Pure cactus groups and configuration spaces of points on the circle

Kazuhiro Ichihara and Takatoshi Hama
College of Humanities and Sciences, Nihon University

1 Introduction

The aim of this article is to provide a summary of the results presented in [9, 10] and in the second author's thesis [8], which concern pure cactus groups and configuration spaces of points on the circle.

It is known that the braid group acts naturally on multiple tensor products in a braided monoidal category. As an analogue of this action, Henriques and Kamnitzer introduced the so-called *cactus group* in [11], which acts on coboundary categories associated with braided monoidal categories.

We note that their work was motivated by the study of crystal bases, introduced by Kashiwara in [12]. Although the term "cactus group" was first coined in [11], the group itself had already been studied in earlier works, including [4] and [5].

More precisely, for any integer $n \geq 2$, the *cactus group of degree* n, denoted by J_n, is defined by a presentation with generators $s_{p,q}$ for $1 \leq p < q \leq n$, subject to the following relations:

- $s_{p,q}^2 = e$ for all $1 \leq p < q \leq n$,
- $s_{p,q}s_{m,r} = s_{m,r}s_{p,q}$ if $[p,q] \cap [m,r] = \emptyset$,
- $s_{p,q}s_{m,r} = s_{p+q-r,p+q-m}s_{p,q}$ if $[m,r] \subset [p,q]$,

where $1 \leq m < r \leq n$, e denotes the identity element, and $[p,q]$ is the set $p, p+1, \ldots, q$ for integers $p < q$.

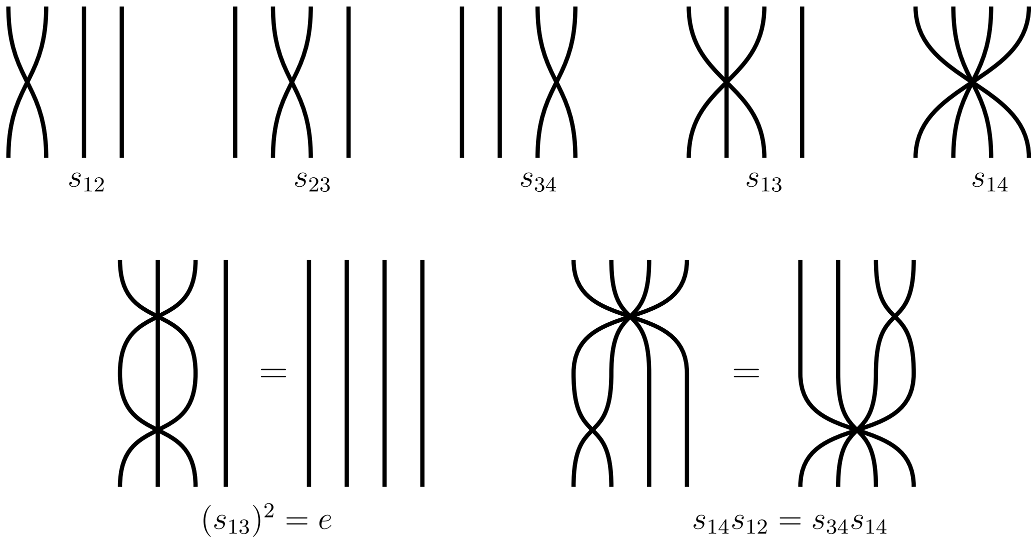

Figure 1: Diagrams for some elements of J_4

Similar to the braid group, elements of the cactus group can be represented by planar diagrams composed of vertical strands. Examples of such diagrams for J_4 are shown in Figure 1.

Owing to this diagrammatic representation, the cactus group J_n admits a natural projection $\pi : J_n \to S_n$ onto the symmetric group S_n of degree n. The kernel of this projection is called the *pure cactus group* of degree n, denoted by PJ_n. For further details, see [11, Subsection 3.1] or [7, Section 1].

Also, in [11, Theorem 9], it is shown that the pure cactus group PJ_n is isomorphic to the fundamental group of the Deligne–Mumford compactification $\overline{M_{0,n+1}(\mathbb{R})}$ of the moduli space of real genus-zero curves with $n+1$ marked points. This space is closely related to the configuration space of $n+1$ points on the circle.

In this article, we report our study the relationship between the pure cactus group and the compactification of the configuration space of points on the circle, focusing on the cases where the degree is three or four.

Throughout this paper, we denote the generators of J_n using the notation s_{23} (without a comma) instead of $s_{2,3}$, for brevity.

2 Configuration spaces of points on S^1

As mentioned above, by [11, Theorem 9], the pure cactus group PJ_n is isomorphic to the fundamental group of the Deligne–Mumford compactification $\overline{M_{0,n+1}(\mathbb{R})}$ of the moduli space of real genus-zero curves with $n+1$ marked points.

For $k \geq 3$, it is natural to regard $M_{0,k}(\mathbb{R})$ as the configuration space of k distinct points on the circle S^1. In this paper, we denote this space by $X(k)$, which can be explicitly described as:

$$X(k) = \mathrm{PGL}(2) \backslash \left\{ (\mathbb{P}^1)^k - \Delta \right\},$$

where $\Delta = (x_1, \ldots, x_k) \mid x_i = x_j$ for some $i \neq j$, and the projective general linear group PGL(2) acts diagonally and freely. That is, there exists a homeomorphism between $M_{0,k}(\mathbb{R})$ and $X(k)$.

On the other hand, a purely combinatorial compactification $\overline{X(k)}$ was introduced by M. Yoshida in [17]. See also [13] and [15] for related work.

