On skein modules of twisted tori

Patrick Kinnear
Fachbereich Mathematik, Universität Hamburg

1 Introduction

Skein modules were introduced by Przytycki [31] and Turaev [37], and enjoy numerous connections to different areas of mathematics including quantum field theory [1], character varieties [4, 33], and cluster theory [2, 28]. It was conjectured by Witten and proved by Gunningham–Jordan–Safronov [13] that skein modules of closed, compact, oriented 3-manifolds are finite-dimensional over $\mathbb{C}(q^{1/2})$. However, the proof is not constructive and the computation of the dimensions of skein modules remains a problem of interest.

The earliest computations of skein modules were for lens spaces and $S^2 \times S^1$ [17, 18]. Since then, skein modules have been computed for integer Dehn surgeries along a trefoil [3], the quaternionic manifold [11], some prism manifolds [27], and some families of hyperbolic manifolds obtained by Dehn filling of knot complements [7]. The dimension was computed for the 3-torus by Carrega [5] and Gilmer [10], and this was generalized to the trivial S^1-bundle over any closed surface in [12, 8]. In [22] the skein modules of mapping tori of the 2-torus were computed, and the purpose of this article is to give an exposé of some of the key ideas used in the proof.

If $\gamma \in \mathrm{Mod}(\mathbb{T}^2) \cong \mathrm{SL}_2(\mathbb{Z}) \cong \langle S, T | S^4 = 1, (ST)^3 = S^2 \rangle$ is a mapping class, we denote the corresponding mapping torus by M_γ. The result of [22] can be stated as follows.

Theorem 1.1. *Let* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. *Then* $\dim \mathrm{Sk}(M_\gamma)$ *is as follows:*

- $|\mathrm{tr}(\gamma)| = 0$*: then* $\dim \mathrm{Sk}(M_\gamma) = 6$.
- $|\mathrm{tr}(\gamma)| = 1$*: then* $\dim \mathrm{Sk}(M_\gamma) = 4$.
- $|\mathrm{tr}(\gamma)| = 2$*: then* γ *is conjugate to* $\pm T^n$*, and*

$$\dim \mathrm{Sk}(M_\gamma) = \begin{cases} 9 + k & n = 2k \\ 6 + k & n = 2k+1 \end{cases}.$$

- $|\mathrm{tr}(\gamma)| > 2$*: then*

$$\dim \mathrm{Sk}(M_\gamma) = |\mathrm{tr}(\gamma)| + 2^{c(\gamma)+1}$$

where

$$c(\gamma) = \#\{m \in \{\gcd(a-1, b, c, d-1), \mathrm{tr}(\gamma)\} : m \text{ even}\}.$$

The above theorem has the advantage that within each case, the formula for the skein module dimension is elementary to state. In [22], a single formula was given which does not require the mapping classes to be separated by the absolute value of their trace.

Theorem 1.2. *Let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Then $\dim \mathrm{Sk}(M_\gamma)$ is given by*

$$\dim \mathrm{Sk}(M_\gamma) = s_\gamma + \frac{\prod_{i=1}^{r_+} a_i^+ + 2^{p_+}}{2} + \frac{\prod_{i=1}^{r_-} a_i^- + 2^{p_-}}{2}$$

where $r_\pm$ is the rank and $a_i^\pm$ are the invariant factors of $\mathrm{Id} \mp \gamma$, and

$$p_\pm = \#\{a_i^\pm \text{ even} : 1 \leq i \leq r_\pm\}$$

and

$$s_\gamma = \begin{cases} 4 & \bar{\gamma} = \bar{\mathrm{Id}} \\ 2 & \mathrm{tr}(\bar{\gamma}) = 0 \mod 2 \text{ and } \bar{\gamma} \neq \bar{\mathrm{Id}} \\ 1 & \mathrm{tr}(\bar{\gamma}) = 1 \mod 2 \end{cases}$$

for $\bar{\gamma}$ the matrix of γ taken mod 2 in each entry.

One advantage of the formula of Theorem 1.2 is that it reflects the structure of the calculation, which we will describe in this article. It was not noticed by the author that the alternative statement, Theorem 1.1, can be obtained from Theorem 1.2, until after the publication of [22].

