Knitted surfaces and their chart description: overview and the case of degree 2

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1 Introduction

In this paper, we give an overview of our papers [9, 10]. We assume that surfaces, embeddings and isotopies are smooth. Let D^2 be a 2-disk and let $I = [0, 1]$. Let Q_n be a set of m interior points of D^2. Let $p : D^2 \times I \to I$ be the natural projection. A *braid of degree* n or an n*-braid* is a 1-manifold b embedded in $D^2 \times I$ such that the restriction map $p|_b : b \to I$ is a covering map of degree n and $\partial b = b \cap \partial(D^2 \times I) = Q_m \times \partial I$. A braided surface or a surface braid is a notion which is given as a 2-dimensional analogous of a braid. Let $D^2 \times B^2$ be a bidisk. Let $p : D^2 \times B^2 \to B^2$ be the natural projection. A *braided surface* of degree n is an oriented surface S properly embedded in $D^2 \times B^2$ such that the restriction map $p|_S : S \to B^2$ is a branched covering map of degree n and ∂S is a closed braid in $D^2 \times \partial B^2$. A braided surface S is *simple* if the associated branched covering map $p|_S : S \to B^2$ is simple, that is, the number of the elements of $p^{-1}(y)$ is n or $n-1$ for each $y \in B^2$. A braided surface S is called a *surface braid* if ∂S is a trivial closed braid in $D^2 \times \partial B^2$. A simple braided surface has a graphical description called a *chart* description. A chart of degree n is a certain finite graph in B^2 with three types of vertices, each of whose edge is oriented and equipped with a label in $\{1, 2, \ldots, n-1\}$. We give [2, 3, 4, 5, 6, 7] for references.

In this paper, we extend the notion of a simple braided surface to a "knitted surface". From a braid b, we obtain a tangle called a "knit" by splicing some crossings of b. A knitted surface is given as a 2-dimensional analogous of a knit, and it is constructed as the trace of deformations of knits. We extend the notion of a chart of a simple braided surface to a chart of a knitted surface. Further, we investigate knitted surfaces of degree 2, using charts of degree 2.

The paper is organized as follows. In Section 2, we review knits and knitted surfaces. In Section 3, we review chart description of knitted surfaces. In Section 4, we observe knitted surfaces of degree 2.

2 Knits and knitted surfaces

2.1 Knits and the Knit monoid D_n

Let n be a positive integer. A *knit* of degree n [1, 8] is a tangle obtained from a braid of degree n in $D^2 \times I$ by splicing some crossings. A knit is constructed as a product of standard generators σ_i and their inverses σ_i^{-1} of the braid group B_n of degree n, and "hook pairs" τ_i which are obtained from $\sigma_i^{\pm 1}$ $(i = 1, \ldots, n-1)$. We consider that "hooks" exist only as hook pairs, always keeping the information of the pairs, and we describe the information of each hook pair by a segment called a "pairing" [9]. We give an equivalence relation to the set of n-knits so that two n-knits are equivalent if they are related by an isotopy of n-knits.

The *knit monoid* of degree n, denoted by D_n, is the set of equivalence classes of n-knits. Let e be the trivial braid. The unit element of D_n is represented by e. The n-knit monoid D_n has the monoid presentation

$$\left\langle \begin{array}{c} \sigma_1, \ldots, \sigma_{n-1}, \\ \sigma_1^{-1}, \ldots, \sigma_{n-1}^{-1}, \\ \tau_1, \ldots, \tau_{n-1} \end{array} \middle| \begin{array}{l} \sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = e, \ \sigma_i\tau_i = \tau_i\sigma_i = \tau_i, \\ \sigma_i\sigma_k = \sigma_k\sigma_i, \ \sigma_i\tau_k = \tau_k\sigma_i, \ \tau_i\tau_k = \tau_k\tau_i \quad (|i-k| > 1), \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, \ \sigma_i\sigma_j\tau_i = \tau_j\sigma_i\sigma_j \quad (|i-j| = 1) \end{array} \right\rangle_{\text{monoid}}.$$

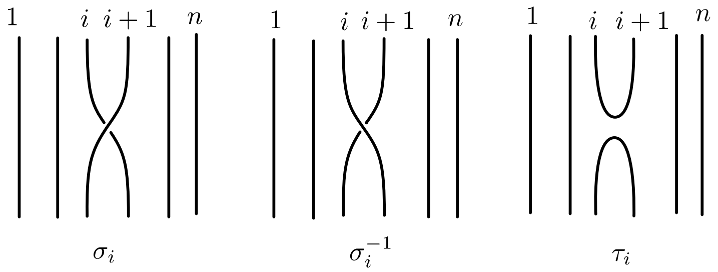

Figure 1: Generators σ_i, σ_i^{-1} and τ_i of the monoid D_n of n-knits.

