Elements preserving the standardness of a parabolic subgroup in spherical-type Artin groups

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Abstract

A classic result by Benardete, Gutierrez and Nitecki shows that if the action of a braid α on the curve complex sends a standard curve into another standard curve, then the standardness is also preserved after applying each factor of the normal form of α. This survey aims to explain how generalize this result into the frame of Artin groups following a result of Digne and Michel.

1 Introduction

The braid group on n strands, originally introduced by Artin in the early 20th century, can be interpreted from multiple perspectives. Topologically, it coincides with the mapping class group of the n-punctured disk—homeomorphisms fixing the boundary, up to isotopy. From an algebraic viewpoint, it belongs to the class of Artin groups, with the standard presentation:

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{if } |i-j|>1 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{if } |i-j|=1 \end{array} \right\rangle.$$

In recent decades, significant progress in understanding braid groups has emerged from the interplay between combinatorial algebra and geometric topology. On one side, Garside theory provides algebraic frameworks for studying normal forms and decision problems. On the other hand, the topological perspective—via the action of the braid group on the *curve complex* of the punctured disk—has offered deep geometric insights.

The curve complex associated to a surface is a simplicial complex whose vertices correspond to isotopy classes of essential simple closed curves, with higher-dimensional simplices representing collections of such curves that can be realized disjointly. The mapping class group of the surface acts naturally on this complex by sending curves to their images under homeomorphisms. This action underlies the Nielsen–Thurston classification, which classifies elements of the mapping class group into three types. In particular, an element g of a braid group is called:

- periodic: some power of g is central;
- reducible: some power of g preserves a nontrivial multicurve;
- pseudo-Anosov: if neither of the above conditions holds. In this case, there exist a pair of transverse measured foliations on the surface that are invariant under the action of g, scaled respectively by a factor $\lambda > 1$ and $1/\lambda$ each time g acts.

In a foundational result from the 1980s, Birman, Lubotzky, and McCarthy [3] showed that every reducible braid g —and in general every reducible mapping class— admits a canonical system of invariant curves. This collection partitions the surface into subsurfaces, each of which is preserved by some power of g, and on which the induced action is either periodic or pseudo-Anosov. This decomposition provides a deep insight into the internal structure of reducible mapping classes.

The algebraic generalization of braid groups are Artin groups. An Artin group is defined by a generating set S and a symmetric matrix $(m_{s,t})_{s,t\in S}$, with relations of the form:

$$A_S = \langle S \mid \underbrace{sts\cdots}_{m_{s,t}} = \underbrace{tst\cdots}_{m_{s,t}},\ m_{s,t} < \infty \rangle.$$

The braid group B_n is the special case where the associated Coxeter group, obtained by adding $s^2 = 1$ for each generator, is the symmetric group, which is finite. Any Artin group associated to a finite Coxeter group is said to have spherical type, and all of them admit a Garside structure. The main question that motivates this survey is the following:

Main question. *How topological properties of the braid group generalize to spherical-type Artin groups?*

In the case of braid groups, vertices of the curve complex correspond to a specific family of subgroups: the irreducible parabolic subgroups. A *standard parabolic subgroup* $A_X \subseteq A_S$ is generated by a subset $X \subseteq S$; in braid groups, when X consists of consecutive generators, this subgroup corresponds to a multicurve encircling the associated punctures. Van der Lek proved in his thesis [14] that every standard parabolic subgroup is an Artin group with the same relations present in the bigger Artin groups where the generators of X are involved. We say that an Artin group (or a standard parabolic subgroup) is *irreducible* if it cannot decompose as a direct product Artin groups. More generally, parabolic subgroups are conjugates of standard ones. If $\alpha \in B_n$, then $\alpha A_X \alpha^{-1}$ corresponds to the image of the original curve under the mapping class represented by α. An irreducible parabolic subgroup is the conjugate of an irreducible standard parabolic subgroup. In braids, it corresponds to a single curve.

Motivated by this analogy, one can define a flag simplicial complex for general Artin groups, mimicking the curve complex. In this complex, the vertices are irreducible parabolic subgroups and two irreducible parabolic subgroups P and Q span an edge if one is contained in the other, or if their intersection is trivial and $pq = qp$ for all $(p, q) \in P \times Q$. In spherical-type Artin groups, the Garside structure allows an efficient criterion: it suffices to check whether the central elements of two such subgroups commute [5].