The dimensions of Theorem 1.2 lead to the observation, due to R. Detcherry, that there does not exist an oriented $(3+1)$-TQFT which assigns skein modules to 3-manifolds and yields the natural actions of mapping class groups on skein modules. Essentially, a TQFT would allow us to calculate traces of mapping class group actions on 3-manifolds, and in particular the dimensions of Theorem 1.2 will appear as traces of the mapping class group of the 3-torus. But in the case of the 3-torus the natural mapping class group action is already well-understood due to work of Carrega [5], and the dimensions of Theorem 1.2 do not appear as traces.

Another implication of Theorem 1.2 is as follows. For the lower bound on the dimension of the skein modules of circle bundles over closed surfaces obtained in [12] Gilmer–Masbaum introduced the so-called *evaluation map*, which takes a Kauffman bracket skein and produces an almost-everywhere-defined $\mathbb{C}$-valued function on the set of even order roots of unity. Their lower bound is obtained by finding skeins which are linearly independent under the evaluation map. Injectivity of the evaluation map would allow one to conclude the lower bound is optimal. However, it was shown in [23], using the computations of Theorem 1.2, that the evaluation map is not injective in general. In the case of circle bundles over a surface, optimality of the bound from [12] was shown by different means in [8].

In the remainder of this article we will explain some key elements of the proof of Theorem 1.2. To complement the paper [22], we will try to emphasise important results used in the proof in different language to that of the original paper. In particular, in [22], results of [15] are used which can be phrased in terms of skein categories. In this article, we avoid mention of skein categories and aim to give a sketch of the main ideas which are accessible to a reader unfamiliar with the skein category construction.

In Section 2 we recall skein modules. As well as the Kauffman bracket skein module, which is the main object of Theorem 1.2, we also introduce and give some remarks on the SL_N and GL_N-skein modules, to which some of our techniques can be extended. In Section 3 we point out the key observations in the proof of Theorem 1.2 in non-categorical terms. For completeness we describe how Theorem 1.1 is obtained from Theorem 1.2. We also remark on generalizations of our calculations to SL_N and GL_N skein theory.

Acknowledgements

The author thanks the organizers of ILDT for their invitation to contribute. The rephrasing of Theorem 1.1 was noticed by the author while discussing possible implications of Theorem 1.2 with colleagues while visiting Japan as a JSPS fellow.

2 Skein modules

In this section we recall the definition of Kauffman bracket skein modules, to which our main results apply. We also describe their generalization to and SL_N and GL_N skein modules, to which some of our techniques can be generalized.

2.1 Kauffman bracket skein modules

The Kauffman bracket skein module $\mathrm{Sk}(M)$ of a 3-manifold M is the $\mathbb{C}(q^{1/2})$-vector space spanned by isotopy classes of framed links embedded in M, modulo the Kauffman bracket relations shown in Fig. 1, which are applied in any embedded ball in M. The equivalence class of a given link is called a *skein*.

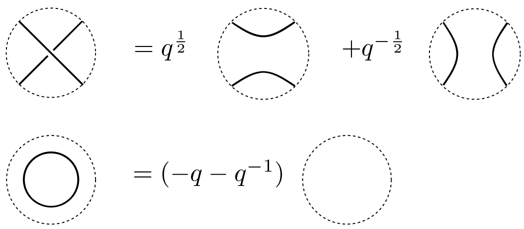

Figure 1: The Kauffman bracket skein relations.

The Kauffman bracket skein module of S^3 is 1-dimensional: given a knot K, this evaluates to a scalar multiple of the empty link, where this scalar is the Kauffman bracket polynomial of K [21]. In this sense, skein modules allow us to generalize the Kauffman bracket polynomial to study links embedded in an arbitrary 3-manifold.

Let us remark that skein modules can be considered as modules over the ring $\mathbb{Z}[q^{\pm 1/2}]$ of Laurent polynomials (indeed this is why they are called skein *modules*) or over $\mathbb{C}$ where $q^{1/2}$ is specialized to some specific complex number, often a root of unity. However in this work, we will only ever consider skein modules over $\mathbb{C}(q^{1/2})$, i.e. when the quantum parameter is generic.

2.2 SL_N and GL_N skein modules

The Kauffman bracket is closely related to the representation theory of quantum SL_2. It follows from work of Rumer–Teller–Weyl [35] that the tensor category generated by the vector representation V of $U_q(\mathfrak{sl}_2)$ is equivalent to the Temperly–Lieb category, whose objects are collections of points and morphisms are planar 1-manifolds connecting them, with the relation that a circle can be erased at the cost of $(-q - q^{-1})$. The category generated by V has a ribbon structure, and the Kauffman bracket can be seen as promoting the Temperly–Lieb diagrammatics to a 3-dimensional calculus for this ribbon category. This allows us to regard Kauffman skeins as embedded 1-manifolds which locally look like morphisms between tensor powers of the vector representation.