2.2 Knitted surfaces

A knitted surface is a surface with no closed components, properly embedded in $D^2 \times B^2$, given as an analogous to a knit in $D^2 \times I$. A knitted surface S is given by using the notion of a "pairing" [10], which is an orientable surface embedded in $D^2 \times B^2$ describing pairings of fold singular points of S. A knitted surface of degree n is constructed as the trace of deformations of knits, as follows. Let $\beta_0 \to \beta_1 \to \beta_2 \to \cdots \beta_m$ be a sequence of knits of degree n presented by a knit monoid presentation consisting of σ_i, σ_i^{-1} and τ_i $(i = 1.\ldots, n-1)$ such that each sequence $\beta_{j-1} \to \beta_j$ satisfies one of the following.

(Case 1) β_{j-1} and β_j is related by an isotopy of n-knits.

(Case 2) β_{j-1} and β_j are related by a transformation of subwords $e \leftrightarrow \sigma_i^{\epsilon}$ or $e \leftrightarrow \tau_i$. We define $\beta_{j-1} \to \beta_j$ to be the transformation such that β_j is obtained from β_{j-1} by band surgery along a band.

(Case 3) β_{j-1} and β_j are related by a transformation of subwords $\tau_i \leftrightarrow \tau_i\tau_i$. We define $\beta_{j-1} \to \beta_j$ to be the transformation such that the part $\tau_i\tau_i$ is obtained from τ_i by adding a disk D whose boundary is the simple closed curve in $\tau_i\tau_i$ and then deleting $\mathrm{Int}(D)$.

A knitted surface S of degree n is constructed by taking the trace of such a sequence $\beta_0 \to \beta_1 \to \beta_2 \to \cdots \beta_m$, that is, when we denote by β^t $(t \in [0,1])$ the isotopy of knits (with attached bands/disks) associated with $\beta_0 \to \beta_1 \to \beta_2 \to \cdots \beta_m$, $S = \cup_{t\in[0,1]}(\beta^t \times \{t\})$. See [9, Sections 3,4] for more precise explanation.

Theorem 2.1 *[9, Theorem 6.1] Every compact surface with no closed components smoothly properly embedded in $D^2 \times B^2$ is smoothly isotopic to some knitted surface.*

This theorem extends the result of Rudolph [11] that every orientable ribbon surface is smoothly isotopic to a braided surface. A *surface-link* is a closed surface smoothly embedded in $\mathbb{R}^4$. A knitted surface S in $D^2 \times B^2$ is called a *2-dimensional knit* if $\partial S = S \cap (D^2 \times \partial B^2)$ is the closure of a trivial braid. In particular, when a 2-dimensional knit is a simple braided surface, it is called a *2-dimensional braid*. For a 2-dimensional knit S of degree n in $D^2 \times B^2$, we embed $D^2 \times B^2$ in $\mathbb{R}^4$ and paste n disks trivially to the boundary of ∂S to obtain a surface-link $\hat{S}$ in $\mathbb{R}^4$. We call $\hat{S}$ the *closure* of a knitted surface S. It is known [5] that every oriented surface-link in $\mathbb{R}^4$ is ambient isotopic to the closure of some 2-dimensional braid.

Theorem 2.2 *[10, Theorem 1.1] Every surface-link in $\mathbb{R}^4$ is ambient isotopic to the closure of some 2-dimensional knit.*

3 Chart description of knitted surfaces

A *BMW chart* or simply a *chart* of degree n of a knitted surface is a finite graph in B^2 such that (1) each edge is either oriented or unoriented, and (2) each edge is equipped with a label in $\{1, 2, \ldots, n-1\}$, and (3) each vertex is one of those given in Figure 2; see [9, Definition 4.5]. When a BMW chart has no unoriented edges, then it is a chart for a simple braided surface. A BMW chart Γ in B^2 has a surface $S(\Gamma)$ in $D^2 \times B^2$ described by transformations of knits. An interior point of an oriented edge of Γ with the label i corresponds to the crossing of a letter σ_i or σ_i^{-1} of a knit describing $S(\Gamma)$, and an interior point of an unoriented edge of Γ with the label i corresponds to a hook pair τ_i of a knit describing $S(\Gamma)$.

Theorem 3.1 *[9, Theorem 4.8] A knitted surface has a BMW chart description. More precisely, for a knitted surface S, there exists a BMW chart Γ such that S and $S(\Gamma)$ are equivalent.*

Surfaces with no closed components properly embedded in $D^2 \times B^2$ can be investigated through BMW charts. We define chart moves or C-moves consisting of T, CI, CII, CIII-moves, which are local modifications for charts [9, Definition 7.1]. Further, we give a set of explicit CI-moves [9, Definition 7.3].

