WEAKER CAN OFTEN BE BETTER

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Abstract. This short note covers the contents of my talk given at RIMS, Kyoto, in October 2024. Some estimates in analytic number theory are best when the variable n or x, say, is sufficiently large. Others may be better for more moderate values of n or x, despite being weaker in the long run. We should pursue weaker results since, in applications, weaker can often be better.

1. Introduction

Suppose you have a function $f(x)$, and you want a bound for $f(x)$ to hold for a certain range of x. What sorts of bounds can you write down? Of all these bounds, which is the best when x is large? Which is the best when x is *very* large?

Suppose that, via a simple argument, you can show that

$$f(x) \geq x, \quad (x > 1). \tag{1.1}$$

Suppose that, via a complicated argument, you can improve this, and show that

$$f(x) \geq 1.01x - 10^{10}x^{0.99}, \quad (x > x_0), \tag{1.2}$$

for some x_0. When x is very large, it is clear that the bound in (1.2) is better than that in (1.1). Indeed, a simple rearrangement shows that this is so whenever $x > 10^{1200}$. This is very large indeed! What is worse, we need to figure out x_0 in order to be able to apply the bound in (1.2). Not only may this be difficult, but even if we accomplish this task, we still have that (1.2) beats (1.1) only for $x > \max\{x_0, 10^{1200}\}$. If the application we have in mind calls for a smaller x, then the weaker bound (1.1) is surely the one to use.

The purpose of this article is to explore the theme above. Instead of asking *which result is stronger* (clearly (1.2 is stronger for sufficiently large x), we really should be asking *which result is better*, where 'better' means 'gives a sharper result in the region of our interest'. It may be that weak bounds can be useful in small ranges, then slightly stronger ones when the range increases, and then the strongest.

2. Three zero-free regions

As an example, consider zero-free regions for the Riemann zeta-function. The problem is to find a function $f(t)$ such that $\zeta(\sigma + it)$ is non-zero for $\sigma > 1 - 1/f(t)$. If the Riemann hypothesis (RH) were true, we could take $f(t) = 2$. There are

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zeroes on the line $\sigma = 1/2$, and RH[1] says that these are the only (non-trivial) ones. Not only is proving $f(t) = 2$ beyond reach, but even proving $f(t) = 10^{100}$ or indeed $f(t) = C$ for any fixed C seems hopelessly difficult. We only know how to exhibit functions $f(t)$ with the property that $f(t) \to \infty$ as $t \to \infty$. The aim of the game is to try to exhibit $f(t)$ that grow as slowly as possible.

The so-called classical zero-free region of $\zeta(s)$, due to de la Vallée Poussin (see [30, Chapter 3]) shows that one may take $f(t) = A_3 \log t$ for some constant A_3 provided that $t \geq 3$. The choice of indexation on this constant A_3 will become clear later. One area of research is to reduce this A_3 as much as possible: the current best value is 5.558691 found in [20]. Much work has been done here: see, e.g., Table 1 in [19].

Littlewood showed that one can instead take $f(t) = A_4 \log t / \log\log t$. Ultimately this is smaller than the corresponding $f(t)$ from the classical zero-free region: hence this is ultimately a wider (and hence better) zero-free region. Recent work by Yang [33] shows that one may take $A_4 = 21.233$ whenever $t \geq 3$. When does this result overtake the classical zero-free region? A simple rearrangement shows that we need $t > \exp(\exp(A_4/A_3)) \approx 6.3 \cdot 10^{19}$. Hence, even though the Littlewood zero-free region is ultimately better than the classical one, the classical region is still useful for certain ranges[2] of t.

Still stronger zero-free regions are known: the strongest is due to Vinogradov and Korobov and allows one to take $f(t) = A_5 (\log t)^{2/3} (\log\log t)^{1/3}$. This result, from 1957, is (slightly less than) a factor of $(\log t)^{1/3}$ better[3] than the classical zero-free region. Building on seminal work by Ford [9, 10], and improvements made in [20], Bellotti [2] showed that one could take $A_5 = 53.989$. A quick calculation shows that this improves on the Littlewood zero-free region when $t \geq \exp(483,000) \approx 10^{2\cdot 10^5}$.

One can summarise the above results[4] as follows.

- For $0 \leq t \leq 10^{12}$ RH is true.
- For $10^{12} \leq t \leq 6.3 \cdot 10^{19}$ the classical zero-free region is best.
- For $6.3 \cdot 10^{19} \leq t \leq 10^{2\cdot 10^5}$ the Littlewood zero-free region is best.
- For $t \geq 10^{2\cdot 10^5}$ the Vinogradov and Korobov zero-free region is best.