In general, any ribbon category $\mathcal{A}$ has a ribbon calculus: morphisms can be depicted as framed graphs in 3 dimensions labelled by objects, with vertices labeled by morphisms from the incoming to the outgoing edges. It is then possible to talk of $\mathcal{A}$-skein module of M as being spanned by isotopy classes of such graphs embedded in M, where we impose the relations which would hold in $\mathcal{A}$ within any ball embedded in M. This very general notion of skein module is implied by the generality of the Reshetikhin–Turaev construction [34]. Taking $\mathcal{A}$ to be the category $\mathrm{Rep}^{\mathrm{f.d.}}U_q(\mathfrak{g})$ of all finite-dimensional type 1 representations, for $\mathfrak{g}$ the Lie algebra of SL_N or GL_N, gives a definition of SL_N- or GL_N skein module.

It is interesting to ask if there are diagrammatic presentations for these skein modules similar to the Kauffman bracket. There was early work in this direction by Kuperburg [24], who showed how to give a calculus based on trivalent graphs (called webs) for the category generated by the fundamental representations (exterior powers of the vector representation) of $U_q(\mathfrak{sl}_3)$. Around the same time, it was noticed in [29] that the HOMFLYPT polynomial can be computed by resolving a knot into a linear combination of trivalent planar graphs, which suggested a relation with the representation theory of $U_q(\mathfrak{sl}_N)$.

These ideas were developed in work of Sikora [36] and Cautis–Kamnitzer–Morrison [6]. In [36], a bracket invariant is defined for graphs which contain only N-valent sources or sinks, corresponding to the tensor category generated by the vector representation V of $U_q(\mathfrak{sl}_N)$ with the sinks and sources corresponding to the projection to and inclusion of the (trivial) N-th exterior power. In [6], a category of trivalent $\mathfrak{sl}_N$-webs is defined, with strands labeled by integers. It is shown that this is equivalent to the category generated by fundamental representations of $U_q(\mathfrak{sl}_N)$, thus giving a diagrammatic calculus for this category.

We therefore have two diagrammatic theories where skeins are embedded graphs which locally look like morphisms in some category of $U_q(\mathfrak{sl}_N)$-representations. It was shown in [30] that the diagram category implicit in [36] is equivalent to the subcategory of $\mathfrak{sl}_N$-webs with strands labeled by the vector representation, and moreover that the skein modules associated to the category of N-valent graphs and the category of $\mathfrak{sl}_N$-webs are isomorphic. Clearly the former skein module embeds in the latter, and the proof of the isomorphism amounts to observing that any exterior power is a retract of a tensor power, i.e. there exists a factorization $\bigwedge^k V \xrightarrow{i} V^{\otimes k} \xrightarrow{r} W = \bigwedge^k V$ of $\mathrm{Id}_{\bigwedge^k V}$. By inserting $\mathrm{Id}_{\bigwedge^k V} = ri$ into a strand labeled by $\bigwedge^k V$ and isotoping r and i to the ends of the strand, it suffices to consider only webs with strands coloured by V, establishing that the inclusion is also surjective.

Since every finite-dimensional representation of $U_q(\mathfrak{sl}_N)$ is a retract of a fundamental representation, an identical argument shows that the skein module defined via $\mathfrak{sl}_N$-webs and the skein module more abstractly associated to $\mathrm{Rep}^{\mathrm{f.d.}} U_q(\mathfrak{sl}_N)$ are isomorphic. We may therefore speak unambiguously of the SL_N-skein module, which admits diagrammatic descriptions based on either the N-valent calculus of [36] or the trivalent one of [6] as preferred.

There is a generalization of $\mathfrak{sl}_N$-webs to $\mathfrak{gl}_N$-webs [26]. As observed in [25, Theorem 6A.1], the proofs of [6] essentially show that $\mathfrak{gl}_N$-webs give a diagrammatic description of the tensor category generated by the fundamental representations of $U_q(\mathfrak{gl}_N)$. Again, since every finite-dimensional representation of $U_q(\mathfrak{gl}_N)$ is a retract of a fundamental one, it follows that these diagrammatics can be used to calculate the GL_N-skein module.