At present, it is not known whether $\overline{X(k)}$ is homeomorphic to $\overline{M_{0,k}(\mathbb{R})}$ in general. However, for $k = 3, 4, 5$, the following are known:

- Both $\overline{X(3)}$ and $\overline{M_{0,3}(\mathbb{R})}$ consist of a single point.
- Both $\overline{X(4)}$ and $\overline{M_{0,4}(\mathbb{R})}$ are homeomorphic to S^1.
- Both $\overline{X(5)}$ and $\overline{M_{0,5}(\mathbb{R})}$ are homeomorphic to the closed non-orientable surface with Euler characteristic -3, namely, the connected sum of five projective planes.

See [17, 1, 6], for example.

This implies that, for $n \leq 4$, the pure cactus group PJ_n is isomorphic to the fundamental group of $\overline{X(n+1)}$, using $\overline{M_{0,n+1}(\mathbb{R})}$ as an intermediate space. Thus, the following question arises naturally:

Question. Can one show that PJ_n is isomorphic to $\pi_1(\overline{X(n+1)})$ without using $\overline{M_{0,n+1}(\mathbb{R})}$?

The following theorem provides an affirmative answer in the case $n = 3$.

Theorem 1 ([9]). *Let $\widetilde{\overline{X(4)}}$ be the universal cover of the space $\overline{X(4)}$, endowed with the action $\widetilde{\Gamma}$ of the fundamental group $\pi_1(\overline{X(4)})$ of $\overline{X(4)}$ as deck* *transformations. Let ${C_3}^{\{2\}}$ be the Cayley complex of the subgroup $J_3^{\{2\}}$ of the cactus group J_3. Then there exists an action Γ of PJ_3 on ${C_3}^{\{2\}}$ and a bijective equivariant map φ from $\widetilde{X(4)}$ to ${C_3}^{\{2\}}$ with respect to the actions of $\widetilde{\Gamma}$ and Γ, i.e., for any $g \in PJ_3$, there exists $\tilde{g} \in \pi_1(\overline{X(4)})$ such that the following diagram commutes;*

$$\begin{array}{ccc} \widetilde{X(4)} & \xrightarrow{\widetilde{\Gamma}_{\tilde{g}}} & \widetilde{X(4)} \\ \varphi \big\downarrow & & \big\downarrow \varphi \\ {C_3}^{\{2\}} & \xrightarrow{\Gamma_g} & {C_3}^{\{2\}} \end{array}$$

The theorem above implies the following immediately.

Corollary 1. *The pure cactus group PJ_3 of degree three is isomorphic to the fundamental group $\pi_1(\overline{X(4)})$ of the compactification $\overline{X(4)}$ of the configuration space of four points on the circle.*

Here, we recall the definition of the Cayley complex. Let G be a group and S a generating set of G. The *Cayley graph of G with respect to the generating set S* is the graph whose vertex set is G, and whose edge set is $\{g, gs \mid g \in G, s \in S\}$. The cell complex obtained by attaching 2-cells to each cycle formed by the relations of G is called the *Cayley complex* of G with respect to S.

We now introduce the subgroup of J_3 used in the theorem above. In general, for each integer $n \geq 2$ and subset $S \subset [2, n]$, let J_n^S be the subgroup of J_n generated by the elements $s_{p,q}$ for $1 \leq p < q \leq n$ such that $q-p+1 \in S$, and defined by the following relations:

- $s_{p,q}^2 = e$ for every $1 \leq p < q \leq n$ satisfying $q - p + 1 \in S$,
- $s_{p,q}s_{m,r} = s_{m,r}s_{p,q}$ for every $1 \leq p < q \leq n$ and $1 \leq m < r \leq n$ satisfying $[p, q] \cap [m, r] = \emptyset$ and $q - p + 1 \in S$,
- $s_{p,q}s_{m,r} = s_{p+q-r,p+q-m}s_{p,q}$ for every $1 \leq p < q \leq n$ and $1 \leq m < r \leq n$ satisfying $[m, r] \subset [p, q]$ and $q - p + 1 \in S$.

For more details, see [7, Section 5]. Under this setting, the subgroup $J_3^{\{2\}}$ of J_3 is defined for $S = \{2\} \subset \{2, 3\}$. It is the subgroup of J_3 generated by

s_{12} and s_{23} (excluding s_{13}), with the presentation:

$$\left\langle s_{12}, s_{23} \,\middle|\, s_{12}^2 = s_{23}^2 = e \right\rangle .$$

It follows that $J_3^{\{2\}}$ is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$.

For the proof of the theorem above, please refer to [9]. The key is the action of PJ_3 on the Cayley complex ${C_3}^{\{2\}}$, which is homeomorphic to the real line $\mathbb{R}$.

Actually, this action on ${C_n}^{[2,n-1]}$ for PJ_n is essentially obtained in [7].

Proposition 1. *Let* $\left({C_n}^{[2,n-1]}\right)^{(0)}$ *denote the 0-skeleton of the Cayley complex* ${C_n}^{[2,n-1]}$*, which is identified with* $J_n^{[2,n-1]}$*. The map*

$$\Gamma_0 : PJ_n \times \left({C_n}^{[2,n-1]}\right)^{(0)} \longrightarrow \left({C_n}^{[2,n-1]}\right)^{(0)}$$

defined by

$$(g,h) \mapsto (\Gamma_0)_g(h) := \begin{cases} gh & \text{if } gh \in {J_n}^{[2,n-1]}, \\ ghs_{1,n} & \text{if } gh \notin {J_n}^{[2,n-1]} \end{cases}$$

induces a group action Γ *of* PJ_n *on* ${C_n}^{[2,n-1]}$.

