Infinitely many virtual knots whose virtual unknotting number equals one and a sequence of n-writhes

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1 Introduction

Satoh and Taniguchi [6] introduced a virtual knot invariant J_n, called the n-writhe, for any non-zero integer n. The n-writhes give the coefficients of several polynomial invariants for virtual knots. They gave a necessary and sufficient condition for a sequence of integers to be that of the n-writhes of a virtual knot as follows.

Theorem 1 ([6]). *The sequence of n-writhes $\{J_n(K)\}_{n\neq 0}$ of a virtual knot K satisfies $\sum_{n\neq 0} nJ_n(K) = 0$. Conversely, for any sequence of integers $\{r_n\}_{n\neq 0}$ with $\sum_{n\neq 0} nr_n = 0$, there exists a virtual knot K such that $J_n(K) = r_n$ for any $n \neq 0$.*

It is well known that local moves in Fig. 1 are unknotting operations for classical knots or virtual knots. The crossing change and the Delta-move are unknotting operations for classical knots, and the virtualization and the forbidden moves are unknotting operations for virtual knots. The unknotting number by the virtualization is called the virtual unknotting number and is denoted by u^v. The details of local moves and unknotting numbers are elaborated in Section 2.

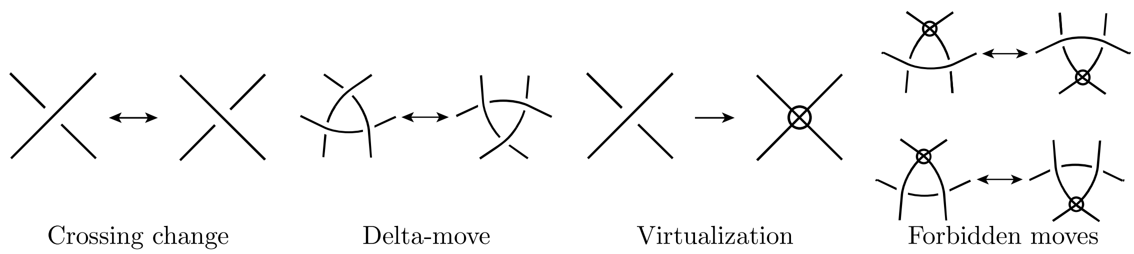

Figure 1: Local moves.

There are some studies on unknotting operations and n-writhes, where the changes of the values of n-writhes by some unknotting operations are calculated. For a crossing change and a Delta move see [6], and for forbidden moves see [5]. In our previous papers (see [2] and [3]), the following results were obtained regarding the virtualization.

Theorem 2 ([2])**.** *For any given non-zero integer n and any given integer m, there exists a virtual knot K such that $u^v(K) = 1$ and $J_n(K) = m$.*

Theorem 3 ([3])**.** *Let $\{r_n\}_{n\neq 0}$ be a sequence of integers. If $\sum_{n\neq 0} nr_n = 0$, then there exists a virtual knot K such that*

$$u^v(K) = 1 \text{ and } J_n(K) = r_n$$

for any $n \neq 0$.

In this paper, we obtain a more effective result.

Theorem 4. *Let $\{r_n\}_{n\neq 0}$ be a sequence of integers. If $\sum_{n\neq 0} nr_n = 0$, then there exist infinitely many virtual knots K_m ($m \in \mathbb{N}$) such that*

$$u^v(K_m) = 1 \text{ and } J_n(K_m) = r_n$$

for any $n \neq 0$.

In Section 2, we review the basic notions of virtual knots and definitions of some virtual knot invariants. In Section 3, we explain the outline of the proof of Theorem 4.

2 Preliminaries

2.1 Virtual knots and local moves

A *virtual knot diagram* is a generalization of a knot diagram and it has virtual crossings as well as real crossings in Fig. 2. We say that two virtual knot diagrams are *equivalent* if one can be obtained from the other by a finite sequence of *generalized Reidemeister moves* in Fig. 3 [1]. A *virtual knot* is an equivalence class of virtual knot diagrams under the generalized Reidemeister moves.

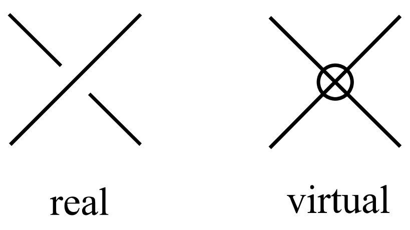

Figure 2: Crossing types.