The following analogy that one would like to find is if we can classify elements on an Artin group in the same manner we do with braids and use this classification in an effective way. We could naturally stablish the following classification:

An element g of an Artin group can be

- **Periodic**, if a power of g is central;
- **Reducible**, if a power of g preserves a non-trivial family of parabolic subgroups under conjugacy;

- **Pseudo-Anosov**, if no power of the braid preserves any parabolic subgroup under conjugacy.

However, if one wants to find an analogue of the canonical reduction system in this setting, there are many non-trivial questions that need to be addressed. For example, what does it mean to decompose a subgroup along a set of disjoint parabolic subgroups? To get a sense of the topological questions that require a Garside-theoretical generalization, we present in [6] an algorithm to compute the canonical reduction system of a braid using Garside theory. One of the results we use in constructing the algorithm is a classic theorem by Benardete, Gutierrez, and Nitecki [2]. To understand this result, one must first understand how the Garside structure works for braids and how it leads to the construction of a normal form.

Let A be a spherical-type Artin group with standard generating set S. To study its structure, we focus on several key elements and properties:

- The positive monoid A^+, consisting of the elements of A that can be expressed as products of generators in S without using inverses.
- A prefix partial order $\preceq$, defined by: $a \preceq b$ if there exists $c \in A^+$ such that $b = ac$. In this case, a is said to be a prefix of b. This order is left-invariant and admits both least common multiples $a \vee b$ and greatest common divisors $a \wedge b$ for any $a, b \in A$.
- The Garside element Δ, which is defined as the least common multiple of the generators: $\Delta = \bigvee_{s \in S} s$. It satisfies the conjugation property $\Delta A^+ \Delta^{-1} = A^+$.
- The simple elements, which are defined as the prefixes of Δ. These elements not only play a central role in the group's structure, but they also generate the entire group.
- The group A is atomic, meaning that the length of a positive element (when written as a product of atoms, i.e., generators) is bounded above — and this bound is in fact uniform for all such expressions.

In parallel, one can define the suffix order $\succeq$ by declaring $a \succeq b$ if $a = cb$ for some $c \in A^+$. In this case, b is a suffix of a. This order also admits unique least common multiples and greatest common divisors, and under it, Δ remains the least common multiple of the atoms. Unless otherwise stated, we will work with the prefix order.

These notions allow us to express any element $x \in A$ in a canonical form. In the prefix setting, we have the left normal form:

$$x = \Delta^p x_1 \cdots x_r,$$

where each $x_i \neq \Delta$ is a simple element and satisfies the condition $x_i x_{i+1} \wedge \Delta = x_i$. This solution to the word problem for braids is attributed to separate works of Garside, Adyan and Elrifai and Morton [1,8,9], while the generalization of these combinatorial properties to spherical-type Artin group is due to Brieskorn and Saito [4].

Given a braid α and a curve C in D_n, we denote by C^α the result of the action of α on C. Embed D_n in the complex plane in such a way that the punctures of the disc lie on the real axis. We say that a curve in D_n is *standard* if it only intersects the real axis in two points. This is the same as saying that it correspond to a standard (irreducible) parabolic subgroup.

Theorem 1 ([2, Theorem 5.7] and [12, Theorem 3.8])**.** *Let C be a standard curve and let $\alpha = \Delta^k x_1 \cdots x_r$ be a braid in left normal form. If C^α is standard, then $C^{\Delta^k x_1 \cdots x_i}$, for $i \in \{1, \ldots, r\}$ is standard.*