3 A presentation of PJ_4

Hereafter, we focus on the cactus group J_4 of degree four. Recall that J_4 has the following presentation:

$$\left\langle \begin{array}{l} s_{12}, s_{23}, s_{34}, \\ s_{13}, s_{24}, s_{14} \end{array} \,\middle|\, \begin{array}{l} s_{12}^2 = s_{23}^2 = s_{34}^2 = s_{13}^2 = s_{24}^2 = s_{14}^2 = e, \\ s_{12}s_{34} = s_{34}s_{12},\ s_{12}s_{13} = s_{13}s_{23},\ s_{23}s_{24} = s_{24}s_{34}, \\ s_{12}s_{14} = s_{14}s_{34},\ s_{23}s_{14} = s_{14}s_{23},\ s_{13}s_{14} = s_{14}s_{24} \end{array} \right\rangle .$$

As noted in the previous section, both $\overline{X(5)}$ and $\overline{M_{0,5}(\mathbb{R})}$ are homeomorphic to the closed non-orientable surface of Euler characteristic -3, which is the connected sum of five projective planes. This implies that PJ_4 admits the following presentation:

$$\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \mid \alpha_1^2\alpha_2^2\alpha_3^2\alpha_4^2\alpha_5^2 = e \rangle \tag{1}$$

On the other hand, another explicit presentation of PJ_4 was obtained purely algebraically in [3, Appendix A], using the Reidemeister–Schreier method. In fact, as [3, Theorem 5.5] states, it was shown that PJ_4 has the following presentation:

$$\langle \alpha, \beta, \gamma, \delta, \epsilon \mid \alpha\gamma\epsilon\beta\epsilon\alpha^{-1}\delta^{-1}\beta\gamma\delta^{-1} = e \rangle \tag{2}$$

This presentation is a simple one-relator presentation with five generators, but it does not appear to be directly related to the presentation described above in the presentation (1). Indeed, as noted in [3, Remark 5.6], it seems difficult to express the generators α_k in terms of the standard generators $s_{i,j}$ of the full cactus group J_4.

In [10], we obtain another presentation of PJ_4.

Theorem 2. *The pure cactus group PJ_4 admits the following presentation:*

$$\left\langle g_1, \dots, g_{10} \;\middle|\; \begin{array}{l} g_1 g_{10}^{-1} g_2^{-1} = g_9 g_5^{-1} g_4 = g_5 g_1 g_6^{-1} \\ = g_8 g_{10} g_7^{-1} = g_8 g_3^{-1} g_4 = g_2 g_9 g_7^{-1} g_6 g_3^{-1} = e \end{array} \right\rangle$$

This presentation can be transformed into the following:

$$\langle g_2, g_4, g_8, g_9, g_{10} \mid g_2 g_9 g_{10}^{-1} g_8^{-1} g_4 g_9 g_2 g_{10} g_8^{-1} g_4^{-1} = e \rangle$$

We remark that the generators g_i given above are explicitly described in terms of the standard generating set $\{s_{ij}\}$ of the cactus group J_4 and are of minimal word length with respect to $\{s_{ij}\}$. That is, each g_i has word length 4 or 5, and there are no elements in PJ_4 whose word length is less than or equal to 3.

We can confirm that the presentation in Theorem 2 is equivalent to the presentation (1), as well as to the presentation (2) given in [3]. See [10] for details.

The key to our proof of Theorem 2 is to consider the action of PJ_4 on the Cayley complex $C_4^{[2,3]}$ of the subgroup $J_4^{[2,3]}$, as given in Proposition 1. In fact, the subgroup $J_4^{[2,3]}$ is generated by $s_{12}, s_{23}, s_{34}, s_{13}, s_{24}$ (excluding s_{14}), and has the following presentation:

$$\left\langle s_{12}, s_{23}, s_{34}, s_{13}, s_{24} \;\middle|\; \begin{array}{l} s_{12}^2 = s_{23}^2 = s_{34}^2 = s_{13}^2 = s_{24}^2 = e, \\ s_{12}s_{34} = s_{34}s_{12},\; s_{12}s_{13} = s_{13}s_{23},\; s_{23}s_{24} = s_{24}s_{34} \end{array} \right\rangle.$$

It follows that the Cayley complex ${C_4}^{[2,3]}$ is isometric to the hyperbolic plane $\mathbb{H}^2$ up to scaling. See Figure 2 for a local picture of the complex, and compare it with Figure 3.

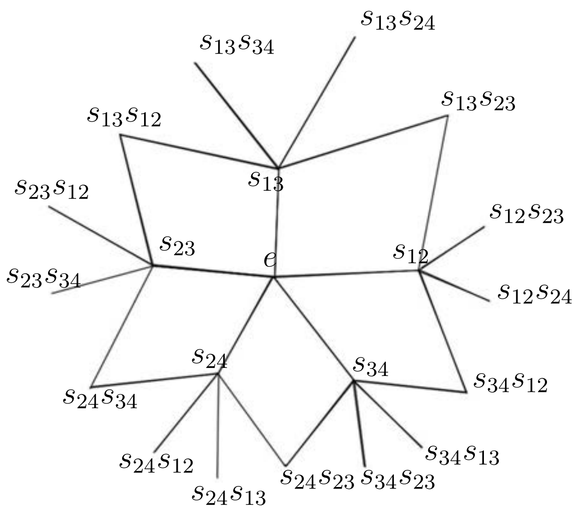

Figure 2: Neighborhood of e in the Cayley complex ${C_4}^{[2,3]}$

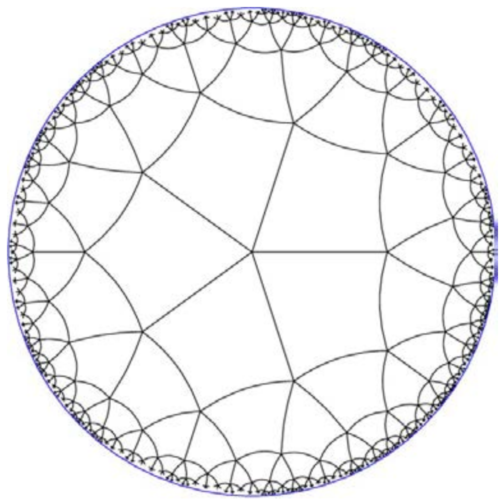

Figure 3: The $\{4,5\}$-tesselation of the hyperbolic plane. ([2, Figure 7])

Here we give an outline of the proof of Theorem 2.