3 Elements of proof

Here we will sketch some key ideas used to prove Theorem 1.2. We will explain how to decompose the skein module $\mathrm{Sk}(M_\gamma)$ into a direct sum, from which the summands of Theorem 1.2 are seen to count the dimensions of each direct summand.

3.1 Parity grading and cutting torus method

The first observation is the following.

Lemma 3.1 ([5]). *For M a 3-manifold,* $\mathrm{Sk}(M)$ *is graded by* $\mathrm{H}_1(M;\mathbb{Z}/2\mathbb{Z})$*.*

The grading, which featured in the computations of [5, 10, 12, 8], is given by viewing skeins as homology 1-cycles with $\mathbb{Z}/2\mathbb{Z}$-coefficients. This is well-defined since the Kauffman bracket skein relations become null in $H_1(M;\mathbb{Z}/2\mathbb{Z})$. The upshot in the case of mapping tori is that any skein wraps the monodromy direction either an even or an odd number of times, i.e.

$$\mathrm{Sk}(M_\gamma) \cong \{\text{even skeins}\} \oplus \{\text{odd skeins}\}. \tag{1}$$

This decomposition can be even further strengthened by the following powerful result.

Lemma 3.2 ([15, Corollary 1.7]). *Let M be a 3-manifold, and fix $\mathbb{T}^2 \subset M$. Then* $\mathrm{Sk}(M)$ *is spanned by skeins which intersect $\mathbb{T}^2$ at most once.*

By fixing a copy of $\mathbb{T}^2$ inside a mapping torus transverse to the monodromy direction, (1) can be strengthened to the following:

$$\mathrm{Sk}(M_\gamma) \cong W_0 \oplus W_1 \tag{2}$$

where W_0 is the space of skeins which wrap the monodromy direction zero times, and W_1 the space of those which wrap once.

3.2 Twisted Hochschild homology

Recall that the skein module of any cylinder $\Sigma\times[0,1]$ has an algebra structure, by stacking skeins in the interval direction, and considered with this structure is known as the skein algebra, denoted $\mathrm{SkAlg}(\Sigma)$.

Considering the map $\mathbb{T}^2\times[0,1]\to M_\gamma$, there is an induced linear map $\mathrm{SkAlg}(\mathbb{T}^2)\to W_0$. Of course, there is a kernel given by identifying a skein with the result of wrapping it around the mapping torus and applying the monodromy. Since the source is an algebra, we can write this kernel as

$$(ab-\gamma(b)a : a,b\in \mathrm{SkAlg}(\mathbb{T}^2))$$

where γ acts in the obvious way on $\mathrm{SkAlg}(\mathbb{T}^2)$. The resulting quotient is denoted

$$\mathrm{HH}_0^\gamma(\mathrm{SkAlg}(\mathbb{T}^2))$$

and called *twisted Hochschild homology*, since for $\gamma=\mathrm{Id}$ this is the usual zeroth Hochschild homology space of an algebra. Clearly this space is isomorphic to W_0.

Similarly, let us fix a marked point $x\in\mathbb{T}^2$ and consider an algebra $E_{(1,1)}$ spanned by framed tangles in $\mathbb{T}^2\times[0,1]$ ending at $\{x\}\times\{0\}$ and $\{x\}\times\{1\}$, modulo the Kauffman bracket relations. As before, the algebra structure is given by stacking, and γ acts in the obvious way, and we see that just as described above we have

$$W_1\cong \mathrm{HH}_0^\gamma(E_{(1,1)}).$$

It is shown in [22, §4.1], using observations about the skein theory of $\mathbb{T}^2$ going back to P. Samuelson, that

$$E_{(1,1)}\cong \mathbb{C}(q^{1/2})[\mathbb{F}_2^2]$$

for $\mathbb{F}_2$ the field with 2 elements. This is a 4-dimensional algebra, and the action of γ can be described explicitly. It is then straightforward to calculate that

$$\dim \mathrm{HH}_0^\gamma(E_{(1,1)})=s_\gamma$$

for s_γ the number appearing in Theorem 1.2.

3.3 Decomposition of W_0

We have now explained the summand s_γ from Theorem 1.2, which counts the dimension of the direct summand $W_1\cong \mathrm{HH}_0^\gamma(E_{(1,1)})$ of the skein module. To explain the other two summands of Theorem 1.2, we must give a further decomposition of the space $W_0\cong \mathrm{HH}_0^\gamma(\mathrm{SkAlg}(\mathbb{T}^2))$. Our starting point is the following important description of the skein algebra.

