Geometry of intrinsic singular points and geodesic circles of piecewise-smooth surfaces

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1 Introduction

Gauss's Theorema egregium (which means remarkable theorem in Latin) states that the Gaussian curvature K of a regular smooth surface M in the 3-dimensional Euclidean space $\mathbb{R}^3$ is determined by the first fundamental form $ds^2 = \sum_{i,j=1,2} g_{ij} du^i du^j$ of M, which means that K is determined by the Riemannian metric of M. This property can be proved to give an explicit representation of K in terms of g_{ij} and their derivatives. Another proof was given by J. Bertrand [2] showing the following formula.

Theorem 1.1 (Bertrand-Puiseux theorem (1848) [2], cf. [1, Theorem 17])**.** *Assume that M is a smooth regular surface. Then, for sufficiently small $r > 0$, the length $L(p;r)$ of the geodesic circle $C(p;r)$ with center $p \in M$ and radius r is represented as*

$$L(p;r) = 2\pi r - \frac{\pi}{3} K(p) r^3 + O(r^4), \tag{1}$$

where $K(p)$ is the Gaussian curvature of M at p.

Integrating (1) with respect to r, one can get the corresponding formula for the area of small geodesic discs. Later A. Gray [3] gave the first five terms of the power series expansion in radius r of the volume of a small geodesic ball in an analytic Riemannian manifold of general dimension. However, if M is only piecewise-smooth, such a formula does not hold in general.

Let $M = \cup_i M_i$ be a piecewise-smooth surface in $\mathbb{R}^3$ (for the definition of piecewise-smooth surfaces, see §2). Since we will discuss local and intrinsic properties of M, it is

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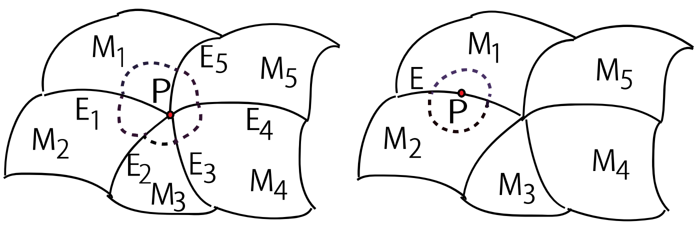

Figure 1: Left: a geodesic circle (dashed curve) with center at an extrinsic vertex $p = M_1 \cap \cdots \cap M_5$ of a piecewise-smooth surface. Right: a geodesic circle (dashed curve) with center at an extrinsic edge point $p \in E = M_1 \cap M_2$ of a piecewise-smooth surface.

not necessary to consider any ambient space that includes M. However, for simplicity, we assume that M is embedded in $\mathbb{R}^3$. For points $p, q \in M$, the distance $d(p,q)$ between p and q is defined as the shortest length of piecewise-smooth curves in M connecting p with q. The geodesic circle with center at p and radius r is the set of all points $q \in M$ satisfying $d(q,p) = r$, which is denoted by $C(p;r)$ and its length is denoted by $L(p;r)$ (Figure 1). The domain $D(p;r)$ bounded by $C(p;r)$ including p is called a geodesic disc with center at p and radius r. We can prove the following result.

Theorem 1.2 ([4, 5]). *For sufficiently small $r > 0$, we have the following.*

(i) Let $p = M_1 \cap \cdots \cap M_N$ be an extrinsic vertex of M. Then,

$$\begin{aligned} L(p;r) =& \Big(\sum_{i=1}^{N} \sigma_i\Big) r - \frac{1}{2} \sum_{i=1}^{N} \Big(k_{gi}^1(0) + k_{gi}^2(0)\Big) r^2 \\ & - \frac{1}{6} \sum_{i=1}^{N} \Big(K_i(p)\sigma_i + (k_{gi}^1)'(0) + (k_{gi}^2)'(0)\Big) r^3 + O(r^4) \end{aligned} \tag{2}$$

holds, where σ_i is the inner angle of M_i at p, $K_i(p)$ is the Gaussian curvature of M_i at p, $k_{gi}^1(r)$ is the signed geodesic curvature of the edge $M_{i-1} \cap M_i$ and $k_{gi}^2(r)$ is that of $M_i \cap M_{i+1}$ for the geodesic polar coordinate (r_i, θ_i) in M_i around p.