First, through brute-force enumeration, we listed all the elements of PJ_4 whose translation lengths are at most 4 with respect to the action on ${J_4}^{[2,3]}$. As a result, we obtained the ten elements $g_1, \ldots, g_{10}$ and their inverses in PJ_4 below.

$$
\begin{aligned}
g_1 &= s_{13}s_{24}s_{12}s_{34}s_{14} \ , & g_2 &= s_{13}s_{24}s_{13}s_{24} \ , \\
g_3 &= s_{13}s_{34}s_{23}s_{12}s_{14} \ , & g_4 &= s_{13}s_{34}s_{13}s_{23}s_{14} \ , \\
g_5 &= s_{23}s_{12}s_{23}s_{13} \ , & g_6 &= s_{23}s_{12}s_{24}s_{12}s_{14} \ , \\
g_7 &= s_{23}s_{34}s_{13}s_{34}s_{14} \ , & g_8 &= s_{24}s_{34}s_{23}s_{34} \ , \\
g_9 &= s_{24}s_{12}s_{23}s_{34}s_{14} \ , & g_{10} &= s_{24}s_{23}s_{13}s_{34}s_{14}
\end{aligned}
$$

In fact, if $d(e, g \cdot e) \leq 4$ holds for some $g \in PJ_4$, then $g = g_i^{\pm 1}$ for some $1 \leq i \leq 10$, and in fact, $d(e, g_i \cdot e) = 4$.

Using these elements, we can construct the Dirichlet polygon D in $\mathbb{H}^2 \cong {C_4}^{[2,3]}$ centered at e for the action of PJ_4. This polygon is illustrated in Figure 4.

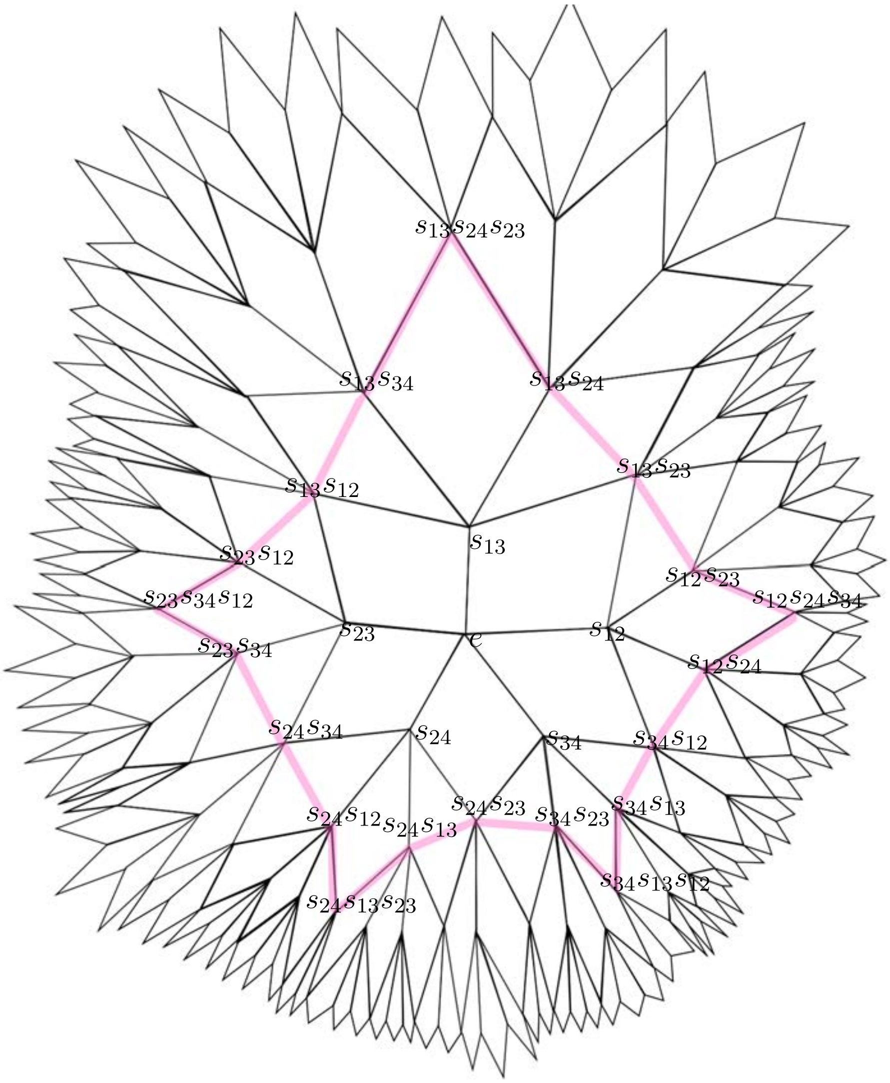

Figure 4:

The polygon D has 20 sides. Note that ten of these sides are not edges of the Cayley complex ${C_4}^{[2,3]}$; rather, they are diagonals of certain 2-cells (quadrangles) in the complex. Also, note that the interior angles around the vertices of D are $\frac{2\pi}{5}$, $\frac{3\pi}{5}$, or $\frac{4\pi}{5}$.

