Zero mean curvature surfaces with planar curvature lines in isotropic 3-space*

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Introduction

The study of minimal surfaces, specifically characterized by vanishing mean curvature and commonly referred to as zero mean curvature surfaces (ZMC surfaces), dates back to the mid 18th century, with significant contributions from J. -L. Lagrange and L. Euler. This field remains to be an area of active research across a variety of geometric spaces to the present day. Various investigations and classifications regarding ZMC surfaces, particularly those featuring planar curvature lines, have been conducted in different classes of surfaces and spaces. Since the 19th century, scholars such as O. Bonnet [6] and L. P. Eisenhart [12] have made substantial contributions to this topic. More recently, the field has been further advanced by many researchers, amongst whom are S. Akamine, J. Cho, and Y. Ogata [3, 9, 10].

In Euclidean 3-space, the types of minimal surfaces with planar curvature lines are restricted to the plane, the Enneper surface, the catenoid, and surfaces belonging to the Bonnet family (refer to Figure 1).

Moreover, there is a burgeoning interest in isotropic 3-space, which is primarily studied by K. Strubecker [20], largely due to its relationship with the geometries of Euclidean 3-space $\mathbb{R}^3$ and Lorentz-Minkowski 3-space $\mathbb{L}^3$ (see [4, 17, 18, 19]). Notably, these geometries can be cohesively described through the consideration of affine hyperplanes in $\mathbb{L}^4$, and furthermore, it is well established that (spacelike) ZMC surfaces within this framework also admits a Weierstrass representation [20] (see also [5, 15] for uniform points of view).

In this survey article, we first establish the foundational surface theory in isotropic 3-space within the 4-dimensional Lorentz-Minkowski space. Subsequently, we present a complete classification of ZMC surfaces with planar curvature lines in $\mathbb{I}^3$, accompanied by their canonical

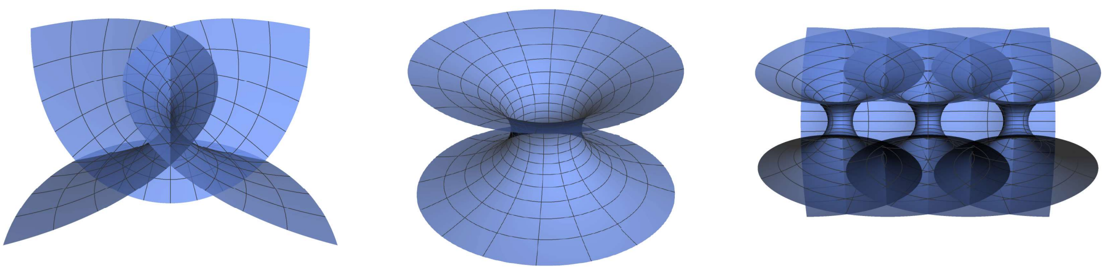


Figure 1: Examples of minimal surfaces with planar curvature lines in $\mathbb{R}^3$: Enneper surface (left), catenoid (middle) and a surface in the Bonnet family.

*This research is based on joint work with Joseph Cho (Handong Global University).

Weierstrass data (see Theorem 2.6). This topic was firstly investigated by K. Strubecker [21], however, we complete the classification by adding in a new member of this class.

1 Surface theory in isotropic 3-space

Firstly, we introduce the isotropic 3-space derived from 4-dimensional Lorentz-Minkowski space. Note that some researchers use an alternative definition, describing it as a Euclidean 3-space equipped with a degenerate metric $ds^2 = dx^2 + dy^2$, often referred to as the *simply isotropic space* (see, for example, [18]). Our point of view coincides with this via a projection (see [8]).

Let $\mathbb{L}^4$ be a Lorentz-Minkowski 4-space equipped with the inner product $\langle \cdot, \cdot \rangle$ of signature $(- + ++)$, and let $\mathcal{L}$ be the light cone, that is,

$$\mathcal{L} = \{X \in \mathbb{L}^4 \mid \langle X, X \rangle = 0\}.$$

For a lightlike vector $\mathfrak{p} \in \mathcal{L}$ called a *point sphere complex*, the *isotropic 3-space* $\mathbb{I}^3$ is defined as

$$\mathbb{I}^3 = \{X \in \mathbb{L}^4 \mid \langle X, \mathfrak{p} \rangle = 0\}.$$

Without loss of generality, we choose $\mathfrak{p} = (1, 0, 0, 1)^t$, and then we can write

$$\mathbb{I}^3 = \{(l, x, y, l)^t \in \mathbb{L}^4\}.$$

We refer to the coordinate space $\{(x, y, l)^t\}$ as isotropic 3-space, and refer to the l-direction as vertical.

