On an R-equivalence relation on the set of quandle colorings

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1 Introduction

A quandle, introduced by Joyce [2], is an algebraic system whose axioms have close relationships with Reidemeister moves for oriented knot diagrams. For each quandle X, we may consider X-colorings of an oriented knot diagram. It is well-known that a Reidemeister move naturally relates the X-colorings of the original diagram to the ones of the deformed diagram, one-to-one onto.

We may deform an oriented knot diagram to itself by a finite sequence of Reidemeister moves. On the other hand, the X-colorings of the diagram related to each other by the sequence are not always the same. Then we have a natural question: which X-colorings of a diagram are related to each other by a finite sequence of Reidemeister moves? To consider the above question, the author introduced the R-equivalence relation on the set of X-colorings of an oriented knot diagram [5].

In this report, we review the notion of R-equivalence relation and determine the R-equivalence classes of colorings of diagrams of the square knot and the granny knot by the dihedral quandle of order 3, for example. We assume that each knot is embedded in $\mathbb{R}^3$, throughout this report.

2 R-equivalence for quandle colorings

In this section, we review an equivalence relation, named R-equivalence relation, on the set of quandle colorings of an oriented knot diagram introduced in [5]. To do it, we start with reviewing the definitions of a quandle and a quandle coloring briefly. More details can be found in [3], for example.

A *quandle* is a set X equipped with a binary operation $* : X \times X \to X$ satisfying the following three axioms.

Q1. For any $x \in X$, $x * x = x$.

Q2. There exists a binary operation $*^{-1} : X \times X \to X$ satisfying $(x * y) *^{-1} y = (x *^{-1} y) * y = x$ for any $x, y \in X$.

Q3. For any $x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

For example, the cyclic group $\mathbb{Z}/n\mathbb{Z}$ equipped with a binary operation $*$ given by

$$x * y = 2y - x$$

becomes a quandle for each positive integer $n \geq 3$. We call it the *dihedral quandle* of order n and write as R_n.

Let X be a quandle and D an oriented knot diagram. An X-*coloring* of D is a map $\mathscr{C}$ from the set of all arcs of D to X satisfying the condition depicted in Figure 1 at each crossing of D. In the figure, x, y and $x*y$ denote elements of X assigned to correspondent arcs by $\mathscr{C}$ and we call them the *colors* of the arcs. We note that a constant map from the set of all arcs of D to X obviously satisfies the condition for an X-coloring. We thus call it a *trivial* X-*coloring* of D. The R_3-colorings of the diagram TK of the trefoil knot are depicted in Figure 2, for example. We note that the trivial colorings of TK are $\mathscr{C}_0$, $\mathscr{C}_1$ and $\mathscr{C}_2$.

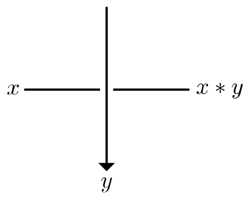

Figure 1: The condition for an X-coloring.

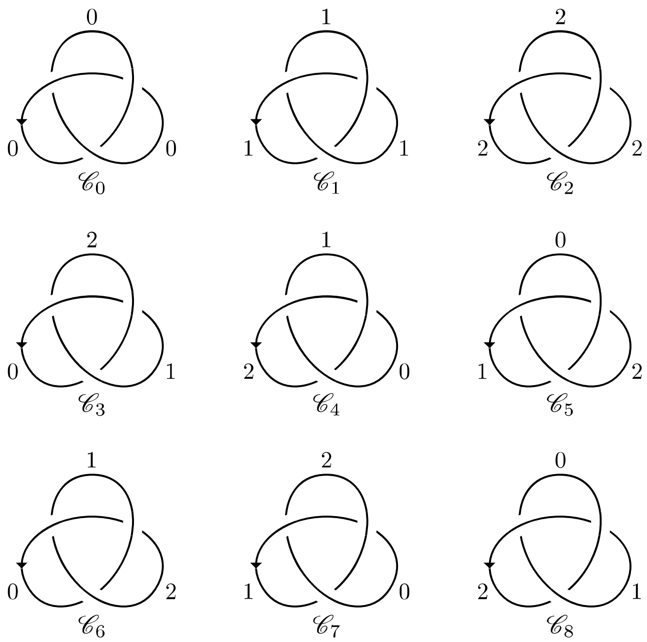

Figure 2: The R_3-colorings of the diagram TK of the trefoil knot.