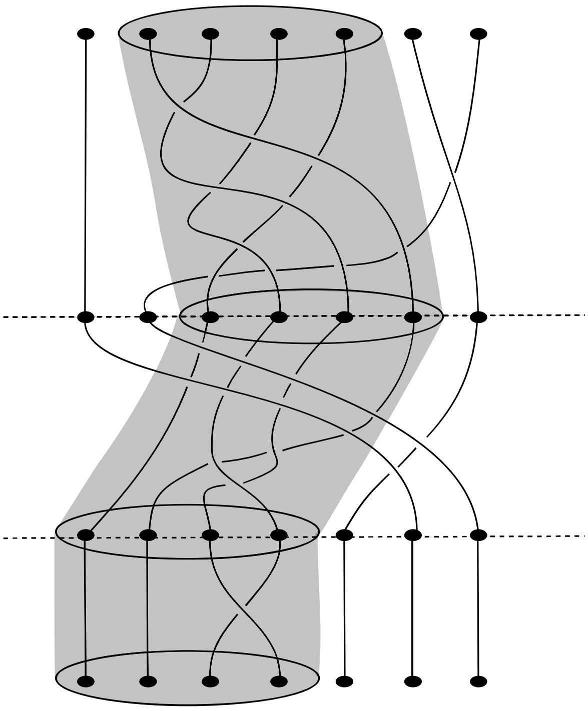

Figure 1: An example of Theorem 1. The dashed lines separate the factors of the braid normal form.

For the algorithm presented in [6], this theorem plays a crucial role. The main idea is as follows: given a reducible braid α, we aim to find a conjugate β that admits a standard canonical reduction system. To do this, we consider specific conjugates that preserve standard systems of curves and identify which of them has a standard canonical reduction system.

At times, we need to manipulate braids by altering their normal forms. A classical operation on braids is known as cycling [8]. If a braid α is written in left normal form as $\alpha = \Delta^k x_1 \cdots x_N$, then its cycling, denoted $c(\alpha)$, is the braid:

$$c(\alpha) = \Delta^k x_2 \cdots x_N (\Delta^k x_1 \Delta^{-k}).$$

In essence, we cyclically permute the simple factors of the normal form, conjugating appropriately by powers of Δ.

What we want is the following: if a braid preserves a standard curve, then its cycling should also preserve a standard curve. This is captured in the following corollary:

Corollary 2. *Let α be a braid. If C is a standard curve preserved by α, then $C^{\Delta^k x_1 \Delta^{-k}}$ is a standard curve preserved by $c(\alpha)$.*

Proof. Let $\alpha = \Delta^k x_1 \cdots x_N$ be the left normal form of α. By Theorem 1, the fact that C^α is standard implies that

$$C_1 := C^{\Delta^k x_1}$$

is also a standard curve. Therefore, x_1 sends C^{Δ^k} to C_1, and so the conjugating element used in the cycling operation, $\Delta^k x_1 \Delta^{-k}$, sends C to $(C_1)^{\Delta^{-k}}$, which remains a standard curve.

Finally, observe that

$$\left(C^{\Delta^k x_1 \Delta^{-k}}\right)^{c(\alpha)} = \left(C^{\Delta^k x_1 \Delta^{-k}}\right)^{(\Delta^k x_1^{-1} \Delta^{-k})\alpha(\Delta^k x_1 \Delta^{-k})} = C^{\alpha(\Delta^k x_1 \Delta^{-k})} = C^{\Delta^k x_1 \Delta^{-k}}.$$

This proves that $C^{\Delta^k x_1 \Delta^{-k}}$ is preserved by $c(\alpha)$. □

Since the algorithm in [6] was designed to identify which topological properties in the construction of a canonical reduction system can be translated into algebraic ones using Garside theory, the aim of this survey is to take a step further by explaining how Theorem 1 generalizes to spherical-type Artin groups.

2 Elements sending standard parabolic subgroups to standard parabolic subgroup

The main result of this survey is an immediate corollary of an unpublished result by Digne and Michel [7]. Their result was further generalized to Garside groups by Eddy Godelle [11]. A braid that sends a standard curve C_1 to a standard curve C_2 always contains a tube connecting these two curves. The portion of the braid that lies inside this tube is a braid on fewer strands—specifically, the number of punctures enclosed by the standard curve. More formally, the braid can be expressed as $\beta = \beta_i \beta_e$, where β_i consists only of generators moving the punctures enclosed by C_1, and β_e is such that the braid inside the tube connecting C_1 and C_2 is trivial (see Figure 2). We refer to β_e, both in this context and more generally in the setting of Artin groups, as a *ribbon*.