Let $X(u, v) : \Sigma \to \mathbb{I}^3$ be a spacelike conformal immersion over a simply-connected domain Σ with the metric ds^2 given by

$$ds^2 = e^{2\omega}(du^2 + dv^2)$$

for some $\omega : \Sigma \to \mathbb{R}$ and let $n : \Sigma \to \mathcal{L}$ be its *lightlike Gauss map* of X.

Then the mean curvature H and Hopf differential Q are

$$H = 2e^{-2\omega}\langle X_{z\bar{z}}, n \rangle \quad \text{and} \quad Q dz^2 = \langle X_{zz}, n \rangle dz^2,$$

respectively, where $z = u+iv$. Thus, we have the Gauss-Weingarten equations and Gauss-Codazzi equations

$$\begin{cases} X_{zz} = 2\omega_z X_z + Q\mathfrak{p} \\ X_{z\bar{z}} = \dfrac{1}{2} e^{2\omega} H \mathfrak{p} \\ n_z = -HX_z - 2e^{-2\omega} Q X_{\bar{z}} \end{cases} \quad \text{and} \quad \begin{cases} \omega_{z\bar{z}} = 0 \\ H_z = 2e^{-2\omega} Q_{\bar{z}}, \end{cases}$$

respectively.

Since we only consider (spacelike) ZMC surfaces, we suppose $H = 0$ and omit 'spacelike' for the sake of simplicity. Then, since Q is holomorphic, we may choose $M = 0$ and $L = -N = -1$ without loss of generality. Now the compatibility condition for X is

$$\omega_{uu} + \omega_{vv} = 0.$$

2 ZMC surfaces with planar curvature lines

As mentioned in [17], $\mathbb{I}^3$ is closely related to the geometries of both $\mathbb{R}^3$ and $\mathbb{L}^3$. For example, when considering the mean curvature of spacelike graphs in each of these spaces, the corresponding mean curvature formulas are nearly identical. The only difference is that for the cases of $\mathbb{R}^3$, $\mathbb{I}^3$ and $\mathbb{L}^3$, some signs are modified to $+$, 0 and $-$, respectively. In fact, we can find such features in some of our equations about the planar curvature line conditions. Thus the study of $\mathbb{I}^3$ is important because its geometry is expected to be the simplest.

The u-curvature lines of a surface $X(u, v)$ are *planar* if X_u, X_{uu}, X_{uuu} are linearly dependent. Then we have the following lemma, which is an analogue of previous research [9, 10]:

Lemma 2.1. *For an umbilic-free ZMC surface in* $\mathbb{I}^3$*, the following are equivalent:*

- *u-curvature lines are planar.*
- *v-curvature lines are planar.*
- $\omega_{uv} + \omega_u\omega_v = 0$.

Therefore, the solutions of the following system of partial differential equations for the conformal factor ω correspond to every ZMC surface with planar curvature lines:

$$\begin{cases} \omega_{uu} + \omega_{vv} = 0 \\ \omega_{uv} + \omega_u\omega_v = 0. \end{cases} \tag{1}$$

Starting with viewing the second equation as logarithm derivatives of ω_u or ω_v, we can solve (1):

Lemma 2.2. *Except for planes, without loss of generality, we have the following four solutions to* (1) *as follows:*

1. $e^\omega = \dfrac{1}{\alpha}e^{-\alpha u}$,
2. $e^\omega = \dfrac{\beta}{2}(u^2 + v^2)$,
3. $e^\omega = \dfrac{\beta}{\alpha^2}(\cosh\alpha u - \cos\alpha v)$ *and*
4. $\omega \equiv c$,

where $\alpha, \beta \in \mathbb{R}_{>0}$ *and* $c \in \mathbb{R}$.

Next, we compute their canonical Weierstrass-data; it is known that there is the Weierstrass representation for ZMC surfaces in $\mathbb{I}^3$:

Fact 2.3 ([8], [20, Equation 8.31])**.** *Any spacelike ZMC surface* $X : \Sigma \to \mathbb{I}^3$ *over a simply-connected domain* Σ *can be locally represented as*

$$X = \mathrm{Re}\int (1, -i, h)^t\, \eta$$

for some meromorphic function h and holomorphic 1-form η with holomorphic $h^2\eta$. Then X has the induced metric

$$ds^2 = |\eta|^2$$

with the Hopf differential

$$Q dz^2 = \frac{1}{2}\,\eta\, dh.$$

We call (h, η) the Weierstrass data.

