Transcendence of continued fractions related to Stern polynomials

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Let $t > 1$ and $k > 0$ be any fixed integers. Dilcher and Ericksen [DE] introduced Type-1 Stern polynomials $a(n; z)$ defined by $a(0; z) = 0$, $a(1; z) = 1$, and for $n > 0$

$$a(2n; z) = za(n : z^t), \quad a(2n + 1; z) = a(n + 1; z^t) + a(n; z^t).$$

and found a new class of continued fractions stated below. Define the sequence

$$\alpha_n = \frac{2^{nk} - (-1)^n}{2^k + 1}.$$

The subsequence of polynomials $(a(\alpha_n; z))$ generates the finite continued fractions

$$\frac{a(\alpha_{n+1}; z)}{a(\alpha_n; z^{t^k})} = a(2^k - 1; z) + \frac{a(2^k; z^{t^k})}{a(2^k - 1; z^{t^k}) +} \frac{a(2^k; z^{t^{2k}})}{a(2^k - 1; z^{t^{2k}}) + \cdots +} \frac{a(2^k; z^{t^{(n-1)k}})}{a(2^k - 1; z^{t^{(n-1)k}})}.$$

The polynomial $a(\alpha_{n+1}; z)$ is an extension of $a(\alpha_n; z)$, and so the limmit

$$H_k(z) = \lim_{n\to\infty} a(\alpha_n; z)$$

defines a power series converges in the unit circle. Letting n tend to infinity, we obtain the infinite continued fraction

$$\frac{H_k(z)}{H_k(z^{t^k})} = a(2^k - 1; z) + \frac{a(2^k; z^{t^k})}{a(2^k - 1; z^{t^k}) +} \frac{a(2^k; z^{t^{2k}})}{a(2^k - 1; z^{t^{2k}}) +} \frac{a(2^k; z^{t^{3k}})}{a(2^k - 1; z^{t^{3k}}) + \cdots},$$

For example,

$$\frac{H_1(z)}{H_1(z^{t^k})} = 1 + \frac{z^t}{1} + \frac{z^{t^2}}{1} + \frac{z^{t^3}}{1} + \cdots,$$

$$\frac{H_2(z)}{H_2(z^{t^k})} = 1 + z^t + \frac{z^{t^2+t^3}}{1+z^{t^3}} + \frac{z^{t^4+t^5}}{1+z^{t^5}} + \frac{z^{t^6+t^7}}{1+z^{t^7}} + \cdots,$$

$$\frac{H_3(z)}{H_3(z^{t^k})} = 1 + z^t + z^{t^2} \frac{z^{t^3+t^4+t^5}}{1+z^{t^5}+z^{t^4+t^5}} + \frac{z^{t^6+t^7+t^8}}{1+z^{t^8}+z^{t^7+t^8}} + \cdots.$$

We state our results.

Theorem 1. For any algebraic number z with $0 < |z| < 1$ the numbers $H_k(z)$ and $H_k(z^{t^k})$ are algebraically independent.

Corollary 1. The continued fraction $H_k(z)/H_K(z^{t^k})$ is trancsendental.

Our proof of the theorem is a simple application of Mahler's method in trancsendental number theory developed in [N] and also in [A].

References

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