The elements $g_1, \dots, g_{10}$ and their inverses define the side identifications of D in $\mathbb{H}^2 \cong {C_4}^{[2,3]}$. Moreover, we can confirm that a complete hyperbolic surface is constructed after the identifications. Then, by virtue of the famous Poincaré Polygon Theorem (cf. [16, 14]), it follows that $g_1, \dots, g_{10}$ generate PJ_4, and the compositions of generators $g_3 g_6^{-1} g_7 g_9^{-1} g_2^{-1}$,

$g_3g_8^{-1}g_4^{-1}$, $g_5g_9^{-1}g_4^{-1}$, $g_5g_1g_6^{-1}$, $g_8g_{10}g_7^{-1}$, $g_{10}g_1^{-1}g_2$ form a complete set of relations. Thus, we obtain the following presentation of PJ_4 as desired: See [10] for details.

Finally, we give an explanation of how to find the correspondence between the following two presentations:

$$\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \mid \alpha_1^2\alpha_2^2\alpha_3^2\alpha_4^2\alpha_5^2 = e \rangle,$$

$$\left\langle g_1, \ldots, g_{10} \;\middle|\; \begin{array}{l} g_3g_6^{-1}g_7g_9^{-1}g_2^{-1} = g_3g_8^{-1}g_4^{-1} = g_5g_9^{-1}g_4^{-1} \\ = g_5g_1g_6^{-1} = g_8g_{10}g_7^{-1} = g_{10}g_1^{-1}g_2 = e \end{array} \right\rangle$$

Viewing the fundamental polygon D, we see that the painted regions in Figure 5 give five Möbius bands embedded in $C_4^{[2,3]}/PJ_4$. After removing them, we can rebuild the remaining parts to obtain a sphere with five holes. See Figure 6. Then we can confirm that $C_4^{[2,3]}/PJ_4$ is homeomorphic to the connected sum $\sharp_5\mathbb{RP}^2$ of five real projective planes. Setting the closed curve α_2 in $\sharp_5\mathbb{RP}^2$ shown in Figure 6, we obtain a correspondence $\alpha_2 \longmapsto g_8^{-1}g_4g_9g_2g_{10} \in PJ_4$ by considering a lift of α_2 in $C_4^{[2,3]}$. In the same way, we can obtain the corresponding elements in PJ_4 for α_1, α_3, α_4, α_5.

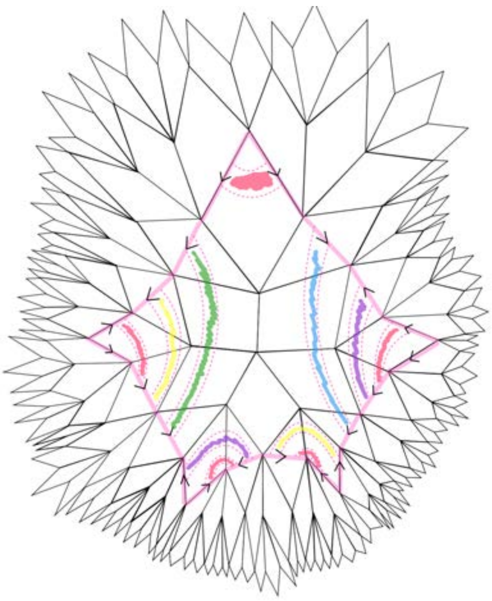

Figure 5: Five Möbius bands

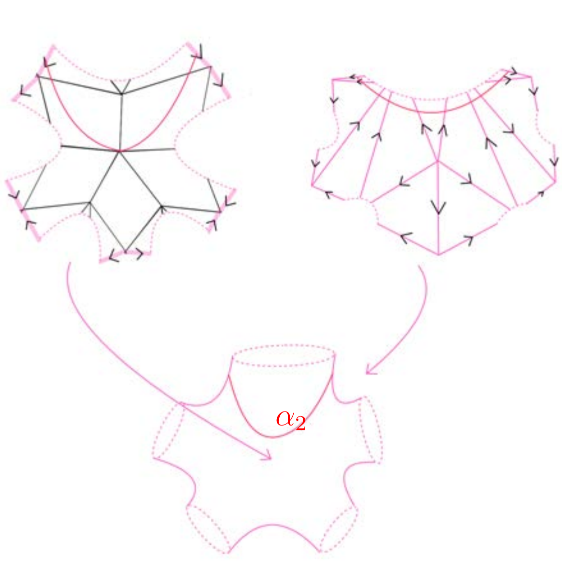

Figure 6: Finding the generator α_2

4 PJ_4 and $\overline{X(5)}$

As a consequence of the previous section, it can be proved that PJ_4 is isomorphic to the fundamental group of the connected sum of five projective planes, which is in turn isomorphic to $\pi_1(\overline{X(5)})$. In this section, we provide an alternative proof by constructing a homeomorphism directly between $C_4^{[2,3]}/PJ_4$ and $\overline{X(5)}$.