Remark 2.4. Spacelike ZMC surfaces in $\mathbb{I}^3$ are often referred to as *minimal* surfaces, as you can see in [8, 18, 20]. However, we need to care about the direction of the variation. If we take the variation along the point sphere complex $\mathfrak{p}$, then ZMC surfaces are minimal. While if we take the variation along the lightlike Gauss map n, then ZMC surfaces are maximal. This preference well reflects the role of isotropic 3-space as a geometry between Euclidean and Lorentz-Minkowski geometries.

Since we supposed $ds^2 = e^{2\omega}(du^2 + dv^2)$ and $Q = -\frac{1}{2}$, we can find Weierstrass data of solutions for (1) by solving

$$|\eta|^2 = e^{2\omega} dz d\bar{z} \quad \text{and} \quad \eta\, dh = -dz^2.$$

Choosing initial conditions so that the Weierstrass data is simple, we can recover the canonical Weierstrass data and complete the classification.

Remark 2.5. Compared to the Weierstrass representations of minimal surfaces in Euclidean space and maxfaces in Lorentz-Minkowski space, the metric of ZMC surfaces in $\mathbb{I}^3$ only depends on the holomorphic data η. Therefore, we can directly find the canonical Weierstrass data only from the metric.

By applying homotheties in $\mathbb{I}^3$, we may normalize η, and the scale of h also can be simplified by applying isometries in $\mathbb{I}^3$. Furthermore, we consider coordinates changes without loss of generality, and we conclude as follows:

Theorem 2.6. *Let $X : \Sigma \to \mathbb{I}^3$ be a ZMC immersion with planar curvature lines. Then X must be a piece of one, and only one, of the following:*

- *plane* $(0,\, dz)$,
- *trivial Enneper-type surface* $(z,\, dz)$,
- *catenoid* $(e^z,\, e^{-z}dz)$,
- *Enneper-type surface* $\left(z^{-1},\, z^2 dz\right)$, *or*
- *Bonnet-type surface* $\left(\coth z,\, \sinh^2 z dz\right)$,

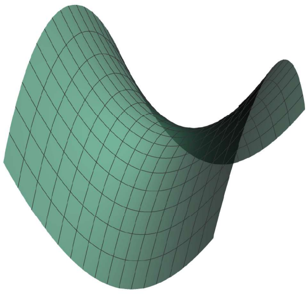
up to the isometries and homotheties of $\mathbb{I}^3$ and Lorentz boosts in the ambient space (see Figure 2).

For the classification theorems in other spaces, we have the Bonnet family up to isometries and homotheties. However, in the case of isotropic 3-space, we only have exactly five surfaces by additionally considering the Lorentz boost in $\mathbb{L}^4$.

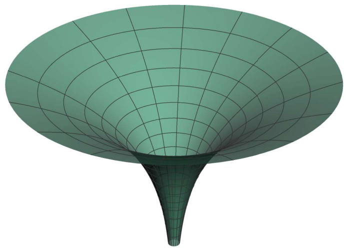
Remark 2.7. Incomplete points on the five surfaces are points where the metric vanishes, that is, zero points of η. They are branch points and the Enneper-type surface in $\mathbb{I}^3$ has such a point at the center. On the Bonnet-type surface, such points appear periodically.

Remark 2.8. Without Lorentz boosts in the ambient space, the classification in Theorem 2.6 only differs at the Bonnet-type surface. Without Lorentz boosts, it is a family given by the Weierstrass data

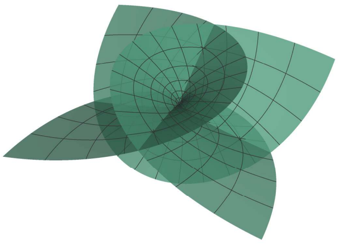
$$(h,\, \eta) = \left(\frac{\alpha}{\beta} \coth \frac{\alpha}{2} z,\; 2\frac{\beta}{\alpha^2} \sinh^2 \frac{\alpha}{2} z dz\right). \tag{2}$$



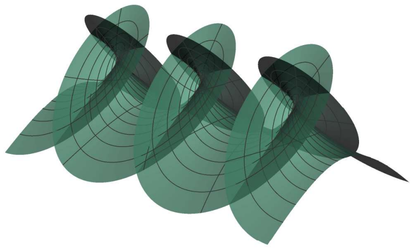
(a) Trivial Enneper-type surface



(b) Catenoid



(c) Enneper-type surface



(d) Bonnet-type surface

Figure 2: ZMC surfaces with planar curvature lines in $\mathbb{I}^3$ other than planes.