It is well-known that, for each quandle X, a Reidemeister move yields a one-to-one correspondence between the set of X-colorings of a diagram D and that of a diagram

D' obtained from D by the Reidemeister move. Indeed, for each X-coloring $\mathscr{C}$ of D, we have a unique X-coloring $\mathscr{C}'$ of D' which assigns the same colors with $\mathscr{C}$ for the arcs unrelated to the deformation and consistent colors for the others as depicted in Figure 3. We note that the axioms of a quandle guarantee the existence and uniqueness of $\mathscr{C}'$. A planar isotopy also yields a one-to-one correspondence between the set of X-colorings of a diagram D and that of a diagram D' obtained from D by the planar isotopy, assigning the same colors to the correspondent arcs. In conclusion, if a diagram D' is obtained from a diagram D by a finite sequence of Reidemeister moves and planar isotopies, we have a unique X-coloring $\mathscr{C}'$ of D' corresponding to an X-coloring $\mathscr{C}$ of D. We say in this situation that $\mathscr{C}'$ is obtained from $\mathscr{C}$ by the sequence, or $\mathscr{C}$ is related to $\mathscr{C}'$ by the sequence. Obviously, the number of X-colorings gives us an invariant of oriented knots. We call this number the *X-coloring number* and write as $\mathrm{col}_X(D)$.

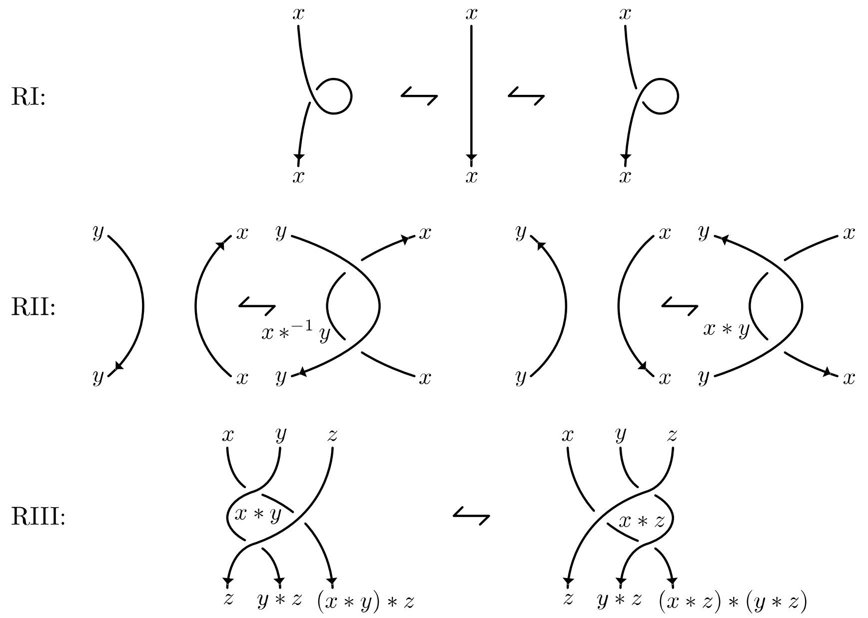

Figure 3: Reidemeister moves relate X-colorings of the diagrams uniquely.

Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be X-colorings of a diagram. We say that $\mathscr{C}_2$ is *R-equivalent* to $\mathscr{C}_1$ if $\mathscr{C}_2$ is obtained from $\mathscr{C}_1$ by a finite sequence of Reidemeister moves and planar isotopies. Obviously, R-equivalence yields an equivalence relation on the set of X-colorings of the diagram. It is easy to see that, for each quandle X and diagram D, a trivial X-coloring of D is not R-equivalent to any other X-colorings of D.

Let us consider which R_3-colorings of TK depicted in Figure 2, are R-equivalent to each other, for example. As stated above, $\mathscr{C}_0$, $\mathscr{C}_1$ and $\mathscr{C}_2$ are respectively not R-equivalent to any other colorings. Since $\mathscr{C}_4$ and $\mathscr{C}_5$ are respectively obtained from $\mathscr{C}_3$ by the $\frac{2\pi}{3}$- and $\frac{4\pi}{3}$-rotations of the plane on which the diagram written, $\mathscr{C}_3$, $\mathscr{C}_4$ and $\mathscr{C}_5$ are R-equivalent to each other. Similarly, $\mathscr{C}_6$, $\mathscr{C}_7$ and $\mathscr{C}_8$ are R-equivalent to each other. Furthermore, since $\mathscr{C}_8$ is obtained from $\mathscr{C}_3$ by a finite sequence of Reidemeister moves and planar isotopies as illustrated in Figure 4, $\mathscr{C}_3$ and $\mathscr{C}_8$ are R-equivalent to each other. Thus, all non-trivial colorings are R-equivalent to each other. Since the number of elements of each R-equivalence classes is 1, 1, 1 and 6, we can divide the R_3-coloring number as