Let K be a virtual knot and D a virtual knot diagram of K. Then, D is regarded as the image $f(\mathbb{S}^1)$ of a generic immersion $f : \mathbb{S}^1 \to \mathbb{R}^2$. A *Gauss diagram* of D is the preimage of D with chords, each of which connects the preimages of each real crossing. We specify over/under information of each real crossing on the corresponding chord by directing the chord toward the under path and assign each chord with the sign of the crossing in Fig. 4. In what follows, we suppose that virtual knot diagrams are oriented.

It is well-known that there exists a bijection from the set of virtual knots to the set of equivalence classes of their Gauss diagrams modulo the *generalized Reidemeister moves*

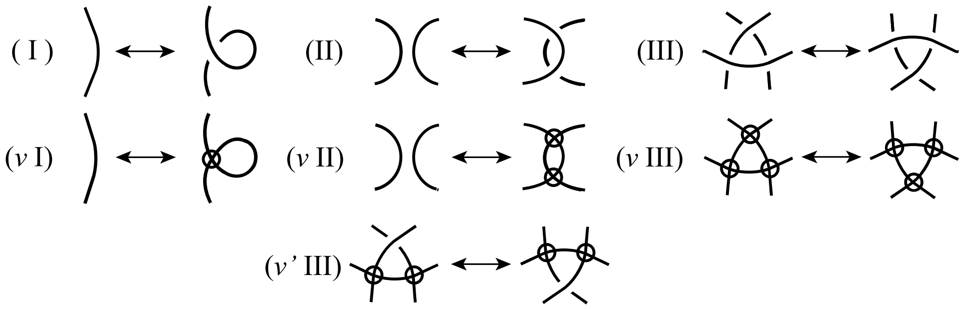

Figure 3: Generalized Reidemeister moves.

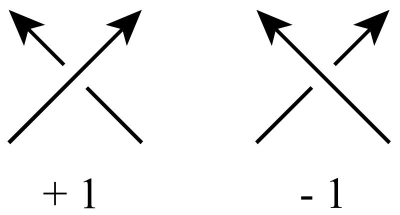

Figure 4: The signs of real crossings.

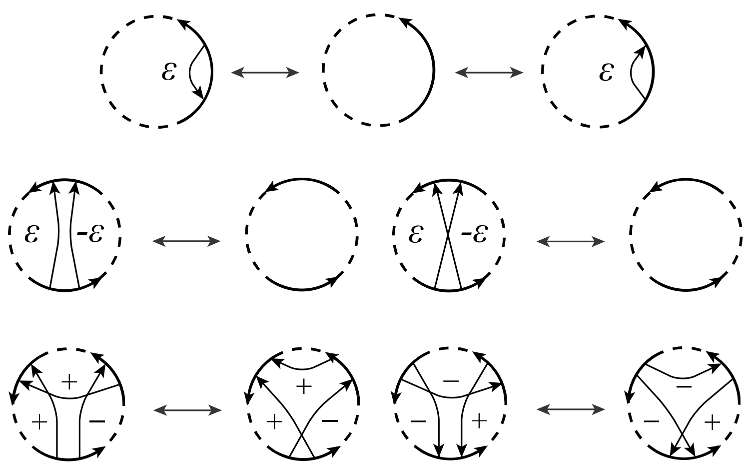

Figure 5: Generalized Reidemeister moves of Gauss diagrams.

of Gauss diagrams as shown in Fig. 5. We identify a virtual knot with an equivalence class of Gauss diagrams.

A local modification on a virtual knot diagram is called a *local move*. Generalized Reidemeister moves are local moves. Let us fix a local move. If any virtual knot diagram is transformed into a trivial knot diagram by a finite sequence of the local moves and generalized Reidemeister moves, then the local move is called an *unknotting operation* of virtual knots. The local move on virtual knot diagrams in Fig. 6 is called a *virtualization*. It is obvious that a virtualization is an unknotting operation.

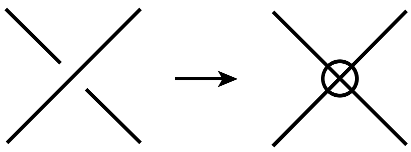

Figure 6: Virtualization.

For an unknotting operation, the minimum number of the operations needed to transform a diagram of a virtual knot K into a trivial knot diagram is called the *unknotting number* of K by the unknotting operation. When we operate the local move, we are allowed to do generalized Reidemeister moves before or after the operation. The unknotting number of a virtual knot K by the virtualization is called the *virtual unknotting number* and denoted by $u^v(K)$. We identify a virtual knot K and the Gauss diagram G associated with K. When we consider the Gauss diagram, we use the notation $u^v(G)$ instead of $u^v(K)$, that is, $u^v(G)$ means $u^v(K)$ where K is a virtual knot whose Gauss diagram is G.

