Then at least one of them satisfies the inequality*

$$\left|\alpha-\frac{p}{q}\right|\le\frac{1}{\left(\sqrt{5}+\frac{4-5\sqrt{5}+\sqrt{61}}{2q^2}\right)q^2}=\frac{1}{\sqrt{5}q^2+\frac{4-5\sqrt{5}+\sqrt{61}}{2}}. \tag{3.1}$$

The next theorem we give shows that the function $\sqrt{5}+\frac{4-5\sqrt{5}+\sqrt{61}}{2q^2}$ is best improvement for $\sqrt{5}$ in Theorem 3.8.

Theorem 3.9 (Hančl, Nair 2024, [8]). *Suppose* $\alpha = [0; 1, 1 + \frac{8}{3\sqrt{5}+\sqrt{61}}]$ *or* $\alpha = [0; 2 + \frac{8}{3\sqrt{5}+\sqrt{61}}]$. *Then for each of the three consecutive convergents* $\frac{p_0}{q_0}$, $\frac{p_1}{q_1}$ *and* $\frac{p_2}{q_2}$ *of the number* α *we have*

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{\left(\sqrt{5} + \frac{4-5\sqrt{5}+\sqrt{61}}{2q^2}\right)q^2}.$$

These two numbers or their integer translates are the only numbers that give us a non-strict inequality in (3.1).

In 2021, we gave a general improved result for Lagrange numbers, by replacing Lagrange numbers L by $L(x) = \frac{L}{2}\left(1 + \sqrt{1 + \frac{4}{L^2x^2}}\right)$. We also have a description of all numbers with the asymptotic irrationality measure $L(x)$. We start with a basic theorem which gives very good approximations to a simple periodic continued fraction by rationals. The main result is as follows.

Theorem 3.10 (Hančl, 2021, [5]). *Let* k *be a positive integer. Assume that* $b_0, b_1, \ldots, b_{k-1} \in \mathbb{Z}^+$. *Set* $\beta_{1,i} = [\overline{b_i, b_{z(i+1)}, \ldots, b_{z(i+k-1)}}]$ *and set* $\beta_{2,i} = [\overline{b_i, b_{z(i-1)}, \ldots, b_{z(i-k+1)}}]$ *where* $z(x) = x - k\left[\frac{x}{k}\right]$ *for all* $x \in \mathbb{Z}$. *Then for every large* $n \in \mathbb{N}_0$ *we have*

$$(-1)^n(\beta_{1,0} - \frac{p_n}{q_n}) = \frac{1}{\frac{q_n^2(\beta_{1,z(n+1)}+\beta_{2,z(n+1)}-b_{z(n+1)})}{2}\left(1 + \sqrt{1 + \frac{4(-1)^{n+1}}{q_n^2(\beta_{1,z(n+1)}+\beta_{2,z(n+1)}-b_{z(n+1)})(\beta_{1,0}+\beta_{2,0}-b_0)}}\right)}.$$

Here $(\frac{p_n}{q_n})_{n\in\mathbb{N}_0}$ *are the partial fractions of* $\beta_{1,0}$.

Theorem 3.11 (Hančl, 2021, [5]). *Let* $k \in \mathbb{N}_0$, *let* $z(x)$ *be as in Theorem 3.10, and let* $b_0, b_1, b_2, \ldots, b_{k-1} \in \{1, 2\}$ *be such that the chains of 1's and 2's have even length in the double sequence* $\{\overline{b_0, b_1, \ldots, b_{k-1}}\}$. *Set* $M = \{\{\overline{b_0, b_1, \ldots, b_{k-1}}\}, \{\overline{b_{k-1}, \ldots, b_1, b_0}\}\}$. *Let*

$$\begin{aligned} L &= [\overline{b_1, b_2, \ldots, b_{k-1}; b_0}; \overline{b_1, b_2, \ldots, b_{k-1}, b_0}] \\ &= \max_{i\in\{0,1,\ldots,k-1\}}([\overline{b_i, b_{z(i+1)}, \ldots, b_{z(i+k-1)}}] + [\overline{b_i, b_{z(i-1)}, \ldots, b_{z(i-k+1)}}] - b_i) \end{aligned}$$

be a Lagrange number. Also let α *be equivalent to* $\beta = [0; \overline{b_1, b_2, \ldots, b_{k-1}, b_0}]$ *with* $0 < \alpha < 1$. *Then* $I_\alpha(x) = L(x) = \frac{L}{2}\left(1 + \sqrt{1 + \frac{4}{L^2x^2}}\right)$ *in only one of the follwing cases:*

$$\begin{aligned} \alpha &= [0; 1, 1, x_1], L = [y_1, 2; 2; 1, 1, x_1], \{y_1, 2, 2, 1, 1, x_1\} \in M \\ \alpha &= [0; 2, y_2], L = [y_2, 2; 2; 1, 1, x_2], \{y_2, 2, 2, 1, 1, x_2\} \in M \\ \alpha &= [0; 2, x_3], L = [y_3, 2; 2; 1, 1, x_3], \{y_3, 2, 2, 1, 1, x_3\} \in M \\ \alpha &= [0; 1, 1, y_4], L = [y_4, 2; 2; , x_4], \{y_4, 2, 2, x_4\} \in M \end{aligned}$$

where $x_j, y_j \in (0, \infty), j \in \{1, 2, 3, 4\}$ *have simple continued fraction expansions whose terms belong to the set* $\{1, 2\}$. *Otherwise* $I_\alpha(x) > L(x)$ *for all large* $x \in \mathbb{N}_0$.

