Generalized Bessel functions corresponding to the lattice point problems generalized circles

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1 Introduction

Firstly, as subject of this paper, let us consider the lattice point problems, which have been studied for a long time. In particular, the geometric subject to look at this time is the closed curves p-circle $\{x \in \mathbb{R}^2 |\ |x_1|^p + |x_2|^p = r^p\}$ generalizing a circle for positive real number p, as shown in Figure 1.

If p is 2, it is a circle, and for p greater than 2 it expands outward and for p less than 2 it depresses inward. Particularly, the case $p = \frac{2}{3}$ is well known as *the astroid*.

In view of the fact that the Bessel functions appear in the process of solving the circle problem, the functions corresponding to the p-circle problems, that is, the generalized Bessel functions for p, are determined. Then, the series representation of certain functions, which is the key to solving the problems, is given as the main result.

As an introduction, we will discuss the specifics of the problems and previous studies. $R_p(r)$ is the number of lattice points in the p-circle of radius r at the center of the origin. In addition, since a lattice point and area of the unit square at the center of that point are in one-to-one correspondence, we can consider the error term $P_p(r) := R_p(r) - \frac{2}{p}\frac{\Gamma^2(\frac{1}{p})}{\Gamma(\frac{2}{p})}r^2$ through the approximation of Figure 1. Note that the second term on the right-hand side is the area of the p-circle and $\Gamma(s)$ is the gamma function.

Then, the subject of problems is order of growth of P_p as the radius is infinitely large. Specifically, the problem is to find values such that the order of the Landau symbols $\mathcal{O}$ and Ω match with respect

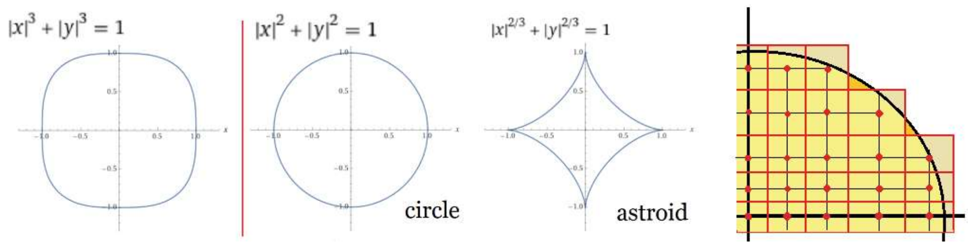

Figure 1: Examples of the p-circle and the approximation by unit squares.

to P_p. Note that, for the functions f and g, $f(t) = \mathcal{O}(g(t))$ and $f(t) = \Omega(g(t))$ respectively mean $\limsup_{t\to\infty} |\frac{f(x)}{g(x)}| < +\infty$ and $\liminf_{t\to\infty} |\frac{f(t)}{g(t)}| > 0$.

In the case of a circle, that is $p = 2$, Hardy[3] derived $\lim_{r\to\infty} r^{-\frac{1}{2}}|P_2(r)| = \infty$ in 1915, and conjectured the following[4].

$$P_2(r) = \mathcal{O}(r^{\frac{1}{2}+\varepsilon}) \qquad \text{for any small } \varepsilon > 0. \tag{1.1}$$

This means that the infimum of P_2 evaluation is a half order, but this conjecture remains unresolved to this day. As latest results, it turns that the evaluation holds with ε greater than $\frac{105}{824}(= 0.127\cdots)$ by Bourgain and Watt[1] in 2017.

On the other hand, in the cases $p > 2$, there is a well-known method. Let us consider the representation of P_p([7], (3.57)), which is decomposed by second main term Ψ by E. Krätzel, and a remainder term Δ of less than $\frac{2}{3}$ order at most.

$$P_p(r) = \Psi(r;p) + \Delta(r;p).$$

Then, the second main term Ψ is defined as a series consisting of generalized Bessel functions by Krätzel([7], (3.55), Definition 3.3).

$$\Psi(r;p) := 8\sqrt{\pi}\Gamma(1+\frac{1}{p})\sum_{n=1}^{\infty}\left(\frac{r}{\pi n}\right) J^{(p)}_{\frac{2}{p}}(2\pi nr) \qquad \text{for } r > 0,$$

$$J^{(p)}_{\nu}(r) := \frac{2}{\sqrt{\pi}\Gamma(\nu+1-\frac{1}{p})}\left(\frac{r}{2}\right)^{\frac{p\nu}{2}}\int_0^1 (1-t^p)^{\nu-\frac{1}{p}}\cos rt\, dt \qquad \text{for } r > 0,\ \nu > \frac{1}{p}-1.$$

But, note that these are different from the generalization of the Bessel functions stated subsequently in this paper.