Theorem 3.2 *[9, Theorem 7.5] Two knitted surfaces of degree n are equivalent if their presenting charts are related by a finite number of C-moves consisting of T, CII, CIII-moves and CI-moves of type (A1)–(F3).*

4 Knitted surfaces of degree 2

In this section, we consider knitted surfaces of degree 2.

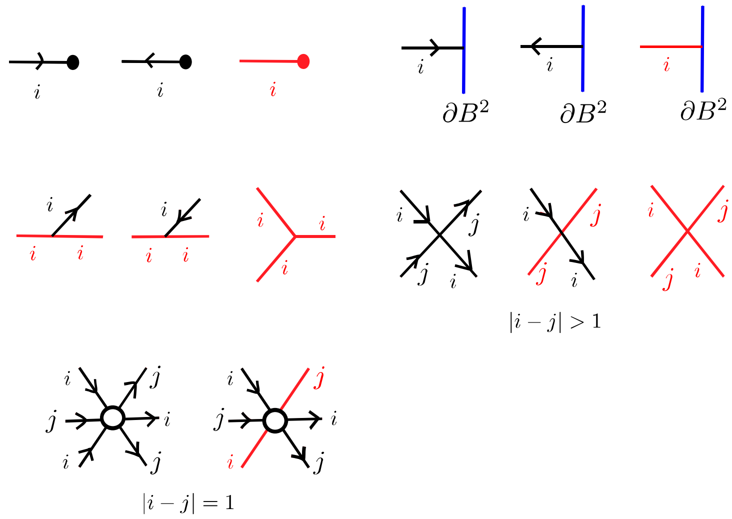

Figure 2: Vertices of a BMW chart.

4.1 Main results

For a knitted surface S of degree $2n$ satisfying certain boundary conditions, we consider another type of the closure of S, called the "plat closure" of S, which is a surface-link obtained from S by pasting several surfaces. For the case of degree 2 such that ∂S is the closure of σ_1^m, the plat closure of S is obtained by pasting an annulus or a Möbius band B with m twist such that $\partial B = \partial S$.

A *standard 2-sphere* (respectively a *standard torus*) in $\mathbb{R}^4$ is the boundary of a 3-ball (respectively an unknotted solid torus) in $\mathbb{R}^3 \times \{0\}$, and a *standard positive* (respectively *negative*) *projective plane* is the surface given in Figure 41 (1) (respectively (2)) in [9]. These surfaces are called *standard surfaces*. A surface-knot in $\mathbb{R}^4$ is called *trivial* if it is a connected sum of a finite number of standard surfaces in $\mathbb{R}^4$, and a surface-link is *trivial* if it is a split union of a finite number of trivial surface-knots.

Theorem 4.1 *[10, Theorem 1.2] The plat closure of any knitted surface of degree 2 is a trivial surface-link, and any trivial surface-link is ambient isotopic to the plat closure of a knitted surface of degree 2.*

As a corollary of Theorem 4.1, we have the following.

Corollary 4.2 *[10, Corollary 1.3] The closure of any 2-dimensional knit degree 2 is a trivial surface-link, and any trivial surface-link is ambient isotopic to the closure of some 2-dimensional knit of degree 2.*

4.2 Key Theorem

For a BMW chart, we call an oriented (respectively unoriented) edge a *σ-edge* (respectively a *τ-edge*. A σ-/τ-edge of a chart is called a *free σ-/τ-edge* if its endpoints are vertices of degree one. A σ-edge of a chart is called a *half σ-edge* if its endpoints consists of a vertex of degree one and either a boundary point or a trivalent vertex connected with a pair of τ-edges. When a half σ-edge has an orientation toward (respectively from) the vertex of degree 2, we call it *positive* (respectively *negative*). We denote by T a standard torus, and by P_+ (respectively P_-) a standard positive (respectively negative) projective plane.

Theorem 4.3 *[9, Theorem 8.1] Let F be a trivial surface-knot, which is a connected sum of g copies of T and m copies of P_+ and n copies of P_-. Then F is the plat closure of the knitted surface presented by a* 2*-chart Γ such that Γ consists of* $\min\{m, n\}$ *copies of free σ-edges, g copies of free τ-edges, and $|m - n|$ copies of positive (respectively negative) half σ-edges if $m \geq n$ (respectively $m < n$).*

The second part of Theorem 4.1 is a corollary of Theorem 4.3. The first part of Theorem 4.1 is shown by considering the “normal form” of knitted surface of degree 2, which is given using charts of degree 2 [10].

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