2.2 Invariants for virtual knots

We review the definition of n-writhes and their properties

Definition 5 ([6])**.** *Let K be a virtual knot with a Gauss diagram G consisting of an oriented circle $\mathbb{S}^1$ together with signed, oriented m chords connecting $2m$ points on $\mathbb{S}^1$. Let $c = \overrightarrow{PQ}$ be the chord in G with the sign ε where c is oriented from P to Q. We assign the signs $-\varepsilon$ and ε to the endpoints P and Q, respectively. For a chord $c = \overrightarrow{PQ}$ in a Gauss diagram G, the specified arc γ of c is the arc in $\mathbb{S}^1$ with endpoints P and Q oriented from P to Q along the orientation of $\mathbb{S}^1$ as shown in Fig. 7. The* index *of c is the sum of the signs of all the points on γ except P and Q. We denote it by $i(c)$. For an integer n, the n-*writhe *of G is*

$$J_n(G) = \sum_{i(c)=n} \varepsilon(c).$$

*The integer $J_n(G)$ defines an invariant of K for $n \neq 0$. It is called the n-*writhe *of K and is denoted by $J_n(K)$.*

And it holds Theorem 1 in Section 1.

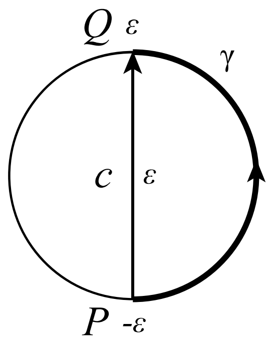

Figure 7: The specified arc in the circle of a Gauss diagram.

We review the definition of the first intersection polynomial. The endpoints of a chord of a Gauss diagram G divide the circle $\mathbb{S}^1$ into two arcs. Let $\alpha \subset \mathbb{S}^1$ be the arc from the tail of the chord to the head of it along the orientation of $\mathbb{S}^1$ and $P(\alpha)$ the set of endpoints of the chords of G in the interior of α. For an endpoint $\rho \in P(\alpha)$, we denote by $\varepsilon(\rho)$ the sign of ρ, and by $\tau(\rho)$ the other endpoint of the chord incident to ρ. Let $\overline{\alpha}$ be the complementary arc of $\alpha \subset \mathbb{S}^1$, and α and $\beta \subset \mathbb{S}^1$ arcs for distinct chords a and b of G, respectively. A numerical value $S(\alpha, \beta)$ is defined by

$$S(\alpha, \beta) = \sum_{\rho \in P(\alpha), \tau(\rho) \in P(\beta)} \varepsilon(\rho).$$

We say that the chords a and b of G are *linked* if their endpoints appear on $\mathbb{S}^1$ alternately, and otherwise *unlinked*. Then the intersection number $\alpha \cdot \beta$ of the arcs α and β is calculated as below.

1. If a and b are unlinked, then $\alpha \cdot \beta = S(\alpha, \beta)$ and
2. if they are linked for $\varepsilon, \delta \in \pm$ as shown in Fig. 8, then it holds that $\alpha \cdot \beta = S(\alpha, \beta) + \frac{1}{2}(\varepsilon + \delta)$ and $\beta \cdot \alpha = S(\beta, \alpha) - \frac{1}{2}(\varepsilon + \delta)$.

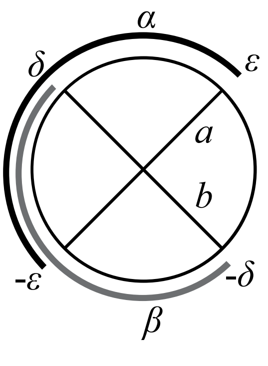

Figure 8: Linked chords a and b.