This representation of P_p and $\mathcal{O}$ - Ω estimates of Ψ, obtained from the asymptotic expansion of Ψ, show that the following important theorem holds.

Theorem 1.1 *([7], Theorem 3.17 A). Let $p > 2$. If $\alpha_p < 1 - \frac{1}{p}$ such that $\Delta(r;p) = \mathcal{O}(r^{\alpha_p})$ exists, then $P_p(r) = \mathcal{O}(r^{1-\frac{1}{p}}), \Omega(r^{1-\frac{1}{p}})$ holds.*

This theorem shows that improving the evaluation for the remainder term Δ is directly related to solving the problem of the cases $p > 2$. As the latest result with this established method, from the Kuba's result[8] which $\mathcal{O}$ order for Δ is approximately $\frac{46}{73}$ in 1993, the cases with at least p greater than $\frac{73}{27}$ have been resolved.

On the other hand, it is difficult to apply this method to solving problems of the cases $0 < p < 2$. The figure *astroid* represented by the case $p = \frac{2}{3}$ is also included in these cases. For example, one of the main problems that cannot be applied is appearance of singular points in the derivative of $x_2(x_1)$. Thus, in order to solve the problem in these unsolved cases such as $0 < p < 2$, we need to use another method.

Now, we turn our attention again to the case $p = 2$, and particularly focus on S. Kuratsubo and E. Nakai's paper in 2022[9]. One of the main results of this paper is *a theorem giving a harmonic-analytic claim that is equivalent to the Hardy's conjecture (1.1)* (see Theorem 7.1 in [9]).

For $\beta > -1$, $s > 0$ and $x \in \mathbb{R}^2$, if we define

$$D_\beta(s:x) := \frac{1}{\Gamma(\beta+1)} \sum_{|m|^2<s} (s-|m|^2)^\beta e^{2\pi i x\cdot m}, \quad \mathcal{D}_\beta(s:x) := \frac{1}{\Gamma(\beta+1)} \int_{|\xi|^2<s} (s-|\xi|^2)^\beta e^{2\pi i x\cdot\xi} d\xi, \tag{1.2}$$

then $D_\beta - \mathcal{D}_\beta$ is a generalization of the circle error term P_2, and this function can be expressed as a series sum of the Bessel function for $\beta > \frac{1}{2}$ ([9], (2.6)).

$$D_\beta(s:x) - \mathcal{D}_\beta(s:x) = s^{\beta+1} 2^{\beta+1} \pi \sum_{n\in\mathbb{Z}^2\setminus\{0\}} \frac{J_{\beta+1}(2\pi\sqrt{s}|x-n|)}{(2\pi\sqrt{s}|x-n|)^{\beta+1}}. \tag{1.3}$$

This equality plays an very important role in various parts in the paper .

From this equality and various properties of the Bessel functions, a relatively crude $\mathcal{O}$ evaluation formula for the error term P_2 is obtained, but for the cases $0 < p < 2$, the current situation is that we have only obtained the two evaluations

$$P_p(r) = \mathcal{O}(r) \qquad \text{if } p > 0,$$
$$P_p(r) = \mathcal{O}(r^{\frac{2}{3}}) \qquad \text{if } 0 < p \leq \frac{1}{2}$$

from the theorem by Krätzel for general convex figures ([7], Theorem 3.1-3.6). Therefore, instead of trying to improve the second main term method of Krätzel, we consider solving the problems by generalizing Kuratsubo and Nakai's method for p.

Thus, in this paper, by replacing $\mathbb{R}^2$-norm by p-norm $|\cdot|_p$ (that is, $|\xi|_p = (|\xi_1|^p + |\xi_2|^p)^{\frac{1}{p}}$ for $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$), we consider $D_\beta^{[p]} - \mathcal{D}_\beta^{[p]}$, which consist of generalizations of Kuratsubo and Nakai's functions (1.2) for p

$$D_\beta^{[p]}(s:x) := \frac{1}{\Gamma(\beta+1)} \sum_{|m|_p^p<s} (s-|m|_p^p)^\beta e^{2\pi i x\cdot m}, \quad \mathcal{D}_\beta^{[p]}(s:x) := \frac{1}{\Gamma(\beta+1)} \int_{|\xi|_p^p<s} (s-|\xi|_p^p)^\beta e^{2\pi i x\cdot\xi} d\xi.$$

Then, as we saw in the circle case, it is clear that $D_\beta^{[p]} - \mathcal{D}_\beta^{[p]}$ are generalizations of P_p by variables.