$$\mathrm{col}_{R_3}(TK) = 9 = 1 + 1 + 1 + 6.$$

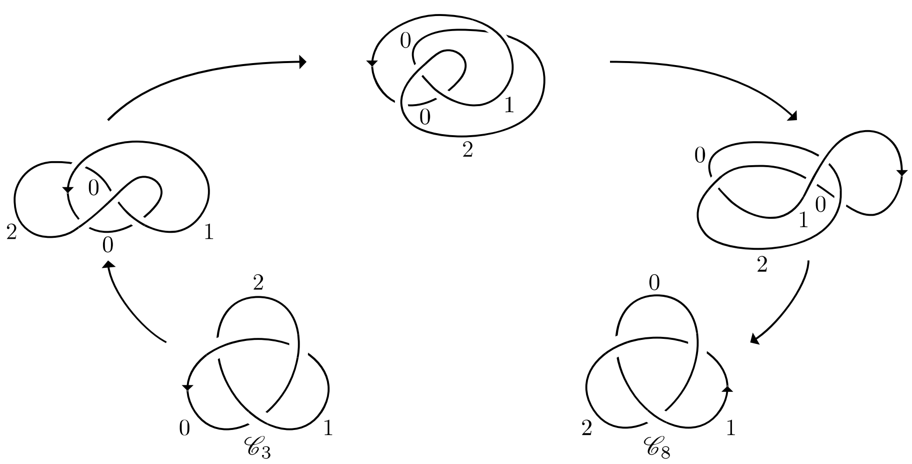

Figure 4: $\mathscr{C}_8$ is obtained from $\mathscr{C}_3$ by a finite sequence of Reidemeister moves and planar isotopies.

Remark 2.1. For some quandle X, let $\mathscr{C}_1$ and $\mathscr{C}_2$ be X-colorings of a diagram D. Furthermore, let $\mathscr{C}_1'$ and $\mathscr{C}_2'$ respectively be X-colorings of D' obtained from $\mathscr{C}_1$ and $\mathscr{C}_2$ by a finite sequence of Reidemeister moves and planar isotopies. Then $\mathscr{C}_1'$ and $\mathscr{C}_2'$ are R-equivalent to each other if and only if $\mathscr{C}_1$ and $\mathscr{C}_2$ are. Therefore, the division of the X-coloring number into the sum of the cardinalities of the R-equivalence classes gives us an invariant of oriented knots.

3 Example

In this section, we determine the R-equivalence classes of the R_3-colorings of the diagram SK of the square knot (depicted in Figure 5) and the ones of the diagram GK of the granny knot (depicted in Figure 18), for example. We first study the R-equivalence classes of the R_3-colorings of SK.

It is routine to check that SK has twenty-seven R_3-colorings depicted in Figure 5. In the figure, $\bigcirc$, $\square$, and $\triangle$ denote some elements of R_3 which are mutually different. Since they are trivial colorings, each coloring of Type A is not R-equivalent to any other colorings as mentioned in Section 2. To investigate the R-equivalence classes of the non-trivial colorings, we consider the deformations 1–4 of SK depicted in Figures 6–9, respectively. Then it is routine to check that any two colorings of Type B (respectively C) are related to each other by a finite sequence of deformations 2, 3 (respectively 1, 4) and their inverses. Therefore, any colorings of Type B (respectively C) are R-equivalent to each other. Furthermore, it is routine to see that any two colorings of Type D or E are related to each other by a finite sequence of deformations 1–3 and their inverses. Therefore, any colorings of Type D or E are R-equivalent to each other.

In fact, the set of non-trivial colorings is divided into the above three R-equivalence classes. To show that, we review the following things. More details can be found in [3], for example.

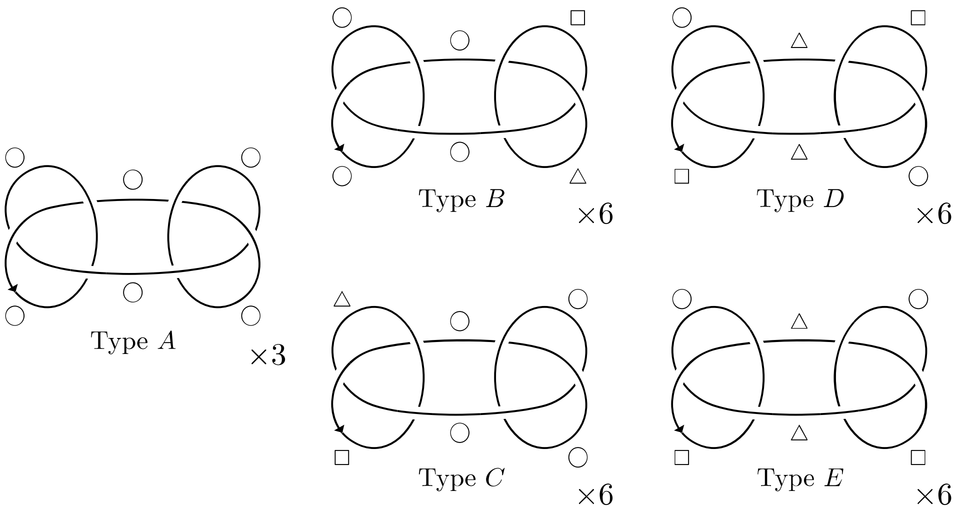

Figure 5: The R_3-colorings of the diagram SK of the square knot.