(ii) Let p be an interior point of an extrinsic edge $E = M_1 \cap M_2$ of M. And let $k_{gi}(p)$ be the geodesic curvature of E at p as a point in the regular surface M_i with respect to the normal pointing into the interior of M_i. Then,

$$L(p;r) = 2\pi r - \big(k_{g1}(p) + k_{g2}(p)\big) r^2 - \frac{\pi}{6}\big(K_1(p) + K_2(p)\big) r^3 + O(r^4) \tag{3}$$

holds.

We can obtain the corresponding formulas of the area of small geodesic discs $D(p;r)$ by integrating (2), (3) with respect to r from $r = 0$ to r.

We should remark That, in the special case where M is a polyhedron and p is a vertex of M, then $2\pi - \sum_{i=1}^{N} \sigma_i$ coincides with the so-called angular defect at p.

In view of Theorems 1.1 and 1.2, we define intrinsic singular points of M as follows.

Definition 1.1 (intrinsic vertex and intrinsic edge [4]). *Let p be a point of a piecewise-smooth surface $M = \cup_i M_i$ in $\mathbb{R}^3$.*

(i) Set $S(p) = \lim_{r \to +0} \dfrac{2\pi r - L(p; r)}{r}$. If $S(p) \neq 0$, then we call p an intrinsic vertex of M.

(ii) Assume that $S(p) = 0$ holds. Set $k_e(p) = \lim_{r \to +0} \dfrac{2\pi r - L(p; r)}{r^2}$. If $k_e(p) \neq 0$, then we call p an intrinsic edge point of M.

(iii) We call p an intrinsic singular point of M if p is either an intrinsic vertex of M or an intrinsic edge point of M.

(iv) Each maximal connected arc that consists of intrinsic edge points of M is called an intrinsic edge of M.

Next we define curvatures of M at each intrinsic singular point.

Definition 1.2 (curvature at intrinsic singular point [4]). *Let p be a point of a piecewise-smooth surface $M = \cup_i M_i$ in $\mathbb{R}^3$.*

(i) We call $S(p)$ the sharpness of M at p.

(ii) Assume that $S(p) = 0$ holds. We call $k_e(p)$ the edge curvature of M at p.

We give two important remarks on the intrinsic curvatures as follows.

Remark 1.1. *(i) If p is an interior point of an extrinsic edge $E = M_a \cap M_b$ of M, then $k_e(p) = k_{ga}(p) + k_{gb}(p)$ holds because of (3).*

(ii) If p is a regular point of M, then $S(p) = k_e(p) = 0$ because of (1).

By using the curvatures defined above, we obtain a simple representation of the celebrated Gauss-Bonnet formula for piecewise-smooth surfaces. For simplicity, we give the formula for closed surfaces as follows.

Theorem 1.3 (Gauss-Bonnet type theorem [4]). *Let M be a closed (that is, compact and without boundary) piecewise-smooth surface in $\mathbb{R}^3$ with Euler characteristic $\chi(M)$. Denote the union of all intrinsic edges of M by $\tilde{E}$. Then, it holds that*

$$\int_M K \, dA + \int_{\tilde{E}} k_e \, ds + \sum_{p \in M} S(p) = 2\pi \chi(M), \tag{4}$$

where dA is the area element of M and ds is the line element at regular points of $\tilde{E}$.

Note that in the formula (4), the summation of $S(p)$ is a finite sum because only at intrinsic vertices of M, the sharpness S does not vanish. We also remark that, regarding Gauss-Bonnet type formulas for surfaces with singular points, ones for fronts are known[6].

Here we give two simple examples.

Example 1.1. *(i) Let M be the surface of a cube. If p is a vertex of M, then*

$$L(p;r) = \frac{3}{2}\pi r$$

holds.

(ii) Let M be the closed surface that consists of the surface of a right circular cylinder with radius R and with finite height covered by two flat discs with radius R. If p is a point in one of the two edges of M, then

$$L(p;r) = 2\pi r - \frac{1}{R}r^2 + O(r^4)$$

holds.

We give another application of the curvatures defined above. A piecewise-smooth surface M is said to be developable if it is isometric to a planar region Ω (that is, there exists a Lipschitz continuous bijective mapping F from M onto Ω which preserves the length of each curve). Since developable surfaces can be constructed by bending a flat sheet, they are important in manufacturing objects from sheet metal, cardboard, and plywood. If M is smooth, then it is locally developable if and only if the Gaussian curvature of M vanishes everywhere.