Definition 6. *[4] Let D be a diagram of a virtual knot K with m crossings c_i for $1 \leq i \leq m$, G the Gauss diagram of D, γ_i the arc oriented from the tail of c_i to its head along the orientation of $\mathbb{S}^1$, and $\overline{\gamma_i}$ the other arc associated with c_i. The following Laurent polynomials $W_D(t)$, $f_{01}(D)$ and $I_D(t)$ are defined by*

$$W_D(t) = \sum_i \varepsilon_i (t^{\gamma_i \cdot \overline{\gamma_i}} - 1),$$

$$f_{01}(D) = \sum_{i,j} \varepsilon_i \varepsilon_j (t^{\gamma_i \cdot \overline{\gamma_i}} - 1) \text{ and}$$

$$I_D(t) = f_{01}(D) - w_D W_D(t),$$

where ε_i is the sign of c_i and $w_D = \sum_{i=1}^m \varepsilon_i$ is the writhe of D. The polynomial $I_D(t)$ is an invariant of K which is called the first intersection polynomial *and denoted by $I_K(t)$. The n-th coefficient of $W_D(t)$ is equal to the value of $J_n(D)$ and $\gamma_i \cdot \overline{\gamma_i}$ is equal to the value of the index $i(c_i)$.*

Notation 7. *Let G_1 and G_2 be Gauss diagrams having two chords c_0 and c_0' respectively. If c_0 and c_0' have the same sign and $i(c_0) = i(c_0') = 0$, then the* vertex connected sum $G_1 \natural G_2$ *with respect to c_0 and c_0' is the Gauss diagram obtained by removing the interiors of regular neighborhoods of the head of c_0 and the tail of c_0' from the diagrams and connecting them as shown in Fig. 9.*

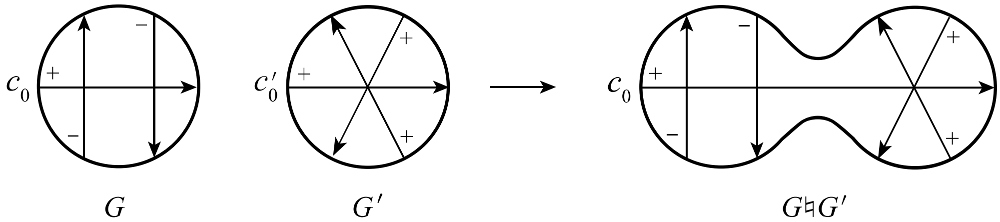

Figure 9: The vertex connected sum $G \natural G'$.

3 Outline of the proof of Theorem 4.

We prepare a notation and some lemmas for the outline of the proof of Theorem 4.

Notation 8. *Let G be a Gauss diagram with the chords c_i for $0 \leq i \leq s$, and γ_i the arc associated with the chord c_i. We define the sets of the numbers $M_{k\ell}$ for $k, \ell = 0, 1$ on the Gauss diagram G as follows:*

$$M_{11}(G) = \{ i \mid \gamma_i \text{ has both the head of } c_0 \text{ and the tail of it} \},$$
$$M_{10}(G) = \{ i \mid \gamma_i \text{ has the head of } c_0 \text{ but doesn't have the tail of it} \} \text{ and}$$
$$M_{01}(G) = \{ i \mid \gamma_i \text{ doesn't have the head of } c_0 \text{ but has the tail of it} \},$$
$$M_{00}(G) = \{ i \mid \gamma_i \text{ has neither the head of } c_0 \text{ nor the tail of it} \}.$$

Lemma 9. *Let L be a virtual knot with the Gauss diagram H as shown in Fig. 10. Let K be a virtual knot and G be a Gauss diagram of K with the chords c'_i for $0 \le i \le s$. The chords c'_i has the sign ε'_i. Let γ'_i be the arc of G associated with c'_i. Then, for the virtual knot $L \natural K$ with the vertex connected sum $H \natural G$ with respect to the chords c_0 and c'_0, we obtain*

$$\begin{aligned}
&I_{L\natural K}(t) = I_L(t) + I_K(t) \\
&+ \sum_{i \in M_{11}(G)} \varepsilon'_i (t^{\gamma'_i \cdot \overline{\gamma'_i}} - 1)(t^{-1} - 1) + \sum_{i \in M_{10}(G)} \varepsilon'_i (t^{\gamma'_i \cdot \overline{\gamma'_i}} - 1)(t^{-1} - 1) \\
&+ \sum_{i \in M_{01}(G)} \varepsilon'_i (t^{\gamma'_i \cdot \overline{\gamma'_i}} - 1)(t - 1) + \sum_{i \in M_{00}(G)} \varepsilon'_i (t^{\gamma'_i \cdot \overline{\gamma'_i}} - 1)(t - 1).
\end{aligned}$$

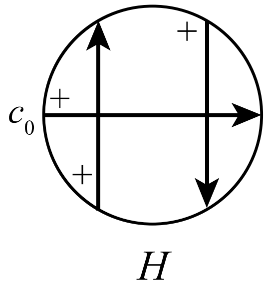

Figure 10: The Gauss diagram H.