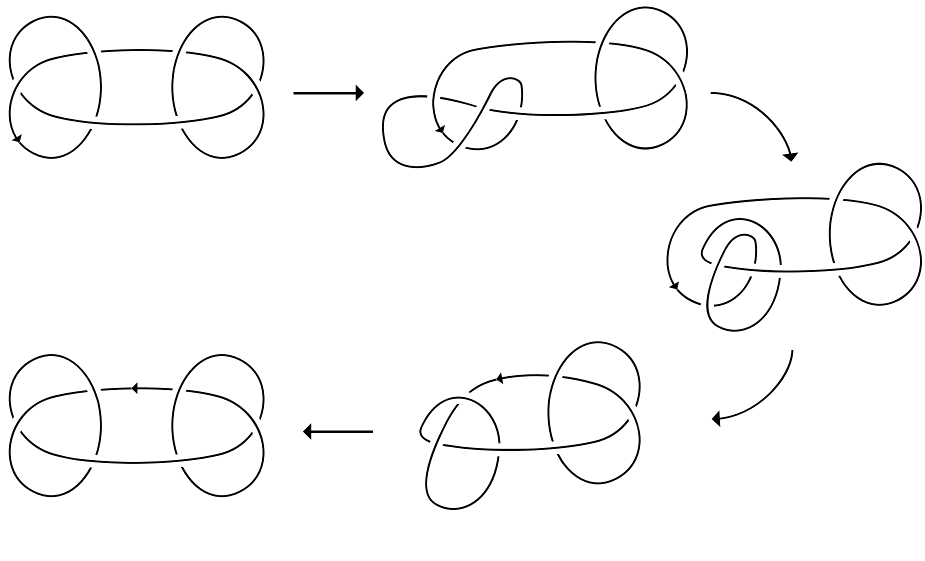

Figure 6: Deformation 1 of SK.

Let X be a quandle, D an oriented knot diagram and $\mathscr{C}$ an X-coloring of D. A *region coloring* of D is a map $\mathscr{R}$ from the set of all regions of D to X satisfying the condition depicted in Figure 10 around each arc of D. In the figure, y denotes the element of X assigned to correspondent arc by $\mathscr{C}$ and x and $x * y$ denote the ones assigned to

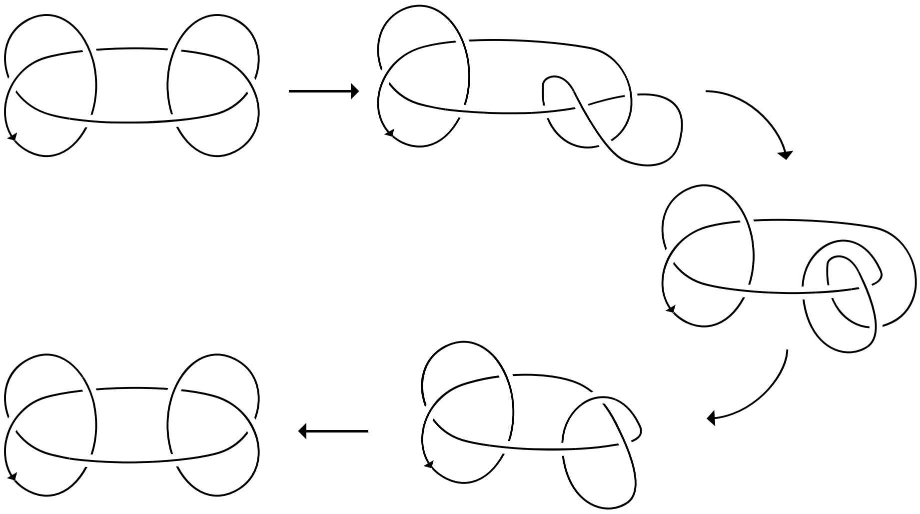

Figure 7: Deformation 2 of SK.

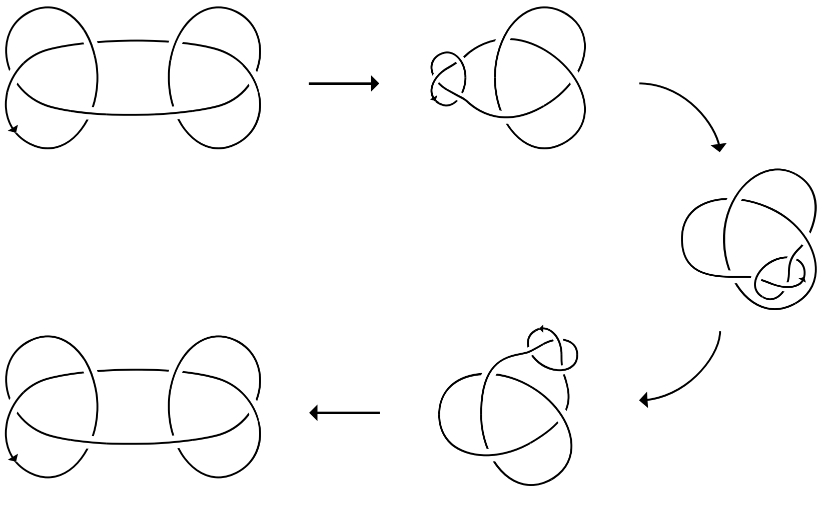

Figure 8: Deformation 3 of SK.