Theorem 1.4 ([5]). *Let M be a piecewise-smooth surface in $\mathbb{R}^3$. Then, it is locally developable if and only if $S(p) = k_e(p) = K(p) = 0$ for all $p \in M$.*

This article is organized as follows. In §2, we recall the definition of piecewise-smooth surface and so-called geodesic polar coordinates for smooth surfaces. In §3, we give a kind of local form of Theorem 1.2, that is, we give an expansion of the length of geodesic circles in an smooth angular domain M_i around a vertex. We also give a representation formula for the geodesic curvatures of the boundary arcs of M_i. Then, we give an idea of the proof of Theorem 1.2. The proofs of Theorems 1.3, 1.4 are omitted.

2 Preliminaries

First we recall the definition of piecewise-smooth surfaces in $\mathbb{R}^3$. For our subject to study in this paper, it is sufficient to consider only compact oriented surfaces without self-intersection. Let $M = \cup_{i=1}^{\mu} M_i$ be a 2-dimensional oriented compact connected topological submanifold of $\mathbb{R}^3$ possibly with boundary, where each M_i is a 2-dimensional simply connected compact C^∞ submanifold of $\mathbb{R}^3$ whose boundary is C^∞-regular possibly except finitely many points, and $M_i \cap M_j = \partial M_i \cap \partial M_j$, $(i, j \in \{1, \cdots, \mu\}$, $i \neq j)$. Such M is called a piecewise-smooth surface, each singular point of $\cup_{i=1}^{\mu} \partial M_i$ is called a vertex (more precisely, an extrinsic vertex) of M, and each maximal C^∞-regular arc in $\cup_{i=1}^{\mu} \partial M_i$ is called an edge (more precisely, an extrinsic edge) of M.

Next we recall geodesic polar coordinates of a smooth surface S in $\mathbb{R}^3$ (cf. [7]). Let P_0 be a point in S. Take two unit tangent vectors $\boldsymbol{e}_1, \boldsymbol{e}_2$ of S at P_0 which are orthogonal to each other. Then a sufficiently small neighborhood U of P_0 in S has a so-called geodesic polar coordinate system (r, θ) that satisfies the following properties. There exists a C^∞ diffeomorphism $p = p(r, \theta)$ from an open disc $D := \{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2 \mid 0 \le \theta < 2\pi, 0 \le r < r_0\}$ where $r_0 > 0$ onto U, such that p satisfies the following properties.

(i) $p(0, \theta) = P_0$.

(ii) For each fixed $\rho \in (0, r_0)$ and $\theta \in [0, 2\pi)$, each arc $\{p(r, \theta) \mid 0 \le r \le \rho\}$ is a geodesic in S with arc-length parameter r.

(iii) The geodesic $\{p(r, \theta) \mid 0 \le r \le \rho\}$ mentioned in (ii) has angle θ with $\boldsymbol{e}_1$ at P_0.

Set

$$h(r, \theta) = |p_\theta(r, \theta)|, \tag{5}$$

where $|p_\theta(r, \theta)|$ denotes the norm of $p_\theta(r, \theta)$. It is known that the following properties hold.

Lemma 2.1. *(i)* $|p_r| = 1$, *(ii)* $p_r \cdot p_\theta = 0$,
(iii) $\lim_{r \to +0} \frac{h}{r} = 1$, *(iv)* $\lim_{r \to +0} h_r = 1$,
(v) $\lim_{r \to +0} h = 0$, *(vi)* $K = -\frac{h_{rr}}{h}$.
(vii) The first fundamental form of S is given by $ds^2 = dr^2 + h^2 d\theta^2$ in U.

Using (5), (i) and (ii) in Lemma 2.1, the following equalities are easily shown.

Lemma 2.2. *(i)* $p_{\theta\theta} \cdot p_r = -hh_r$, *(ii)* $p_{\theta\theta} \cdot p_\theta = hh_\theta$,
(iii) $p_{rr} \cdot p_\theta = 0$, *(iv)* $p_{rr} \cdot p_r = 0$,
(v) $p_{\theta r} \cdot p_\theta = hh_r$, *(vi)* $p_{\theta r} \cdot p_r = 0$.

3 Outline of the proof of Theorem 1.2

In this section, first we compute the geodesic curvatures of the boundary curves of an angular domain M_i, and by using them we compute the lengths $L_i(p; r)$ of geodesic circles $C_i(p; r)$ in M_i. And then, we will give an idea to obtain Theorem 1.2.