Given an Artin group of spherical type A_S, a subset $X \subset S$, and an element $t \in S \setminus X$, we define

$$r_{X,t} = \Delta_X^{-1} \Delta_{X \cup \{t\}}.$$

For any X and t, both $r_{X,t}$ and its inverse $r_{X,t}^{-1}$ are called *elementary ribbons*. Note that conjugation by $r_{X,t}$ always sends the subset X to a subset $Y \subseteq X \cup \{t\}$. Thus, we say that $r_{X,t}$ is an *elementary X-ribbon-Y*, and similarly $r_{X,t}^{-1}$ is an elementary Y-ribbon-X. More generally, we say that an element is an *X-ribbon-Y* if it can be written as a product $r_1 \cdots r_m$, where each r_i is an elementary X_{i-1}-ribbon-X_i with $X_0 = X$ and $X_m = Y$. Thanks to results of Paris and Godelle [10,13], we know that any element that conjugates a standard parabolic subgroup A_X into a standard parabolic subgroupA_Y can be written as the product of an element of A_X and an X-ribbon-Y. We denote the set of all such elements by $\mathrm{Conj}(X, Y)$.

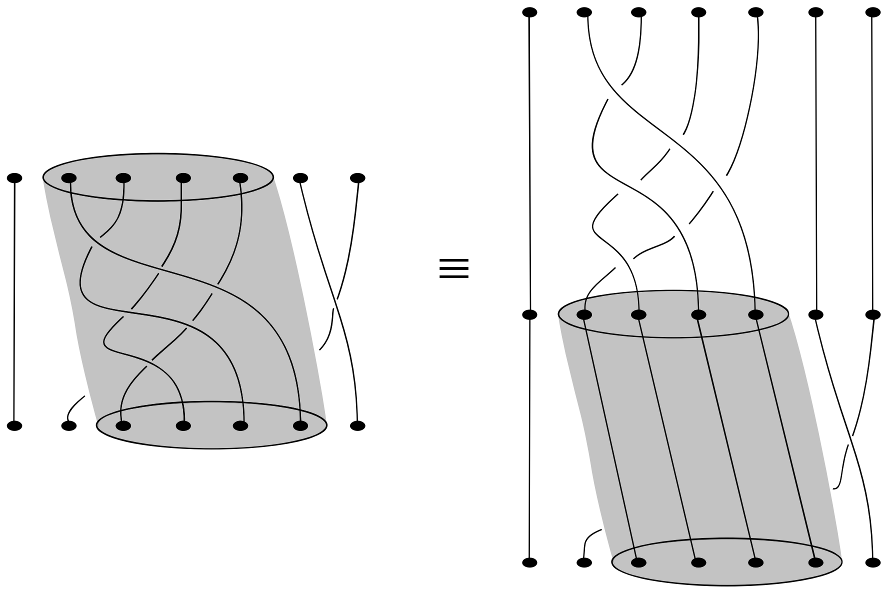

Figure 2: How a braid sending a standard curve to a standard curve decomposes as a braid where the crossings only take place inside the first curve and a ribbon.

Theorem 3. *Let A_S be an Artin group of spherical type and let A_X be a standard parabolic subgroup. Assume that $g \in A_S$ with normal form $g = \Delta^P x_1 \cdots x_r$ is such that $g^{-1}A_X g$ is a standard parabolic subgroup. Then each*

$$(\Delta^p x_1 \ \cdots x_i)^{-1} A_X (\Delta^p x_1 \ \cdots x_i),$$

for $i \in \{1, \dots, r\}$ is a standard parabolic subgroup A_{X_i} and each $x_i \in \mathrm{Conj}(X_{i-1}, X_i)$ and it is positive, where $X_0 = X$.

Proof. Since the conjugation by Δ^p always sends a standard parabolic subgroup to a standard parabolic subgroup, we can suppose without loss of generality that $p = 0$, so g is positive. We also know that g lives in $\mathrm{Conj}(X, Y)$. In [7, Proposition 6.6] Digne and Michel proved that the factors $x'_1, \cdots, x'_{r'}$ in the normal form of a positive X-ribbon-Y are positive such that every x'_i is a X_{i-1}-ribbon-X_i where $X_0 = X$, as described in the statement of the theorem. Since g is the product of an element in g and a X-ribbon-Y, we have that $x_i = g_i x'_i$, where $g_i \in (x'_1 \cdots x'_{i-1})^{-1} A_X (x'_1 \cdots x'_{i-1}) = A_{X_{i-1}}$ and $x'_i = 1$ for $i > r'$. Thus $x_i \in \mathrm{Conj}(X_{i-1}, X_i)$. □

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