There is an important point to note with the above estimates. The size of the constants A_3, A_4, A_5 increases in line with the strength of the result. If, e.g., $A_4 < A_3$ then the bound $A_4 \log t / \log\log t$ would *always* be better $A_3 \log t$. The fact that $A_4 > A_3$ is what gives rise to the results being better in different ranges. One should suspect that this would be true in general. The stronger the result, the 'harder' it is to prove, and hence the 'harder' it is to obtain good[5] explicit constants.

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[1]Indeed, as noted in my talk, the Clay Institute offered \$1,000,000 for the resolution RH. Actually, this award is only paid if one *proves* RH; I know of no award for the disproof of the Riemann hypothesis. Inspired by Yuta Suzuki's talk about work in [4] I am happy to record an award here. If someone can prove the Nnameir hypothesis (that not all non-trivial zeroes lie on the critical line) then I will pay \$0,000,001.

[2]We know that the Riemann hypothesis is true for values of $t \leq 3 \cdot 10^{12}$ from [24].

[3]There does not seem to be a known path to victory to improve even the exponent of $(\log\log t)$. Indeed, even showing some $f(t) = o\left((\log t)^{2/3} (\log\log t)^{1/3}\right)$ seems very difficult.

[4]See also the discussion in [33] for another 'intermediate' zero-free region.

[5]Indeed, I remember Martin Huxley once telling me that if a proof required N steps, then, as a rule-of-thumb, one should assume that the implied constant is at least 2^N.

3. More zero-free regions

But this is not the end of the tale for zero-free regions. There are results in between those of Littlewood and Vinogradov–Korobov. As an example, (6.15.1) in Titchmarsh [30] shows that one make take $f(t) = A_{4.5}(\log t)^{3/4}(\log\log t)^{3/4}$. I am not aware of any explicit work done on $A_{4.5}$. It is unlikely that any such work would be numerically useful, particular because the estimates for A_4 and A_5 are so good. In that sense, such an intermediary result risks being 'squeezed out' by neighbouring results.

Nevertheless, the concept of exploring different bounds is a useful one. Remember the numbering of the constants? Since no bound with constant A_7 is likely to appear any time soon, what about a bound with A_2? That is, what about a zero-free region *weaker* than the classical one? Is there such a region? Yes, in fact, there are two of which I am aware.

The first comes from elementary proofs of the prime number theorem (PNT). One can prove the PNT, with elementary means, and obtain an error term. Let $\psi(x) = \sum_{p^m \leq x} \log p$ be the sum over all prime powers not exceeding x. The PNT is the statement that $\psi(x) \sim x$. The best error term obtained via elementary means is due to Sampath and Srinivasan [28], namely

$$|\psi(x) - x| \leq x \exp\left(-\frac{1}{40}(\log x)^{1/6}\right), \quad (x \geq \exp(\exp(40))). \tag{3.1}$$

Since elementary proofs are a thin subset of 'all proofs', one would expect that other means should furnish better bounds than (3.1). Indeed, this is so: see [8] and [15] for explicit versions of (3.1) with better powers of $\log x$ on the right-hand side.

But what does this have to do with zero-free regions? Pintz [22] shows the connection between error terms $R(x)$ in $\psi(x) - x = O(xR(x))$ and zero-free regions of $\zeta(s)$. See also [25] and [14] for recent explicit work on this front. All of this shows that the error term in (3.1) yields a zero-free region of the shape $f(t) = A_2 \log^5 t$ for some constant A_2. With a little work, one could find the constant A_2. If A_2 is sufficiently small, then this zero-free region would be better than the classical one, but only for a finite range of t. However, this is unlikely ever to be useful. Even if A_2 were as small as 10^{-5}, this 'elementary zero-free region' beats the classical one only when $t < 7.2 \cdot 10^{11}$ — and we know that RH is true up to this height!

Now, if you think that the 'elementary zero-free region' is bad, get a load of the next one. The easiest zero-free region to prove for the Riemann zeta-function is due to an argument of Landau. This is often the first zero-free region proved in books on the subject: see, e.g., just after (3.6.4) in [30]. This shows that one may take $f(t) = A_1 \log^9 t$. Even proving $A_1 = 10^{-10}$ still only gives a zero-free region beating the classical one beneath the height to which we know RH is true. Therefore, computing A_2 or A_1 would be a pointless exercise.

While the calculation of the *values* of these constants is silly, their *generalisation* is an exciting prospect. For example, an explicit zero-free region for Dirichlet L-functions, or Vinogradov–Korobov strength, was only proved recently by Khale [16]. I am not aware of a Littlewood-style zero-free region for $L(s,\chi)$. Moreover, the current classical region performs well when qt (here q is the modulus of the character χ) is relatively large. This often requires some minimal value of t. But what about, for example, zero-free regions around height $t = 1/100$ for $q \approx 10^{10}$? This is beyond the range for which we know partial versions of the Generalised

Riemann Hypothesis are true — see, e.g., [23]. Perhaps a weaker-than-classical zero-free region could give better results here.