Theorem 3.3 ([9, Theorem 2])**.** *There is an algebra isomorphism*

$$\mathrm{SkAlg}(\mathbb{T}^2)\cong A^{\mathbb{Z}/2\mathbb{Z}}$$

where

$$A=\mathbb{C}(q^{1/2})\langle X^{\pm1},Y^{\pm1}\rangle/(XY-q^{\frac{1}{2}}YX)$$

and $\mathbb{Z}/2\mathbb{Z}$ acts by inverting X and Y.

The zeroth Hochschild homology of an algebra is a Morita invariant. It is well-known that there is a Morita equivalence between A^W and the so-called *smash product algebra* $A\#W$. Moreover, it is known that the zeroth Hochschild homology of a smash product has a direct sum decomposition. Tracing through the γ-twisted versions of these statements carefully is the content of [22, §3.1], yielding the decomposition

$$\mathrm{HH}_0^{\gamma}(A^{\mathbb{Z}/2\mathbb{Z}}) \cong \bigoplus_{w \in \mathbb{Z}/2\mathbb{Z}} \mathrm{HH}_0^{\gamma w}(A)_{\mathbb{Z}/2\mathbb{Z}}.$$

Here the Hochschild homologies on the right are further twisted by a sign, and the subscript indicates that we take coinvariants with respect to the natural $\mathbb{Z}/2\mathbb{Z}$ action on this space. The two direct summands here explain the appearance of the two remaining summands in the formula of Theorem 1.2.

It is explained in [22, §3.2] that $\mathrm{HH}_0^{\gamma w}(A)$ is isomorphic to the vector space supported on the torsion part of $\mathrm{coker}(\mathrm{Id} - \gamma w)$ where $\mathrm{Id} - \gamma w$ acts on $H_1(\mathbb{T}^2) \cong \mathbb{Z}^2$. We recall that the cokernel of a map of $\mathbb{Z}$-modules has a canonical form:

$$\mathrm{coker}(\mathrm{Id} - \gamma w) \cong \mathbb{Z}^{2-r_w} \oplus \bigoplus_{i=1}^{r_w} \mathbb{Z}/a_i^w\mathbb{Z}$$

where r_w is the rank of the map and the integers a_i^w are the so-called *invariant factors* of the map.

The natural $\mathbb{Z}/2\mathbb{Z}$-action on this space is by negation in each coordinate. The orbits are generically of cardinality 2, and the dimension of the space of coinvariants is therefore approximately $\frac{1}{2}\prod_i a_i^w$. Whenever a_i^w is even, there will be a fixed-point for the $\mathbb{Z}/2\mathbb{Z}$-action, so that the approximate dimension needs to be corrected and the true dimension is

$$\dim \mathrm{HH}_0^{\gamma w}(A)_{\mathbb{Z}/2\mathbb{Z}} = \frac{\prod_{i=1}^{r_w} a_i^w + 2^{p_w}}{2}$$

where p_w counts the number of even invariant factors. This explains the remaining terms of the formula of Theorem 1.2.

3.4 Derivation of Theorem 1.1

Firstly, we note that two mapping tori M_γ, M_ϕ are oriented-diffeomorphic if and only if γ and ϕ are conjugate in $\mathrm{SL}_2(\mathbb{Z})$ [16, Theorem 2.6].

We recall (see, e.g. [20, Chapter 7]) that:

- $|\mathrm{tr}(\gamma)| = 0$: then γ is conjugate to $\pm S$ for $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
- $|\mathrm{tr}(\gamma)| = 1$: then γ is conjugate to $\pm E^{\pm 1}$ for $E = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$.
- $|\mathrm{tr}(\gamma)| = 2$: then γ is conjugate to $\pm T^n$ for $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $n \in \mathbb{Z}$.

- $|\mathrm{tr}(\gamma)| > 2$: then conjugacy classes are classified via continued fractions in a way we do not recall here.

We recall the following elementary facts about the invariant factors of an integer matrix P. The invariant factors satisfy $a_i | a_{i+1}$ and can be calculated by $a_i = d_i/d_{i-1}$ where d_i is the *i-th determinant divisor*, the greatest common divisor of the determinants of all $i \times i$ minors of P (we take the convention that $d_0 = 1$). Moreover if P is full rank, then $\prod_i a_i = d_r = |\det(P)|$.