Lemma 10. *Let M be a virtual knot with the Gauss diagram I as shown in Fig. 11. Let K be a virtual knot and G a Gauss diagram of K with the chords d_i for $0 \le i \le s$. The chord d_i has the sign δ_i. Let ζ_i be the arc of G associated with d_i Then, for the virtual knot $K \natural M$ with the vertex connected sum $G \natural I$ with respect to the chords d_0 and d'_0, we obtain*

$$\begin{aligned}
&I_{K\natural M}(t) = I_K(t) + I_M(t) \\
&- \sum_{i \in M_{11}(G)} \delta_i (t^{\zeta_i \cdot \overline{\zeta_i}} - 1)(t - 1) - \sum_{i \in M_{10}(G)} \delta_i (t^{\zeta_i \cdot \overline{\zeta_i}} - 1)(t^{-1} - 1) \\
&- \sum_{i \in M_{01}(G)} \delta_i (t^{\zeta_i \cdot \overline{\zeta_i}} - 1)(t - 1) - \sum_{i \in M_{00}(G)} \delta_i (t^{\zeta_i \cdot \overline{\zeta_i}} - 1)(t^{-1} - 1).
\end{aligned}$$

Lemma 11. *Let K be a virtual knot and G a Gauss diagram of K with the chords d_i for $0 \le i \le s$. The chord d_i has the sign δ_i. Let ζ_i be the arc of G associated with d_i. Then for a natural number m, we obtain*

$$\begin{aligned}
&I_{\underbrace{L\natural \ldots \natural L}_{m} \natural K \natural \underbrace{M \natural \ldots \natural M}_{m}}(t) = I_K(t) + m \sum_{i \in M_{11}(G)} \delta_i (t^{\zeta_i \cdot \overline{\zeta_i}} - 1)(t^{-1} - t) \\
&+ m \sum_{i \in M_{00}(G)} \delta_i (t^{\zeta_i \cdot \overline{\zeta_i}} - 1)(t - t^{-1}).
\end{aligned}$$

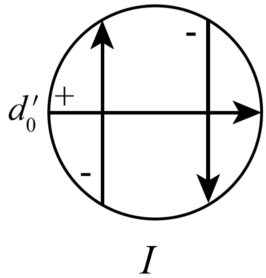

Figure 11: The Gauss diagram I.

Here, we explain the outline of the proof of Theorem 4. the virtual knot K is the virtual knot with the Gauss diagram $\mathcal{G}$ which we constructed in [3]. Let $\mathcal{G}_m$ be the Gauss diagram obtained from the Gauss diagrams H, $\mathcal{G}$ and I by the vertex connected sum with respected to their horizontal chords as shown in Fig. 12.

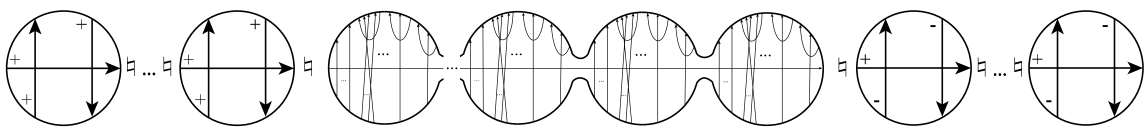

Figure 12: A Gauss diagram $\mathcal{G}_m$

Let $\mathcal{K}_m$ be the virtual knot with the Gauss diagram $\mathcal{G}_m$. We see that it holds that $u^v(K_m) = 1$ and $J_n(K_m) = r_n$ from the construction. From Lemma 11, we have

$$\begin{aligned} &I_{\mathcal{K}_m}(t) = I_{\mathcal{K}}(t) \\ &+ m(t^{-1} - t) \sum_{u=A}^{-2} \sum_{v=u}^{-2} -r_u \left(t^{v+1} - 1\right) + m(t - t^{-1}) \sum_{u=2}^{B} \sum_{v=2}^{u} -r_u \left(t^{v-1} - 1\right) \\ &= I_{\mathcal{K}}(t) + m(-r_A)t^A + \cdots + m(-r_B)t^B, \end{aligned}$$

where A be the minimum degree of $W_{\mathcal{K}_m}(t)$ and B the maximum degree of it. It holds that $I_{\mathcal{K}_m}(t) \neq I_{\mathcal{K}_s}(t)$ for $m \neq s$. Therefore $\mathcal{K}_m \nsim \mathcal{K}_s$ for $m \neq s$. This completes the proof of Theorem 4.

□

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