For still more exotic L-functions, such as Rankin–Selberg L-functions, nothing beyond the generalisation of the classical zero-free region is known — see, e.g., [13]. As such, hope springs eternal, even for the Landau bound with A_1. This is the easiest bound to establish, but may be useful in complicated settings. One could even imagine settings where weak bounds are the only ones we could prove.

4. Weaker may be all there is!

Consider the problem of representing integers as the sum of two squares. Let $R(x)$ be the number of $n \leq x$ such that n can be written as $n = a^2 + b^2$. Landau and Ramanujan showed that $R(x) \sim Cx/\sqrt{\log x}$ for an explicit value of C. Since the number of primes up to x is $\sim x/\log x$, there are more sums-of-squares than primes. Hence, in the absence of some sort of arithmetic obstruction, one would think that problems with sums-of-squares should be easier than problems with primes.

This is certainly true when it comes to proving the existence of a prime or sum-of-square in a short interval. Even under RH all we know is that there is a prime in the interval $(n, n + n^{1/2+\epsilon})$. Without assuming any hypotheses one can show that there is a square-free number in $(n, n + cn^{1/4})$ for some constant c. Bambah and Chowla [1] showed that one may take $c = 2^{3/2} + \epsilon$. Uchiyama[6] [32] refined their method and, in just two pages, proved that one can take $c = 2^{3/2}$, and that this[7] result holds for all $n \geq 1$.

At the RIMS meeting I offered a prize[8] for anyone who could improve this result in any capacity. Could one show, for example, that for sufficiently large n there are sums-of-squares in intervals of the form $(n, n + cn^{1/4})$ for some constant $c < 2^{3/2}$? Given that Maynard [18] has shown the existence of sums-of-squares in *some* very short intervals, and given that $(n, n + cn^{1/4})$ is well short of the conjectured $(n, n + n^{\epsilon})$ there is certainly hope that Uchiyama's result can be improved.

The point to make here is that the simple, elementary argument by Uchiyama (which uses nothing more than introductory calculus), gives the best result. Probably stronger results are true, but, for now, this weak result really is the best.

5. Conclusion

There are many other examples of weaker results being more 'useful' than stronger results. Just in the theory of $\zeta(s)$ alone there are results on bounds on $\zeta(1/2 + it)$ (see [21] and [12]) for a list of such results), zero-density estimates (see [3, 5]), and explicit estimates on moments [27, 6, 7].

Slightly further afield, there are explicit bounds of varying strengths known for the error term in the prime number theorem, sums of arithmetic functions (for example, the Möbius function), and sums of Dirichlet characters. A priceless source of information for such results is the website maintained by Olivier Ramaré [26].

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[6]At the RIMS meeting, I enjoyed discussing Uchiyama's work with his former student, and mainstay of Japanese number theory, Koichi Kawada.

[7]It may be possible to refine Uchiyama's constant slightly, but, as shown in [31], one could not hope to reduce it beyond $c = 2.41\ldots$; note that $2^{3/2} = 2.82\ldots$.

[8]The prize was even more generous than that for proving the Nnameir hypothesis from §2, namely, a handy (and complementary!) toothbrush from my hotel lobby.

There are numerous examples from a number field setting where numerically explicit versions of the prime ideal theorem are hard to obtain: simpler versions are often preferred, as in the work by Lee [17] and by Garcia and Lee [11]. Indeed, during the RIMS conference Wataru Takeda made me aware of his work in number fields [29], which requires a kind of explicit Bertrand postulate... much easier to prove than a prime number theorem with explicit error term!

In all these examples one really seeks the result that is best in the range of interest. As such it is important to furnish as many explicit results as possible, even if they are not the strongest asymptotically. This allows a lot of mathematicians to contribute to this area of research. These contributions can range from purely computational to purely theoretical. Since these contributions also require intimate knowledge of the literature (and this brings its own rewards), this programme of research is a rich one, indeed.

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I thank RIMS for their hospitality and for providing a stimulating working environment for this meeting. I am very grateful to Maki Nakasuji and Takashi Taniguchi for organising[9] such a wonderful conference. An extra serving of gratitude is due to Maki Nakasuji: we were enjoying our mathematical discussion at the train station so much that we missed[10] the train! It is always a delight to visit Japan, to reconnect with old friends, and to see the excellent cohort of young mathematicians coming through the ranks. I enjoyed everything from stimulating mathematical conversions to light-hearted banter with so many colleagues that I cannot list them all here. But I wish to single out (to 'name and fame' as it were!) just one: thank you to Kohji Matsumoto for choosing excellent restaurants!

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[9] Indeed, to express my thanks during the talk, and to embrace the brave new world of AI, I asked ChatGPT to compose a haiku. This read

In Kyoto's embrace
Maki and Takashi led
Laughter blooms like spring.

[10] One of the many enjoyable things about visiting Japan is that the train system is so reliable that another train was able to get me to the airport with plenty of time to spare.

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