3 Continuous deformation

In the Euclidean and Lorentz-Minkowski spaces, there exist continuous deformations among ZMC with planar curvature lines. Here, we consider a continuous deformation to be continuous with respect to a parameter if the surface converges uniformly component-wise with respect to the parameter on compact subdomains. In order to construct a deformation of ZMC surfaces in $\mathbb{I}^3$, we need to solve (1) so that every solution has the same initial conditions.

To deform from the Bonnet-type surface to the trivial Enneper-surface via ZMC surfaces with planar curvature lines, we need to take the path $\beta = \dfrac{\alpha^2}{2}$ in (2) and take the limit as $\alpha \to 0$. Between the Enneper surface and catenoid via Bonnet surface, there exists a one parameter family of surfaces along θ in the following Weierstrass data:

Theorem 3.1. *There exists a continuous deformation consisting exactly of the ZMC surfaces with planar curvature lines in isotropic 3-space, and the Weierstrass data is*

$$h_{(r,\theta)} = \frac{2e^{2i\theta}\cos\theta}{e^{-r\cos\theta z}(\cos\theta + 1) - \sin\theta}$$

and

$$\eta_{(r,\theta)} = \frac{e^{-2i\theta}(\cos\theta\sinh(r\cos\theta z) - \cosh(r\cos\theta z) + \sin\theta)}{r\cos^2\theta}\,dz$$

for some $r \in (0,\infty)$ *and* $\theta \in \left(0, \frac{\pi}{2}\right)$ *(see Figure 3).*

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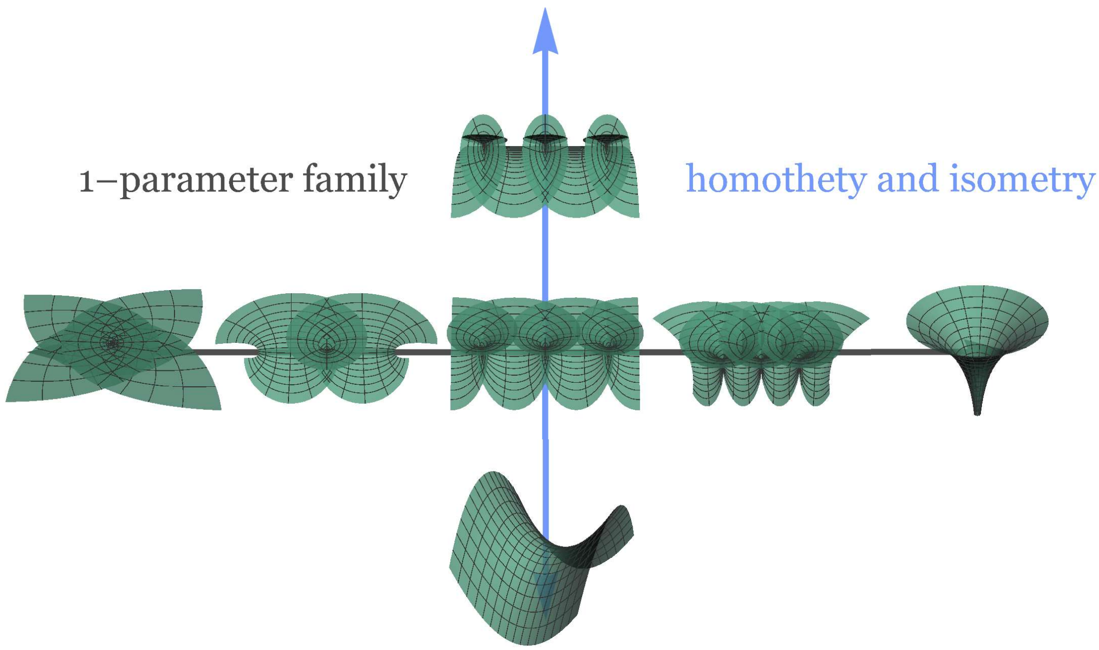

Figure 3: Continuous deformations of ZMC surfaces with planar curvature lines in $\mathbb{I}^3$.

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