2 Main results

As the goal of this paper, in order to derive the series representation of $D_\beta^{[p]} - \mathcal{D}_\beta^{[p]}$, that is a generalization of (1.3), we need to appropriately generalize the Bessel functions.

We note that $\mathcal{D}_\beta^{[p]}$ are in the form of Fourier transforms for p-radial functions. The property of p-radial is generalized one of spherical symmetry, and for a function F on $\mathbb{R}^2$, we say that F is p-radial if there exists a function ϕ on nonnegative real numbers satisfying $F(x) = \phi(|x|_p)$ on $\mathbb{R}^2$.

The Fourier transform of a function F which is p-radial and integrable on $\mathbb{R}^2$ can be expressed as follows from elementary variable transformations.

$$\hat{F}(\xi) = p\Gamma^2(\frac{1}{p}) \int_0^\infty J_0^{[p]}(2\pi r\xi)\phi(r)rdr \qquad \text{for } \xi \in \mathbb{R}^2, \tag{2.1}$$

$$\text{with } J_0^{[p]}(\eta) := \frac{1}{\Gamma^2(\frac{1}{p})}\left(\frac{2}{p}\right)^2 \int_0^1 \cos(\eta_1 t^{\frac{1}{p}})\cos(\eta_2(1-t)^{\frac{1}{p}})t^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}-1}dt \qquad \text{for } \eta \in \mathbb{R}^2.$$

In the case $p = 2$, this is called Hankel transform of order zero([2]). The function $J_0^{[p]}$ defined here is a generalization of the Bessel function of zero order for p. However, note the difference in the definition region between each other.

Furthermore, following the property that a Bessel function of a certain order can be expressed by integration of Bessel function of a different order([10], Lemma 4.13)

$$J_{\alpha+\beta+1}(t) = \frac{t^{\beta+1}}{2^\beta\Gamma(\beta+1)}\int_0^1 J_\alpha(ts)s^{\alpha+1}(1-s^2)^\beta ds \qquad \text{for } \alpha > -\frac{1}{2},\ \beta > -1,\ t > 0,$$

we define the generalized Bessel function of non-negative real order by taking $J_0^{[p]}$ as the nucleus and adding a positive order as follows([5], (2.4), Definition 2.5).

$$J_\omega^{[p]}(x) := \begin{cases} \frac{1}{\Gamma^2(\frac{1}{p})}\left(\frac{2}{p}\right)^2 \int_0^1 \cos(x_1 t^{\frac{1}{p}})\cos(x_2(1-t)^{\frac{1}{p}})t^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}-1}dt & \text{if } \omega = 0, \\ \frac{|x|_p^\omega}{p^{\omega-1}\Gamma(\omega)}\int_0^1 J_0^{[p]}(\tau x)\tau(1-\tau^p)^{\omega-1}d\tau & \text{if } \omega > 0. \end{cases} \tag{2.2}$$

Then, the main result of this paper, a generalization of the display (1.3) is following theorem.

Theorem 2.1 *([5], Theorem 1.3).* *Let $p > 0$. If $\beta > -1$ satisfies that $\mathcal{D}_\beta^{[p]}(1 : x)$ is integrable on $\mathbb{R}^2$, then*

$$D_\beta^{[p]}(s : x) - \mathcal{D}_\beta^{[p]}(s : x) = s^{\beta+\frac{2}{p}}p^{\beta+1}\Gamma^2(\frac{1}{p})\sum_{n\in\mathbb{Z}^2\setminus\{0\}} \frac{J_{\beta+1}^{[p]}(2\pi\sqrt[p]{s}(x-n))}{(2\pi\sqrt[p]{s}|x-n|_p)^{\beta+1}} \qquad \textit{for } s > 0,\ x \in \mathbb{R}^2.$$

Furthermore, under this assumption, since $|x-n|_p \geq \frac{1}{2}$ for $x \in \mathbb{T}^2(:= (-\frac{1}{2}, \frac{1}{2}]^2)$ and $n \neq 0$, the series converges absolutely for $x \in \mathbb{T}^2$.

Hereafter, we will outline the proof of this theorem.

Firstly, note that $\mathcal{D}_\beta^{[p]}(1 : x)$ is in the form of the Fourier transform of function $U_{\beta,1}^{[p]}$

$$U_{\beta,1}^{[p]}(x) = \begin{cases} (1-|x|_p^p)^\beta & \text{if } |x|_p < 1, \\ 0 & \text{if } |x|_p \geq 1, \end{cases}$$

and by the generalized Hankel transform of zero-order (2.1) and the definition of $J_\omega^{[p]}$(2.2), thereby following Lemma is obtained.