Theorem 3. *Let $\overline{X(5)}$ denote the compactification of the configuration space $X(5)$ of five points on the circle. Let ${C_4}^{[2,3]}$ be the Cayley complex of the subgroup $J_4^{[2,3]}$ of the cactus group J_4. Then $\overline{X(5)}$ is homeomorphic to the quotient space of ${C_4}^{[2,3]}$ under the action of the pure cactus group PJ_4 of degree four.*

Corollary 2. *The pure cactus group PJ_4 of degree four is isomorphic to the fundamental group $\pi_1(\overline{X(5)})$, where $\overline{X(5)}$ is the compactification of the configuration space of five points on the circle.* □

We give a brief review of the cell complex structure of $\overline{X(5)}$ studied in [1, 17]. Each 2-cell of the complex $\overline{X(5)}$ corresponds to a connected component of $X(5)$, that is, a configuration of five distinct points, say 0, 1, 2, 3, 4, on S^1. In the following, for example, the 2-cell corresponding to the configuration of points $0, 1, 2, 3, 4$ in clockwise order is coded by $[1, 2, 3, 4]$. Note that $[1, 2, 3, 4]$ and $[4, 3, 2, 1]$ refer to the same 2-cell. Then, there are twelve 2-cells in $\overline{X(5)}$:

$$\begin{array}{cccccc} [1234], & [2134], & [4123], & [1324], & [2341], & [1243], \\ [3124], & [3412], & [3241], & [2314], & [2431], & [3142]. \end{array}$$

Two 2-cells are adjacent in $\overline{X(5)}$ if one is obtained from the other by switching a pair of adjacent points. See Figure 7 for an example. Note that the 2-cells [1234] and [4123] = [3214] are adjacent; this corresponds to switching the points 0 and 4.

Then, the entire cell complex structure and its dual of $\overline{X(5)}$ are visualized in Figure 8. In the figure, the 12 pentagons represent the 2-cells of $\overline{X(5)}$ (i.e., the connected components of $X(5)$).

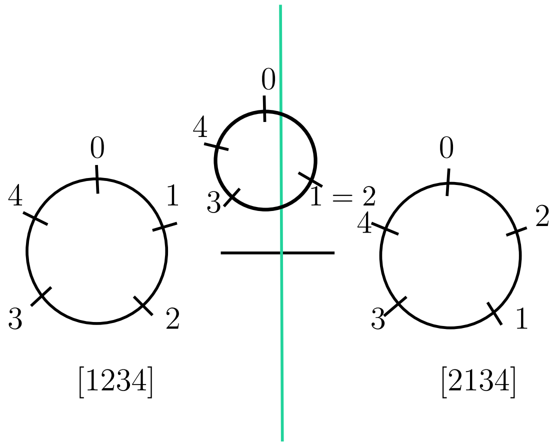

Figure 7: Adjacent 2-cells; [1234] and [2134]

In the following proof, we consider the dual cell complex structure of $\overline{X(5)}$. That is, the vertices (0-cells) are labeled by [1234], [2134], and so on.

Proof of Theorem 3. We consider the cell complex structure on the surface $F := C_4^{[2,3]}/PJ_4$ induced naturally from $C_4^{[2,3]}$. See Figure 4. There are 15 2-cells (quadrangles) and 30 1-cells (edges).

Set the following correspondence φ between $(\overline{X(5)})^0$ and $(C_4^{[2,3]}/PJ_4)^0$.

$$
\begin{array}{rcl}
[1234] & \longmapsto & [e] \\
[2134] & \longmapsto & [s_{12}] \\
[4123] & \longmapsto & [s_{13}] \\
[1324] & \longmapsto & [s_{23}] \\
[2341] & \longmapsto & [s_{24}] \\
[1243] & \longmapsto & [s_{34}] \\
[3124] & \longmapsto & [s_{13}s_{23}] \\
[3412] & \longmapsto & [s_{13}s_{24}] \\
[3241] & \longmapsto & [s_{13}s_{34}] \\
[2314] & \longmapsto & [s_{13}s_{12}] \\
[2431] & \longmapsto & [s_{24}s_{23}] \\
[3142] & \longmapsto & [s_{13}s_{24}s_{23}]
\end{array}
$$

Let us show that this correspondence induces a well-defined, bijective cell map between $\overline{X(5)}$ and $F = C_4^{[2,3]}/PJ_4$.

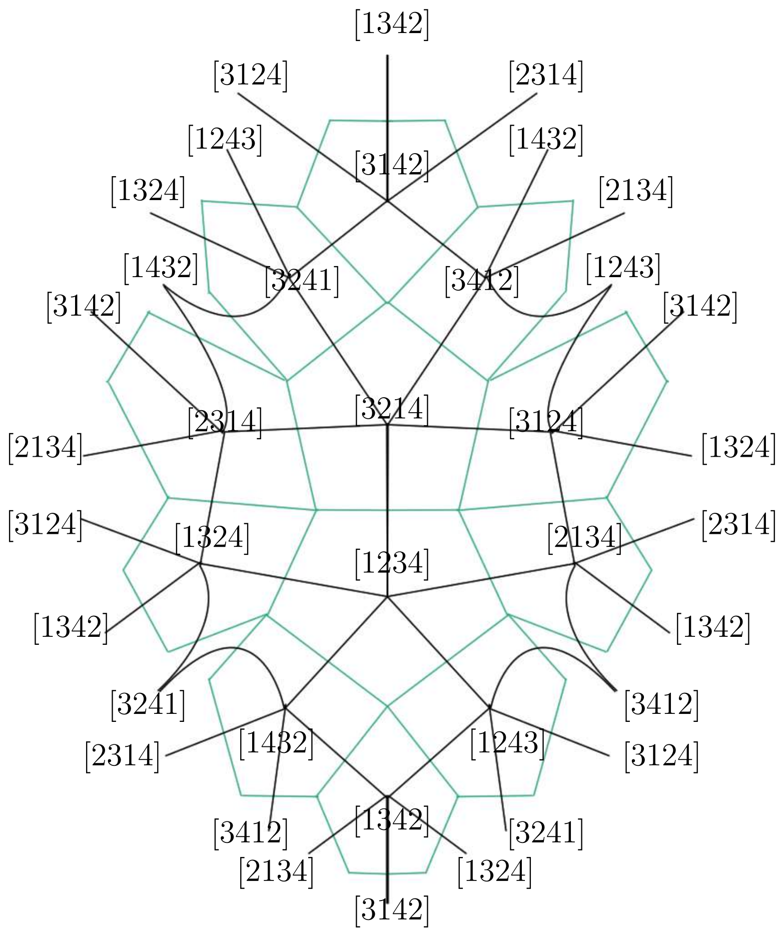

Figure 8: Cell complex structure of $\overline{X(5)}$ and its dual.