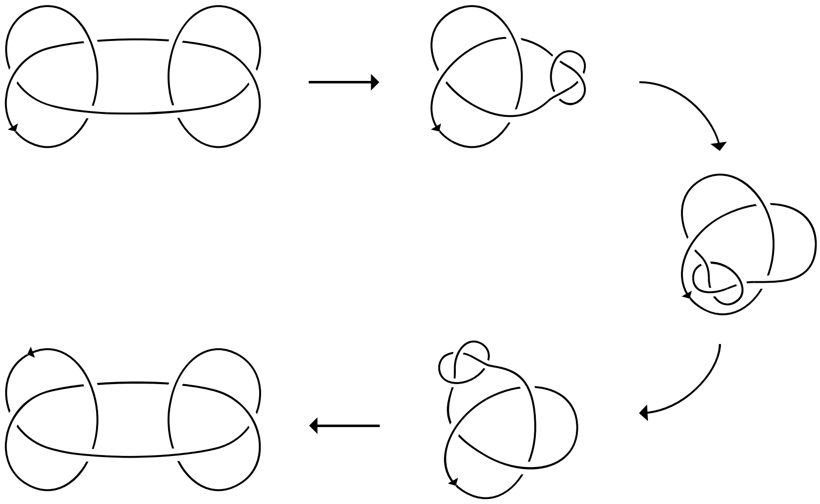

Figure 9: Deformation 4 of SK.

correspondent regions by $\mathscr{R}$. We call x and $x * y$ the *colors* of the regions. The pair $(\mathscr{C}, \mathscr{R})$ of the X-coloring $\mathscr{C}$ and the region coloring $\mathscr{R}$ is called a *shadow coloring* of D by X. In a similar way to X-colorings, if a diagram D' is obtained from a diagram D by a finite sequence of Reidemeister moves and planar isotopies, we have a unique shadow coloring $(\mathscr{C}', \mathscr{R}')$ of D' corresponding to a shadow coloring $(\mathscr{C}, \mathscr{R})$ of D. We also say in this situation that $(\mathscr{C}', \mathscr{R}')$ is obtained from $(\mathscr{C}, \mathscr{R})$ by the sequence, or $(\mathscr{C}, \mathscr{R})$ is related to $(\mathscr{C}', \mathscr{R}')$ by the sequence. We note that $\mathscr{C}'$ is the X-coloring obtained from $\mathscr{C}$ by the sequence.

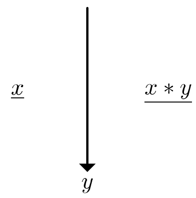

Figure 10: The condition for a region coloring.

Remark 3.1. Let x_0 be an element of X. Then there is a unique region coloring of D whose color of the unbounded region is x_0 with respect to any X-coloring of D.

Let X be a quandle and A an abelian group. A map $f : X^3 \to A$ is said to be a *quandle* 3*-cocycle* if f satisfies the following two conditions.

QC1. For any $x, y, z, w \in X$,

$$\begin{aligned} &f(x,y,z) + f(x*z, y*z, w) + f(x,z,w) \\ &= f(x*y, z, w) + f(x,y,w) + f(x*w, y*w, z*w). \end{aligned}$$

QC2. For any $x, y \in X$, $f(x,x,y) = f(x,y,y) = 0$.

For example, a map $f : {R_p}^3 \to \mathbb{Z}/p\mathbb{Z}$ given by

$$f(x,y,z) = (x-y)\frac{y^p + (2z-y)^p - 2z^p}{p}$$

is a quandle 3-cocycle of R_p for each odd prime integer p [1, 4]. We call it the *Mochizuki 3-cocycle*.

Suppose that $f : X^3 \to A$ is a quandle 3-cocycle of a quandle X. Let $(\mathscr{C},\mathscr{R})$ be a shadow coloring of an oriented knot diagram D by X. For each crossing c of D, whose arcs and regions are colored by $(\mathscr{C},\mathscr{R})$ as depicted in Figure 11, we define a *local weight* $W_f(c,\mathscr{C},\mathscr{R})$ of $(\mathscr{C},\mathscr{R})$ at c by

$$W_f(c,\mathscr{C},\mathscr{R}) = \varepsilon \cdot f(x,y,z),$$

where $\varepsilon = +1$ if c is positive, otherwise $\varepsilon = -1$. Take the sum

$$W_f(D,\mathscr{C},\mathscr{R}) = \sum_c W_f(c,\mathscr{C},\mathscr{R})$$

of the local weights over all crossings of D and call it the *weight* of $(\mathscr{C},\mathscr{R})$. It is easy to see that if a shadow coloring $(\mathscr{C}',\mathscr{R}')$ of D' is obtained from $(\mathscr{C},\mathscr{R})$ by a finite sequence of Reidemeister moves and planar isotopies, then we have $W_f(D',\mathscr{C}',\mathscr{R}) = W_f(D,\mathscr{C},\mathscr{R})$ (see Figure 12).

