Growth rates in a hyperbolic group

Koji Fujiwara
Graduate School of Science, Department of Mathematics, Kyoto University

1 Introduction

This note is based on the talk I gave at the conference Intelligence of Low-dimensional Topology, held at RIMS in May 2025. The talk focused on the structure of the set of growth rates of finitely generated groups. In particular, I emphasized an infinite family of groups arising as fundamental groups of hyperbolic manifolds of fixed dimension. Here, I aim to provide some background and terminology, primarily based on the papers [FS] and [F].

Let G be a finitely generated group with a finite generating set S. We always assume that $S = S^{-1}$. Let $B_n(G, S)$ be the set of elements in G whose word lengths are at most n with respect to the generating set S. Let $\beta_n(G, S) = |B_n(G, S)|$. The *exponential growth rate* of (G, S) is defined to be:

$$e(G, S) = \lim_{n\to\infty} \beta_n(G, S)^{\frac{1}{n}}.$$

A finitely generated group G has *exponential growth* if there exists a finite generating set S such that $e(G, S) > 1$. The group G has *uniform exponential growth* if there exists $c > 1$, such that for every finite generating set S, $e(G, S) \geq c$.

Given a finitely generated group G, we define:

$$e(G) = \inf_{|S|<\infty} e(G, S),$$

where the infimum is taken over all the finite generating sets S of G. Since there are finitely generated groups that have exponential growth but do not have uniform exponential growth [W], the infimum, $e(G)$, is not always realized by a finite generating set for a finitely generated group G.

Given a finitely generated group G, we further define the following set in $\mathbb{R}$:

$$\xi(G) = \{e(G, S) | |S| < \infty\},$$

where S runs over all the finite generating sets of G. The set $\xi(G)$ is always countable.

The focus of the talk is on understanding the structure of this set. It has a striking resemblance to the structure of the set of volumes of hyperbolic 3-manifolds, which we mention at the end of this note.

The author is supported by Grant-in-Aid for Scientific Research (20H00114, 25H00588).

2 Hyperbolic groups

The notion of hyperbolic groups was introduced by Gromov [G]. It has been a central concept in geometric group theory.

A non-elementary hyperbolic group contains a non-abelian free group, hence, it has exponential growth. In fact, a non-elementary hyperbolic group has uniform exponential growth [K].

Important examples of hyperbolic groups include: free groups of finite rank at least two, the fundamental groups of closed, orientable surfaces of genus at least two, and the fundamental groups of closed Riemannian manifolds of negative sectional curvature.

Recently it is proved that $\xi(G)$ of a non-elementary hyperbolic group G is well-ordered, hence, in particular, has a minimum, [FS].

Theorem 2.1 (Hyperbolic groups)**.** *Let G be a non-elementary hyperbolic group. Then $\xi(G)$ is a well-ordered set. For each $r \in \xi(G)$, there are only finitely many finite generating sets S of G with $r = e(G, S)$, upto $Aut(G)$.*

This result is new even for free groups and closed surface groups. The argument is long and intricate. It relies heavily on the hyperbolicity of G, and also makes essential use of the deep fact that G is equationally Noetherian, [S], [RW], a notion we will explain in the next section.

3 Acylindrically hyperbolic groups

The result on hyperbolic groups has been extended in [F]. To explain the results, we need two definitions, acylindricity of group actions and equational Noetherianity of a group.

3.1 Acylindrical actions

To generalize the properness of a group action, Bowditch [Bo] introduced the following definition.

Definition 3.1 (acylindrical action)*.* An action by isometries of a group G on a metric space X is *acylindrical* if for any $\epsilon > 0$, there exist $R = R(\epsilon) > 0$ and $N = N(\epsilon) > 0$ such that for all $x, y \in X$ with $d(x, y) \geq R$, the set

$$\{g \in G | d(x, g(x)) \leq \epsilon, d(y, g(y)) \leq \epsilon\}$$

contains at most N elements.

A group G is called an *acylindrically hyperbolic* group, [O], if it acts on some δ-hyperbolic space X such that the action is acylindrical and non-elementary. Here, we say the action is *elementary* if the limit set of G in the Gromov boundary ∂X has at most two points. See [G] for the definitions of δ-hyperbolic spaces and their boundaries. If the action is non-elementary, it is known that G contains hyperbolic isometries on X. Non-elementary hyperbolic groups and non-virtually-abelian mapping class groups are examples of acylindrically hyperbolic groups, [Bo]. There are many other examples.

3.2 Equational Noetherianity

Let G be a group and $F(x_1, \cdots, x_\ell)$ the free group on $X = \{x_1, \cdots, x_\ell\}$. For an element $s \in F(x_1, \cdots, x_\ell)$ and $(g_1, \cdots, g_\ell) \in G^\ell$, let $s(g_1, \cdots, g_\ell) \in G$ denote the element after we substitute every x_i with g_i and x_i^{-1} by g_i^{-1} in s. Given a subset $S \subset F(x_1, \cdots, x_\ell)$, define

$$V_G(S) = \{(g_1, \cdots, g_\ell) \in G^\ell | s(g_1, \cdots, g_\ell) = 1 \text{ for all } s \in S\}.$$

S is called *a system of equations* (with X the set of variables), and $V_G(S)$ is called *the algebraic set* over G defined by S. We sometimes suppress G from $V_G(S)$.