Note that we assume $I_\alpha(x) \in F$, where F is the set of greatest functions $f(q)$ satisfying, for every irrational number α, the inequality $\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{f(q)q^2}$ for infinitely many $(p, q) \in \mathbb{Z} \times \mathbb{Z}$. The functions $L(x)$ converge to $\frac{3}{2}\left(1 + \sqrt{1 + \frac{4}{9x^2}}\right)$ as L converges to 3.

In 2016, we give general results which provide asymptotic irrationality measures and estimations for the denominators of the convergents for certain almost periodic simple continued fraction expansions. As an application, we obtain new irrationality measures in the following Theorem. We first need some definitions.

Let $\alpha \in \mathbb{R}$ be given and define a function $J_\alpha \colon \mathbb{N} \to \mathbb{R}$ by

$$J_\alpha(N) = N\|N\alpha\|, \quad \text{where} \quad \|\alpha\| = \min_{n\in\mathbb{Z}}\{|\alpha - n|\} \ .$$

We say that $F(N) = f(N) + \overline{O}(g(N))$ if there is $C \in \mathbb{R}^+$ such that
1. $F(N) > f(N) - C|g(N)|$ holds for all $N > N_0$ with $N \in \mathbb{Z}^+$ and
2. $F(N) < f(N) + C|g(N)|$ holds for infinitely many $N \in \mathbb{Z}^+$.

Theorem 3.12 (Dodulíková, S., Hančl, J., Kolouch, O., Leinonen, M., Leppälä, K., 2016, [4])**.**

$$J_e(N) = \frac{\log(\frac{4}{e}\log N) - \log\log(\frac{4}{e}\log N) + \frac{\log\log(\frac{4}{e}\log N)}{\log(\frac{4}{e}\log N)} + \frac{\log^2\log(\frac{4}{e}\log N)}{\log^2(\frac{4}{e}\log N)} + \overline{O}(\frac{\log\log\log N}{\log^2\log N})}{2\log N}.$$

$$J_{e^2}(N) = \frac{\log(\frac{2}{e}\log N) - \log\log(\frac{2}{e}\log N) + \frac{\log\log(\frac{2}{e}\log N)}{\log(\frac{2}{e}\log N)} + \frac{\log^2\log(\frac{2}{e}\log N)}{\log^2(\frac{2}{e}\log N)} + \overline{O}(\frac{\log\log\log N}{\log^2\log N})}{4\log N}.$$

$$J_{e^{\frac{1}{q}}}(N) = \frac{\log(\frac{4}{e}q\log N) - \log\log(\frac{4}{e}q\log N) + \frac{\log\log(\frac{4}{e}q\log N)}{\log(\frac{4}{e}q\log N)} + \frac{\log^2\log(\frac{4}{e}q\log N)}{\log^2(\frac{4}{e}q\log N)} + \overline{O}(\frac{\log\log\log N}{\log^2\log N})}{2q\log N}.$$

$$J_{e^{\frac{2}{q}}}(N) = \frac{\log(\frac{2q}{e}\log N) - \log\log(\frac{2q}{e}\log N) + \frac{\log\log(\frac{2q}{e}\log N)}{\log(\frac{2q}{e}\log N)} + \frac{\log^2\log(\frac{2q}{e}\log N)}{\log^2(\frac{2q}{e}\log N)} + \overline{O}(\frac{\log\log\log N}{\log^2\log N})}{4q\log N}.$$

$$J_{\tanh\frac{1}{q}}(N) = \frac{\log(\frac{2q}{e}\log N) - \log\log(\frac{2q}{e}\log N) + \frac{\log\log(\frac{2q}{e}\log N)}{\log(\frac{2q}{e}\log N)} + \frac{\log^2\log(\frac{2q}{e}\log N)}{\log^2(\frac{2q}{e}\log N)} + \overline{O}(\frac{\log\log\log N}{\log^2\log N})}{2q\log N}.$$

Let c and d be positive integers. Set

$$a := i\frac{J_{c/d-1}(2i/d)}{J_{c/d}(2i/d)}.$$

Then

$$J_a(N) = \frac{\log(\frac{d}{e}\log N) - \log\log(\frac{d}{e}\log N) + \frac{\log\log(\frac{d}{e}\log N)}{\log(\frac{d}{e}\log N)} + \frac{\log^2\log(\frac{d}{e}\log N)}{\log^2(\frac{d}{e}\log N)} + \overline{O}(\frac{\log\log\log N}{\log^2\log N})}{d\log N}.$$

Set $a(1,2) := [0; 1^2, 2^2, 3^2, 4^2, \ldots]$ or $a(1,1) := [0; 1, 2, 3, 4, \ldots]$ (known as Siegel's constant). Then we have

$$J_{a(1,1)}(N) = \frac{\log(\frac{1}{e}\log N) - \log\log(\frac{1}{e}\log N) + \frac{\log\log(\frac{1}{e}\log N)}{\log(\frac{1}{e}\log N)} + \frac{\log^2\log(\frac{1}{e}\log N)}{\log^2(\frac{1}{e}\log N)} + \overline{O}(\frac{\log\log\log N}{\log^2\log N})}{\log N},$$

and

$$J_{a(1,2)}(N) = \frac{\big(\log(\frac{1}{2e}\log N) - \log\log(\frac{1}{2e}\log N) + \frac{\log\log(\frac{1}{2e}\log N)}{\log(\frac{1}{2e}\log N)} + \frac{\log^2\log(\frac{1}{2e}\log N)}{\log^2(\frac{1}{2e}\log N)} + \overline{O}(\frac{\log\log\log N}{\log^2\log N})\big)^2}{\frac{1}{4}\log^2 N}.$$

4. Data Availibility

Data sharing is not applicable to this article as no new data was created or analysed in this study.

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