Lemma 2.2 *([5], Proposition 3.1). Let $p > 0$ and $\beta > -1$. Then, the following holds.*

$$\mathcal{D}_{\beta}^{[p]}(s:x) = s^{\beta+\frac{2}{p}} p^{\beta+1} \Gamma^2(\frac{1}{p}) \frac{J_{\beta+1}^{[p]}(2\pi \sqrt[p]{s} x)}{(2\pi \sqrt[p]{s}|x|_p)^{\beta+1}} \qquad \textit{for } s > 0,\ x \in \mathbb{R}^2.$$

Next, under the integrable assumption, we obtain the series representation of $D_{\beta}^{[p]}$

$$D_{\beta}^{[p]}(s:x) = \sum_{n \in \mathbb{Z}^2} \mathcal{D}_{\beta}^{[p]}(s:x-n) \tag{2.3}$$

by applying the Fourier inverse transform and Poisson's sum formula.

Lemma 2.3 *(Poisson summmation formula: [10], Theorem 2.4). For a function F integrable on $\mathbb{R}^d$ $(d \in \mathbb{N})$, the series $f(x) := \sum_{m \in \mathbb{Z}^d} F(x+m)$ converges in the L^1-norm of $\mathbb{T}^d (:= (-\frac{1}{2}, \frac{1}{2}]^d)$ and is integrable on $\mathbb{T}^d$, and $\hat{F}(m) = \hat{f}(m)$, that is, the following holds.*

$$f(x) = \sum_{m \in \mathbb{Z}^d} \hat{F}(m) e^{2\pi i x \cdot m} \qquad \textit{for } x \in \mathbb{T}^d.$$

Poisson's sum formula is about periodization of integrable functions, and thus we see that $D_{\beta}^{[p]}$ are periodizing functions consisting of $\mathcal{D}_{\beta}^{[p]}$.

Therefore, under the assumption β, by Lemma 2.2 and the expansion(2.3), derivation of the desired formula is completed.

$$\begin{aligned} D_{\beta}^{[p]}(s:x) - \mathcal{D}_{\beta}^{[p]}(s:x) &= \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} \mathcal{D}_{\beta}^{[p]}(s:x-n) \\ &= s^{\beta+\frac{2}{p}} p^{\beta+1} \Gamma^2(\frac{1}{p}) \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} \frac{J_{\beta+1}^{[p]}(2\pi \sqrt[p]{s}(x-n))}{(2\pi \sqrt[p]{s}|x-n|_p)^{\beta+1}} \quad \text{for } s > 0,\ x \in \mathbb{R}^2. \end{aligned}$$

3 Concluding remarks

Finally, I would like to conclude this paper with some perspectives on future research. By the generalization of Kuratsubo and Nakai's the equality (Theorem 2.1), obtained this time, from now on, we aim to improve $\mathcal{O}$ evaluation of P_p.

Firstly, we need to identify an infimum of β where $\mathcal{D}_{\beta}^{[p]}(1:x)$ is integrable on $\mathbb{R}^2$, which is the assumption of Theorem 2.1. If we obtain uniformly asymptotic evaluations of $J_{\omega}^{[p]}$ on $\mathbb{R}^2$, then this is solved.

On the other hand, by *Van der Corput's Lemma*, that is, asymptotic evaluation method for oscillatory integrals, we have already obtained a uniform evaluation of zero-order as a special case.

In detail, as the conditions to be added, under p such that $\frac{2}{p}$ are natural numbers, asymptotic evaluations of $J_0^{[p]}$ uniformly on $\mathbb{R}^2$ are gained as follow.

Theorem 3.1 *([6], Theorem 1.5; Uniformly asymptotic estimates on* $\mathbb{R}^2$*).*
For the cases such that $\frac{2}{p}$ *are the natural numbers, the following holds uniformly on* $\mathbb{R}^2$*.*

$$J_0^{[p]}(\eta) = \begin{cases} \mathcal{O}(|\eta|_p^{-\frac{1}{2}}) & (p=2), \\ \mathcal{O}(|\eta|_p^{-\frac{p}{2}}) & (\frac{2}{p} \in \mathbb{N}_{\geq 2}), \end{cases} \qquad \text{as } |\eta|_p \to \infty.$$

Thus, by this result, the method via $J_\omega^{[p]}$ is expected to be suitable for problems of *astroid type*, that is, the cases p such that $\frac{2}{p}$ are odd numbers.

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