It is sufficient to prove that each set of vertices that span a cell in $\overline{X(5)}$ is mapped to a set of vertices that span a cell in F.

We list all of the 2-cells in $\overline{X(5)}$:

$$\begin{aligned}
&\langle[3124],[1324],[2314],[2134]\rangle, \quad \langle[3124],[3142],[1342],[1324]\rangle,\\
&\langle[3214],[3142],[1342],[1324]\rangle, \quad \langle[3412],[1243],[3124],[3214]\rangle,\\
&\langle[3412],[2134],[1234],[1243]\rangle, \quad \langle[3412],[1432],[1342],[2134]\rangle,\\
&\langle[3142],[4132],[1432],[3412]\rangle, \quad \langle[3142],[1342],[2134],[4132]\rangle,\\
&\langle[4231],[3241],[1243],[1342]\rangle, \quad \langle[1324],[1234],[1432],[3241]\rangle,\\
&\langle[1432],[3241],[3214],[2314]\rangle, \quad \langle[1342],[1243],[1234],[1432]\rangle,\\
&\langle[1234],[2134],[3124],[3214]\rangle, \quad \langle[1234],[1324],[2314],[3214]\rangle,\\
&\langle[3214],[3412],[3142],[3241]\rangle.
\end{aligned}$$

In the above list, for example, $\langle[3124],[1324],[2314],[2134]\rangle$ indicates a cycle of the vertices in this order. See Figure 8. For instance, let us consider the case of the 2-cell $\langle[3124],[1324],[2314],[2134]\rangle$ in $\overline{X(5)}$. The vertices [3124], [1324], [2314], and [2134] correspond to $[s12]$, $[s_{13}s_{12}]$, $[s_{23}]$, and $[s_{13}s_{23}]$ in $C_4^{[2,3]}$ by φ, respectively.

By the definition of the Cayley graph, we have:

$$\langle[s_{12}],[s_{13}s_{23}]\rangle = \langle[s_{12}],[s_{12}s_{13}]\rangle,\ \langle[s_{13}s_{12}],[s_{23}]\rangle = \langle[s_{23}s_{13}],[s_{23}]\rangle,$$

which are edges in $C_4^{[2,3]}$. The element $s_{13}s_{12}$ is mapped to $s_{12}s_{23}$ under the identification g_5^{-1} as follows.

$$g_5^{-1}s_{13}s_{12} = (s_{23}s_{12}s_{23}s_{13})^{-1}s_{13}s_{12} = s_{12}s_{23}s_{12}s_{13}s_{13}s_{12} = s_{12}s_{23}$$

Therefore, $\langle[s_{12}],[s_{13}s_{12}]\rangle$ becomes a 1-cell in F, since $\{s_{12}, s_{12}s_{23}\}$ is an edge of the Cayley graph of $J_4^{[2,3]}$ and $[s_{13}s_{12}] = [s_{12}s_{23}]$ holds in F. In the same way, since $g_5(s_{13}s_{23}) = s_{23}s_{12}$, the pair $\langle[s_{23}],[s_{13}s_{23}]\rangle$ also forms a 1-cell in F. It follows that the vertices $[s_{12}]$, $[s_{13}s_{12}]$, $[s_{23}]$, and $[s_{13}s_{23}]$ form a cycle in F. By the definition of the Cayley complex, these vertices span a 2-cell: $\langle[s_{12}],[s_{13}s_{12}],[s_{23}],[s_{13}s_{23}]\rangle$ in F.

The following calculations show that the other 2-cells are similarly mapped to 2-cells in F.

$\langle[3124],[3142],[1342],[1324]\rangle$:

$$\begin{aligned}
&\varphi([3124]) = [s_{13}s_{23}] = [g_5^{-1}(s_{23}s_{12})] = [g_1(s_{34}s_{13})]\\
&\varphi([3142]) = [s_{13}s_{24}s_{23}] = [g_3(s_{34}s_{13}s_{12})] = [g_3(g_6^{-1}(s_{23}s_{34}s_{12}))]\\
&\varphi([1342]) = [s_{24}s_{23}] = [g_8^{-1}(s_{23}s_{34})]\\
&\varphi([1324]) = [s_{23}]
\end{aligned}$$

$\langle[3214],[3142],[1342],[1324]\rangle$:

$$\begin{aligned}
&\varphi([3124]) = [s_{13}s_{23}] = [g_1(s_{34}s_{13})]\\
&\varphi([1243]) = [s_{34}]\\
&\varphi([3214]) = [s_{13}s_{34}] = [g_3(s_{34}s_{23})]\\
&\varphi([3142]) = [g_3(s_{34}s_{13}s_{12})]
\end{aligned}$$

$\langle[3412], [1243], [3124], [3214]\rangle$:

$$\begin{aligned}
\varphi([3412]) &= [s_{13}s_{24}] = [g_1(s_{34}s_{12})] \\
\varphi([1243]) &= [s_{34}] \\
\varphi([3124]) &= [s_{13}s_{23}] = [g_1(s_{34}s_{13})] \\
\varphi([3214]) &= [s_{13})]
\end{aligned}$$