In the cases where $|\mathrm{tr}(\gamma)| \leq 2$, the description of the invariant factors given above is sufficient to verify the claimed formula in Theorem 1.1 directly from the formula of Theorem 1.2.

Now assume $|\mathrm{tr}(\gamma)| > 2$. We see that $\mathrm{Id} \mp \gamma$ has full rank since

$$\det(\mathrm{Id} \mp \gamma) = \chi_\gamma(\pm 1) = 2 \mp \mathrm{tr}(\gamma) \neq 0.$$

We write $a_i^\pm, d_i^\pm$ for the invariant factors and determinant divisors respectively of $\mathrm{Id} \mp \gamma$. We have

$$\begin{aligned} \frac{1}{2}\Big(\prod a_i^+ + \prod a_i^-\Big) &= \frac{1}{2}(|\det(\mathrm{Id} - \gamma)| + |\det(\mathrm{Id} + \gamma)|) \\ &= \frac{1}{2}(|2 - \mathrm{tr}(\gamma)| + |2 + \mathrm{tr}(\gamma)|) \\ &= |\mathrm{tr}(\gamma)|. \end{aligned}$$

Let us consider the numbers $p_\pm$ from Theorem 1.2. Writing $a_1^\pm$ as the greatest common divisor of the matrix entries, we notice that $2|a_1^+ \iff 2|a_1^-$. Then since $a_1^\pm | a_2^\pm$, we see $p_+ = 2 \iff p_- = 2$.

Assuming $2 \nmid a_1^\pm$, it follows that $2|a_2^\pm \iff 2|d_2^\pm$. In this case, writing $d_2^\pm = |2 \mp \mathrm{tr}(\gamma)|$, we see $2|a_2^+ \iff 2|\mathrm{tr}(\gamma) \iff 2|a_2^-$, and so $p_+ = 1 \iff p_- = 1$. It then follows that $p_+ = 0 \iff p_- = 0$.

Then we see that

$$2^{p_+ - 1} + 2^{p_- - 1} = 2^{c(\gamma)}$$

for $c(\gamma)$ the number in Theorem 1.1. It is also easy to see that $s_\gamma = 2^{c(\gamma)}$. Then it follows that

$$\begin{aligned} s_\gamma + \frac{\prod_{i=1}^{r_+} a_i^+ + 2^{p_+}}{2} + \frac{\prod_{i=1}^{r_-} a_i^- + 2^{p_-}}{2} &= 2^{c(\gamma)} + 2^{p_+ - 1} + 2^{p_1 - 1} + \frac{1}{2}\Big(\prod_{i=1}^{2} a_i^+ + \prod_{i=1}^{2} a_i^-\Big) \\ &= 2^{c(\gamma)} + 2^{c(\gamma)} + |\mathrm{tr}(\gamma)| \\ &= 2^{c(\gamma)+1} + |\mathrm{tr}(\gamma)| \end{aligned}$$

which verifies the formula of Theorem 1.1 in the case $|\mathrm{tr}(\gamma)| > 2$.

3.5 Generalizations for SL_N and GL_N

Now we discuss the aspects of our calculations which admit a generalization to the SL_N and GL_N cases. The first step in our proof was to decompose the skein module of the mapping torus into skeins which wrap the monodromy an even or an odd number of times, using the grading by $H_1(M_\gamma; \mathbb{Z}/2\mathbb{Z})$ (Lemma 3.1). This grading admits a generalization due to D. Jordan.

Lemma 3.4 ([19, §3]). *For any G, the G-skein module of a 3-manifold M admits a grading by $H_1(M; Z(G)^\vee)$ for $Z(G)^\vee$ the Pontryagin dual of the centre of G.*

Then it follows that for mapping tori,

$$\mathrm{Sk}_{\mathrm{SL}_N}(M_\gamma) \cong \bigoplus_{n=0}^{N-1} \{\text{skeins wrapping the monodromy } n \text{ times} \mod N\}$$

and

$$\mathrm{Sk}_{\mathrm{GL}_N}(M_\gamma) \cong \bigoplus_{n\in\mathbb{N}} \{\text{skeins wrapping the monodromy } n \text{ times}\}.$$

These gradings can be strengthened by a more general form of Lemma 3.2.