Let p be a vertex of a smooth piece M_i. We consider geodesic polar coordinates (r, θ) around p. We use the same notation as in §2. We assume that the inner angle of ∂M_i at p is σ_i. Also we assume that the geodesic circle with center at p and radius $r \in (0, r_0)$ in M_i for sufficiently small $r_0 > 0$ is $C_i(p; r) = \{(r, \theta) \mid \theta_i^1(r) \le \theta \le \theta_i^2(r)\}$. Then the length $L_i(p; r)$ of $C_i(p; r)$ is represented as

$$L_i(p; r) = \int_{\theta_i^1(r)}^{\theta_i^2(r)} h(r, \theta)\, d\theta. \tag{6}$$

Let $E_i^1 = \{(r, \theta_i^1(r)) \mid 0 \le r \le r_0\}$, $E_i^2 = \{(r, \theta_i^2(r)) \mid 0 \le r \le r_0\}$ be parts of the edges of M_i meeting at p. Denote the geodesic curvature of E_i^j at point $(r, \theta_i^j(r))$ for inward-pointing normal by $k_{gi}^j(r)$, $(i = 1, 2)$. By computation using Lemmas 2.1 and 2.2, we obtain the following formula ([4]).

Proposition 3.1. *For $i = 1, 2$, the geodesic curvature $k_{gi}^j(r)$ of E_i^j at point $(r, \theta_i^j(r))$ for inward-pointing normal is represented as follows.*

$$
\begin{aligned}
k_{gi}^j(r) =& (-1)^j (1 + h^2(r, \theta_i^j(r))(\dot{\theta}_i^j(r))^2)^{-3/2} \\
& \times \Big[2h_r(r, \theta_i^j(r))\dot{\theta}_i^j(r) + h_\theta(r, \theta_i^j(r))(\dot{\theta}_i^j(r))^2 \\
& + h(r, \theta_i^j(r))\ddot{\theta}_i^j(r) + h_r(r, \theta_i^j(r))h^2(r, \theta_i^j(r))(\dot{\theta}_i^j(r))^3 \Big],
\end{aligned} \tag{7}
$$

where " $\cdot$ " means the derivative with respect to r.

Using the inner angle and the intrinsic curvatures of ∂M_i and M_i, we can represent the length of the geodesic circle $C_i(p; r)$ in M_i as follows.

Proposition 3.2. *The length $L_i(p; r)$ of $C_i(p; r)$ for small $r > 0$ is represented as*

$$L_i(p; r) = (a_i)_1 r + (a_i)_2 r^2 + (a_i)_3 r^3 + O(r^4),$$

where

$$(a_i)_1 = \sigma_i, \tag{8}$$

$$(a_i)_2 = -\frac{1}{2}\Big(k_{gi}^1(0) + k_{gi}^2(0)\Big), \tag{9}$$

$$(a_i)_3 = -\frac{1}{6}\Big(K_i(p)\sigma_i + (k_{gi}^1)'(0) + (k_{gi}^2)'(0)\Big). \tag{10}$$

Proposition 3.2 is obtained by computation using Lemmas 2.1 and 2.2, (6), and Proposition 3.1 ([4], [5]).

Now, we discuss the proof of Theorem 1.2. Let $p = M_1 \cap \cdots \cap M_N$ be an extrinsic vertex of M. If the geodesic circle $C(p; r)$ in M satisfies

$$C(p; r) = \cup_{i=1}^N C_i(p; r), \tag{11}$$

then, from Proposition 3.2, we obtain Theorem 1.2 (i). However, in general (11) does not hold. By estimating the error term, we obtain Theorem 1.2 (i) ([4], [5]). (ii) of Theorem 1.2 is obtained from (i) as a specific case.

References

[1] M. Berger, A Panoramic View of Riemannian Geometry, Springer-Verlag, Berlin, 2003.

[2] J. Bertrand, Demonstration d'un théorème de M. Gauss, Journal de mathématiques pures et appliquées 1^{re} série, **tome 13** (1848), 80–82.

[3] A. Gray, The volume of a small geodesic ball of a Riemannian manifold, Michigan Math. J. **20** (1973), 329–344.

[4] M. Koiso, Intrinsic singular points and curvatures of piecewise-smooth surfaces, preprint, (2024), 11 pages.

[5] M. Koiso, The length of a small geodesic circle of a piecewise-smooth surface, in preparation.

[6] K. Saji, M. Umehara, and K. Yamada, The geometry of fronts, Ann. of Math., **169** (2009), 491–529.

[7] J. J. Stoker, Differential Geometry, page 177, Wiley-Interscience, 1989.

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