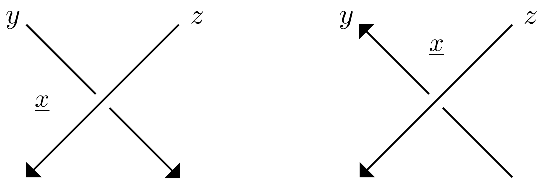

Figure 11: Colors around a positive (left) or negative (right) crossing c.

Let $\mathscr{C}$ and $\mathscr{C}'$ be X-colorings of D which are R-equivalent to each other. Choose and fix an element $x_0 \in X$. Then, in light of Remark 3.1, we have the region coloring $\mathscr{R}$ (respectively $\mathscr{R}'$) with respect to $\mathscr{C}$ (respectively $\mathscr{C}'$) whose color of the unbounded region is x_0. Since the colors of the unbounded regions are preserved under Reidemeister moves and planar isotopies, a finite sequence of Reidemeister moves and planar isotopies which relates $\mathscr{C}$ to $\mathscr{C}'$ relates $(\mathscr{C},\mathscr{R})$ to $(\mathscr{C}',\mathscr{R}')$. Therefore, we have $W_f(D,\mathscr{C},\mathscr{R}) = W_f(D,\mathscr{C}',\mathscr{R}')$.

We are now ready to determine the R-equivalence classes of SK. Let f be the Mochizuki 3-cocycle of R_3. Furthermore, with respect to each R_3-coloring $\mathscr{C}$ of SK, let $\mathscr{R}$ be the region coloring of SK whose color of the unbounded region is 0. Then we have

$$W_f(SK,\mathscr{C},\mathscr{R}) = \begin{cases} 0 & \text{if } \mathscr{C} \text{ is of Type } A,\ D,\ E, \\ 1 & \text{if } \mathscr{C} \text{ is of Type } B, \\ 2 & \text{if } \mathscr{C} \text{ is of Type } C. \end{cases}$$

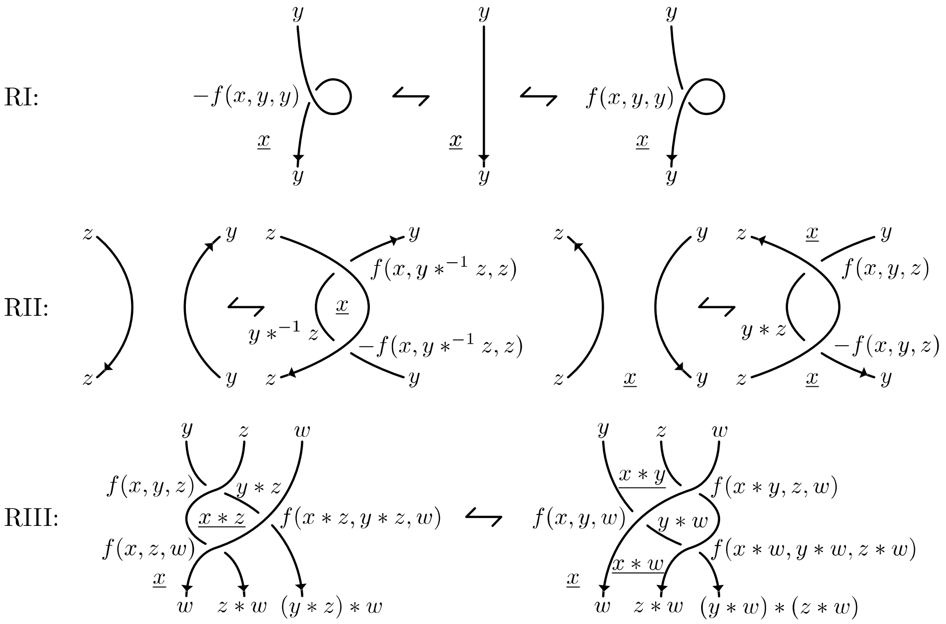

Figure 12: Reidemeister moves do not change the weights of shadow colorings of the diagrams.

Since weights of them are mutually different, a coloring of Type B, one of Type C, and one of Type D or E are not R-equivalent to each other. Therefore, the R-equivalence classes of the R_3-colorings of SK are completely determined.

We next study the R-equivalence classes of the R_3-colorings of GK. We note that we respectively have the deformations 1–4 of GK depicted in Figures 13–16, in a similar manner to SK. Moreover, we have the deformation 5 of GK depicted in Figure 17.

As well as SK, GK has twenty-seven R_3-colorings depicted in Figure 18. Since they are trivial colorings, each coloring of Type A' is not R-equivalent to any other colorings. It is routine to check that any two colorings of Type B' (respectively C') are related to each other by a finite sequence of deformations 2, 3 (respectively 1, 4) and their inverses. Moreover, the deformation 5 relates a coloring of Type B' to one of Type C'. Therefore, any colorings of Type B' or C' are R-equivalent to each other. It is routine to see that any two colorings of Type D' or E' are related to each other by a finite sequence of deformations 1–3 and their inverses. Therefore, any colorings of Type D' or E' are R-equivalent to each other.