$\langle[3412], [2134], [1234], [1243]\rangle$:

$$\begin{aligned}
\varphi([3412]) &= [s_{13}s_{24}] = [g_1(s_{12}s_{34})] \\
\varphi([2134]) &= [s_{12}] \\
\varphi([1234]) &= [e] \\
\varphi([1243]) &= [s_{34}]
\end{aligned}$$

$\langle[3412], [1432], [1342], [2134]\rangle$:

$$\begin{aligned}
\varphi([3412]) &= [s_{13}s_{24}] = [g_2(s_{24}s_{13})] = [g_1(s_{12}s_{34})] \\
\varphi([1432]) &= [s_{24}] \\
\varphi([1342]) &= [s_{24}s_{23}] = [g_{10}(s_{12}s_{24})] \\
\varphi([2134]) &= [s_{12}]
\end{aligned}$$

$\langle[3142], [4132], [1432], [3412]\rangle$:

$$\begin{aligned}
\varphi([3142]) &= [s_{13}s_{24}s_{23}] = [g_2(s_{24}s_{13}s_{23})] \\
\varphi([4132]) &= [s_{13}s_{12}] = [g_4^{-1}(s_{24}s_{12})] \\
\varphi([1432]) &= [s_{24}] = [g_{10}(s_{12}s_{24})] \\
\varphi([3412]) &= [s_{13}s_{24}] = [g_2(s_{24}s_{13})]
\end{aligned}$$

$\langle[3142], [1342], [2134], [4132]\rangle$:

$$\begin{aligned}
\varphi([3142]) &= [s_{13}s_{24}s_{23}] = [g_2(s_{24}s_{13}s_{23})] = [g_2(g_9(s_{12}s_{24}s_{34}))] \\
\varphi([1342]) &= [s_{24}s_{23}] = [g_{10}(s_{12}s_{24})] \\
\varphi([2134]) &= [s_{12}] \\
\varphi([4132]) &= [s_{13}s_{12}] = [g_5(s_{12}s_{23})] = [g_4(s_{24}s_{12})]
\end{aligned}$$

$\langle[4231], [3241], [1243], [1342]\rangle$:

$$\begin{aligned}
\varphi([4231]) &= [s_{23}] \\
\varphi([3241]) &= [s_{13}s_{34}] = [g_4(s_{23}s_{24})] = [g_3(s_{34}s_{23})] \\
\varphi([1243]) &= [s_{34}] \\
\varphi([1342]) &= [s_{24}s_{23}] = [g_8^{-1}(s_{23}s_{34})]
\end{aligned}$$

$\langle[1324],[1234],[1432],[3241]\rangle$:

$$\begin{aligned}\varphi([1324]) &= [s_{23}]\\ \varphi([1234]) &= [e]\\ \varphi([1432]) &= [s_{34}]\\ \varphi([3241]) &= [s_{13}s_{34}] = [g_3(s_{34}s_{23})] = [g_4(s_{24}s_{34})]\end{aligned}$$

$\langle[1432],[3241],[3214],[2314]\rangle$:

$$\begin{aligned}\varphi([1432]) &= [s_{24}]\\ \varphi([3241]) &= [s_{13}s_{34}] = [g_4(s_{24}s_{34})]\\ \varphi([3214]) &= [s_{13}]\\ \varphi([2314]) &= [s_{13}s_{12}] = [g_4(s_{24}s_{12})]\end{aligned}$$

$\langle[1342],[1243],[1234],[1432]\rangle$:

$$\begin{aligned}\varphi([1342]) &= [s_{24}s_{23}]\\ \varphi([1243]) &= [s_{34}]\\ \varphi([1234]) &= [e]\\ \varphi([1432]) &= [s_{24}]\end{aligned}$$

$\langle[1234],[2134],[3124],[3214]\rangle$:

$$\begin{aligned}\varphi([1234]) &= [e]\\ \varphi([2134]) &= [s_{12}]\\ \varphi([3124]) &= [s_{13}s_{23}]\\ \varphi([3214]) &= [s_{13}]\end{aligned}$$

$\langle[1234],[1324],[2314],[3214]\rangle$:

$$\begin{aligned}\varphi([1234]) &= [e]\\ \varphi([1324]) &= [s_{23}]\\ \varphi([2314]) &= [s_{13}s_{12}]\\ \varphi([3214]) &= [s_{13}]\end{aligned}$$

$\langle[3214],[3412],[3142],[3241]\rangle$:

$$\begin{aligned}\varphi([3214]) &= [s_{13}]\\ \varphi([3412]) &= [s_{13}s_{24}]\\ \varphi([3142]) &= [s_{13}s_{34}]\\ \varphi([3241]) &= [s_{13}s_{24}s_{23}]\end{aligned}$$

Thus, each 2-cell in $\overline{X(5)}$ is mapped to a 2-cell in F under φ, and hence φ defines a bijective, cellular map. Consequently, φ is a homeomorphism. $\square$

Acknowledgments

The authors would like to thank Takuya Sakasai for his valuable feedback and Carl-Fredrik Nyberg Brodda for inspiring future research directions.

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College of Humanities and Sciences
Nihon University
3-25-40 Sakurajosui, Setagaya-ku, Tokyo 156-8550
JAPAN
E-mail address: `ichihara.kazuhiro@nihon-u.ac.jp`

Graduate School of Integrated Basic Sciences,
Nihon University
3-25-40 Sakurajosui, Setagaya-ku, Tokyo 156-8550
JAPAN

日本大学文理学部数学科　市原一裕
日本大学大学院総合基礎科学研究科　浜 天星