Lemma 3.5 ([15, Corollary 1.7]). *Let M be a 3-manifold, and fix $\mathbb{T}^2 \subset M$. Then*

1. *The SL_N-skein module is spanned by skeins which intersect $\mathbb{T}^2$ at most $N-1$ times.*
2. *The GL_N-skein module is spanned by skeins which do not intersect $\mathbb{T}^2$.*

This allows us to write the GL_N-skein module of a mapping torus in terms of skeins which do not wrap the monodromy direction. In particular, $\mathrm{Sk}_{\mathrm{GL}_N}(M_\gamma)$ is the twisted Hochschild homology of $\mathrm{SkAlg}_{\mathrm{GL}_N}(\mathbb{T}^2)$. In the case of SL_N, Lemma 3.5 combined with the above decomposition yields

$$\mathrm{Sk}_{\mathrm{SL}_N}(M_\gamma) \cong \bigoplus_{n=0}^{N-1} W_n$$

for W_n the space of skeins wrapping the monodromy direction n times. Then, just as before, we have $W_n \cong \mathrm{HH}_0^\gamma(E_{(n,n)})$ for $E_{(n,n)}$ the algebra of (n,n)-tangles modulo the SL_N-skein relations.

In both the GL_N and SL_N cases we must understand the twisted Hochschild homology of the skein algebra. A generalization of the presentation as an algebra of invariants in the SL_2 case (Theorem 3.3) is given in [14, Corollary 1.12] (see also [14, §4]). Then a direct sum decomposition of the twisted Hochschild homology is possible via the smash product, and is given in [22, Corollary 3.6]. In the GL_1 case, with trivial Weyl group, it is straightforward to calculate the dimensions and show that they agree with those of [32], and this is done in [22, Theorem 3.13]. More generally, giving the dimension for the twisted Hochschild homology of the skein algebra will amount to counting the orbits of (subgroups of) the Weyl group acting on various twisted Hochschild homologies.

In the SL_N case, to understand the whole skein module we will need a description of the algebras $E_{(n,n)}$, and the author does not know of an immediate generalization of the description of $E_{(1,1)}$ given in [22, §4.1].

References

[1] Christian Blanchet, Nathan Habegger, Gregor Masbaum, and Pierre Vogel. Topological Quantum Field Theories derived from the Kauffman bracket. *Topology*, 34(4):883–927, 1995.

[2] Francis Bonahon and Helen Wong. Quantum traces for representations of surface groups in SL_2. *Geom. Topol.*, 15(3):1569–1615, 2011.

[3] Doug Bullock. On the Kauffman bracket skein module of surgery on a trefoil. *Pacific J. Math.*, 178(1):37–51, 1997.

[4] Doug Bullock. Rings of $SL_2(\mathbb{C})$-characters and the Kauffman bracket skein module. *Comment. Math. Helv.*, 72(4):521–542, 1997.

[5] Alessio Carrega. 9 generators of the skein space of the 3-torus. *Algebr. Geom. Topol.*, 17(6):3449–3460, 2017.

[6] Sabin Cautis, Joel Kamnitzer, and Scott Morrison. Webs and quantum skew Howe duality. *Math. Ann.*, 360(1-2):351–390, 2014.

[7] Renaud Detcherry, Efstratia Kalfagianni, and Adam S. Sikora. Kauffman bracket skein modules of small 3-manifolds. *Adv. Math.*, 467:110169, 2025.

[8] Renaud Detcherry and Maxime Wolff. A basis for the Kauffman skein module of the product of a surface and a circle. *Algebr. Geom. Topol.*, 21(6):2959–2993, 2021.

[9] Charles Frohman and Răzvan Gelca. Skein modules and the noncommutative torus. *Trans. Amer. Math. Soc.*, 352(10):4877–4888, 2000.

[10] Patrick M. Gilmer. On the Kauffman bracket skein module of the 3-torus. *Indiana Univ. Math. J.*, 67(3):993–998, 2018.

[11] Patrick M. Gilmer and John M. Harris. On the Kauffman bracket skein module of the quaternionic manifold. *J. Knot Theory Ramifications*, 16(01):103–125, 2007.

[12] Patrick M. Gilmer and Gregor Masbaum. On the skein module of the product of a surface and a circle. *Proc. Amer. Math. Soc.*, 147(9):4091–4106, 2019.