Let f be the Mochizuki 3-cocycle of R_3, again. Furthermore, with respect to each R_3-coloring $\mathscr{C}$ of GK, let $\mathscr{R}$ be the region coloring of GK whose color of the unbounded region is 0. Then we have

$$W_f(GK, \mathscr{C}, \mathscr{R}) = \begin{cases} 0 & \text{if } \mathscr{C} \text{ is of Type } A', \\ 1 & \text{if } \mathscr{C} \text{ is of Type } D' \text{ or } E', \\ 2 & \text{if } \mathscr{C} \text{ is of Type } B' \text{ or } C'. \end{cases}$$

Since weights of them are mutually different, a coloring of Type B' or C', and one of Type D' or E' are not R-equivalent to each other. Therefore, the R-equivalence classes of the

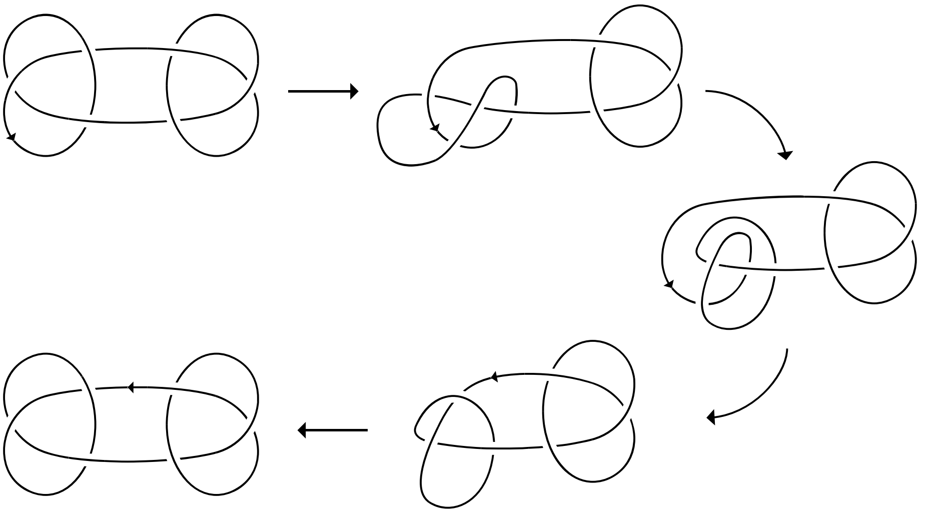

Figure 13: Deformation 1 of GK.

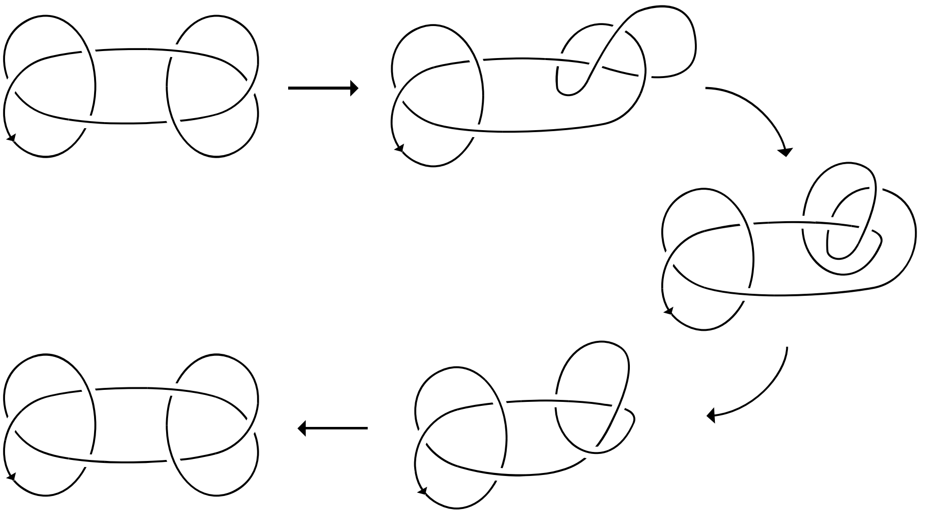

Figure 14: Deformation 2 of GK.

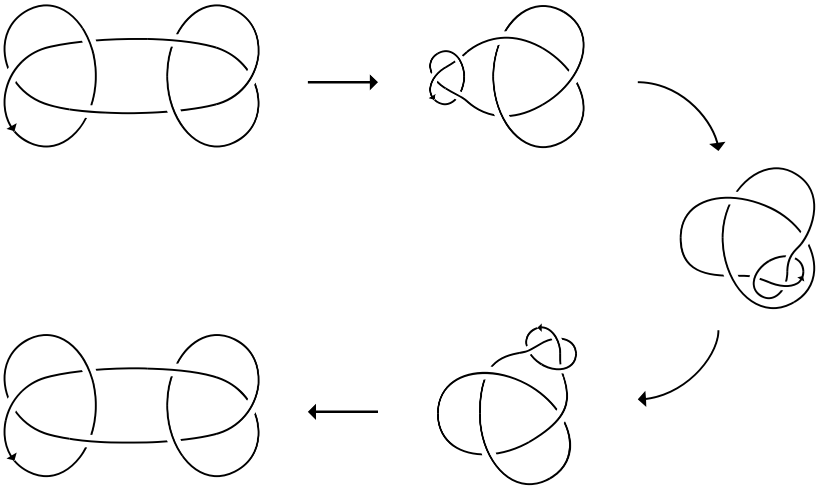

Figure 15: Deformation 3 of GK.

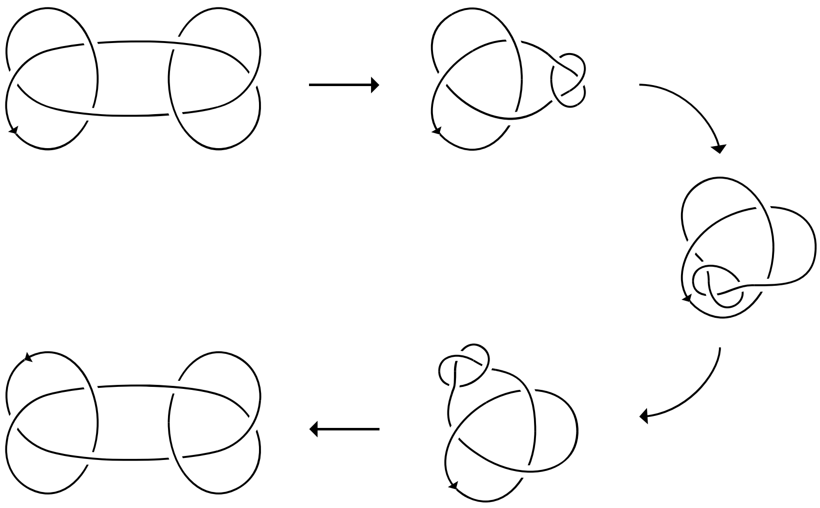

Figure 16: Deformation 4 of GK.

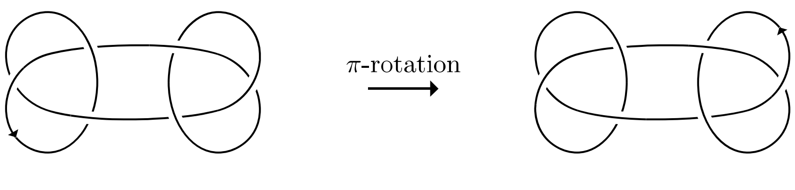

Figure 17: Deformation 5 of GK.

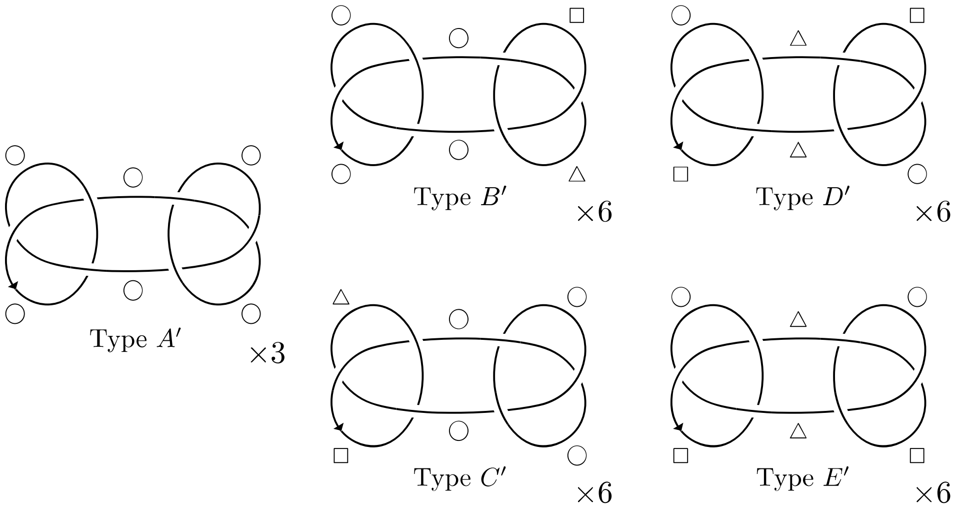

Figure 18: The R_3-colorings of the diagram GK of the granny knot.

R_3-colorings of GK are also determined completely.

As a consequence, we can divide the R_3-coloring numbers of SK and GK as

$$\mathrm{col}_{R_3}(SK) = 27 = 1 + 1 + 1 + 6 + 6 + 12,$$
$$\mathrm{col}_{R_3}(GK) = 27 = 1 + 1 + 1 + 12 + 12.$$

Therefore, in light of Remark 2.1, we obtain the well-known fact that the square knot and the granny knot have different knot types by studying R-equivalence classes of their R_3-colorings. The difference of the divisions of the R_3-coloring numbers of SK and GK is caused by that GK admits the deformation 5 while SK does not. It means that difference of symmetries which the square knot and the granny knot equip is reflected to the R-equivalence classes of the R_3-colorings of SK and GK.

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