[13] Sam Gunningham, David Jordan, and Pavel Safronov. The finiteness conjecture for skein modules. *Invent. Math.*, 232(1):301–363, 2023.

[14] Sam Gunningham, David Jordan, and Monica Vazirani. Quantum character theory, 2023. arXiv:2309.03117 [math.RT]

[15] Sam Gunningham, David Jordan, and Monica Vazirani. Skeins on tori, 2024. arXiv:2409.05613 [math.QA]

[16] Allen Hatcher. Notes on Basic 3-Manifold Topology. 2023.

[17] Jim Hoste and Józef H. Przytycki. The $(2, \infty)$-skein module of lens spaces: a generalization of the Jones polynomial. *J. Knot Theory Ramifications*, 02(03):321–333, 1993.

[18] Jim Hoste and Józef H. Przytycki. The Kauffman bracket skein module of $S^1 \times S^2$. *Math. Z.*, 220(1):65–73, 1995.

[19] David Jordan. Langlands duality for skein modules of 3-manifolds. In *String-Math 2022*, volume 107 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., 2023.

[20] Oleg Karpenkov. Continued fractions and $\mathrm{SL}(2, \mathbb{Z})$ conjugacy classes. Elements of Gauss's reduction theory. Markov spectrum. In *Geometry of Continued Fractions*, volume 26 of *Algorithms Comput. Math.*, pages 67–85. Springer, 2013.

[21] Louis H. Kauffman. State models and the Jones polynomial. *Topology*, 26(3):395–407, 1987.

[22] Patrick Kinnear. Skein module dimensions of mapping tori of the 2-torus. *Quantum Topol.*, 2025.

[23] Edwin Kitaeff. The Gilmer-Masbaum map is not injective on the Kauffman bracket skein module, 2024. arXiv:2410.23153 [math.GT]

[24] Greg Kuperberg. Spiders for rank 2 Lie algebras. *Comm.Math. Phys.*, 180(1):109–151, 1996.

[25] Abel Lacabanne, Daniel Tubbenhauer, and Pedro Vaz. Annular webs and Levi subalgebras. *J. Comb. Algebra*, 7(3):283–326, 2023.

[26] Genta Latifi and Daniel Tubbenhauer. Minimal presentations of gln-web categories, 2021. arXiv:2112.12688 [math.QA]

[27] Maciej Mroczkowski. Kauffman bracket skein module of a family of prism manifolds. *J. Knot Theory Ramifications*, 20(01):159–170, 2011.

[28] Greg Muller. Skein and cluster algebras of marked surfaces. *Quantum Topol.*, 7(3):435–503, 2016.

[29] Hitoshi Murakami, Tomotada Ohtsuki, and Shuji Yamada. HOMFLY polynomial via an invariant of colored plane graphs. *Enseign. Math.*, 44:325–360, 1998.

[30] Anup Poudel. A comparison between SL_n spider categories. *Canad. J. Math.*, pages 1–30, 2025.

[31] Józef H. Przytycki. Skein modules of 3-manifolds. *Bull. Pol. Acad. Sci. Math.*, 39(1-2):91–100, 1991.

[32] Józef H. Przytycki. A q-analogue of the first homology group of a 3-manifold. In *Perspectives on quantization (South Hadley, MA, 1996)*, volume 214 of *Contemp. Math.*, pages 135–144. Amer. Math. Soc., 1998.

[33] Józef H. Przytycki and Adam S. Sikora. On skein algebras and $Sl_2(\mathbb{C})$-character varieties. *Topology*, 39(1):115–148, 2000.

[34] Nicolai Y. Reshetikhin and Vladimir G. Turaev. Ribbon graphs and their invaraints derived from quantum groups. *Comm. Math. Phys.*, 127(1):1–26, 1990.

[35] Yuri Borisovich Rumer, Edward Teller, and Hermann Weyl. Eine für die Valenztheorie geeignete Basis der binären Vektorinvarianten. *Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl.*, 1932:499–504, 1932.

[36] Adam S. Sikora. Skein theory for $SU(n)$–quantum invariants. *Algebr. Geom. Topol.*, 5(3):865–897, 2005.

[37] Vladimir G. Turaev. Conway and Kauffman modules of a solid torus. *J. Sov. Math.*, 52(1):2799–2805, 1990.

Universität Hamburg
Bundesstraße 55
20146 Hamburg
GERMANY
E-mail address: `patrick.kinnear@uni-hamburg.de`