Definition 3.2 (Equationally Noetherian). A group G is *equationally Noetherian* if for every $\ell \geq 1$ and every subset S in $F(x_1, \cdots, x_\ell)$, there exists a finite subset $S_0 \subset S$ such that $V_G(S_0) = V_G(S)$.

Remark 3.3. This definition appears in for example [GrH]. There is another version of the definition that considers $S \subset G * F(x_1, \cdots, x_\ell)$, which is originally in [BMR] and also in [RW]. They are equivalent, see [RW, Lemma 5.1].

Examples of equationally Noetherian groups include finitely generated free groups, [Gu]; linear groups, [BMR]; hyperbolic groups without torsion, [S], as well as possibly with torsion, [RW]; and hyperbolic groups relative to equationally Noetherian subgroups, [GrH].

3.3 Results

We are ready to state one of the main results from [F].

Theorem 3.4 (Well-orderedness for acylindrical actions)**.** *Suppose G acts on a δ-hyperbolic space X acylindrically, and the action is non-elementary. Assume that there exists a constant M such that for any finite generating set S of G, the set S^M contains a hyperbolic element on X. Assume that G is equationally Noetherian. Then, $\xi(G)$ is a well-ordered set.*

In particular, $\inf \xi(G)$ is realized by some S, ie, $e(G) = e(G, S)$.

This results has several applications. We mention one, also from [F].

Theorem 3.5 (Rank-1 lattices)**.** *Let G be one of the following groups:*

1. *A lattice in a simple Lie group of rank-1.*
2. *The fundamental group of a complete Riemannian manifold M of finite volume such that there exist $a, b > 0$ with $-b^2 \leq K \leq -a^2 < 0$, where K denotes the sectional curvature.*

Then $\xi(G)$ is well-ordered.

4 Volume of hyperbolic 3-manifolds

It is interesting to compare our results on the set of growth rates with the following theorem of Thurston (partly due to Jørgensen), see [G2].

Theorem 4.1. *Let V be the set of volumes of complete, hyperbolic 3-dimensional manifolds. Then V is well-ordered. For each $v \in V$, there are only finitely many hyperbolic 3-manifolds whose volume is v.*

At present, we do not know of any implications between this result and ours.

5 Questions

We conclude with a few natural questions (see [F] and [FS] for more).

Question 5.1. Let Σ_g be the closed orientable surface of genus $g \geq 2$. Let G be its fundamental group. Find a finite generating set S such that $e(G, S) = e(G)$. Is it a standard generating set of the surface?

Question 5.2. Find other finitely generated groups G such that $\xi(G)$ is well-ordered. For example, is $\xi(\mathrm{SL}(3, \mathbb{Z}))$ well-ordered? It is known that $\mathrm{SL}(3, \mathbb{Z})$ has uniform exponential growth, [EMO].

References

[BMR] Gilbert Baumslag, Alexei Myasnikov, Vladimir Remeslennikov, Algebraic geometry over groups. I. Algebraic sets and ideal theory. J. Algebra 219 (1999), no. 1, 16-79.

[Bo] Brian H. Bowditch, Tight geodesics in the curve complex. Invent. Math. 171 (2008), no. 2, 281-300.

[EMO] Alex Eskin, Shahar Mozes, Hee Oh, On uniform exponential growth for linear groups. Invent. Math. 160 (2005), no. 1, 1-30.

[F] K. Fujiwara. The rates of growth in an acylindrically hyperbolic group. Groups Geom. Dyn. 19 (2025), no. 1, 109-167.

[FS] K. Fujiwara, Z. Sela. The rates of growth in a hyperbolic group. Invent. Math.233(2023), no.3, 1427-1470.

[G] M. Gromov. Hyperbolic groups. Essays in group theory, 75-263, Math. Sci. Res. Inst. Publ., 8, Springer, New York, 1987.

[G2] M. Gromov, Hyperbolic manifolds (according to Thurston and Jørgensen). Bourbaki Seminar, Vol. 1979/80, 40-53, Lecture Notes in Math., 842, Springer, Berlin-New York, 1981.

[GrH] D. Groves, M. Hull, Homomorphisms to acylindrically hyperbolic groups I: Equationally noetherian groups and families. Trans. Amer. Math. Soc. 372 (2019), no. 10, 7141-7190.

[Gu] Victor S. Guba. Equivalence of infinite systems of equations in free groups and semigroups to finite subsystems. Mat. Zametki, 48(3):321-324, 1986.

[K] M. Koubi, Croissance uniforme dans les groupes hyperboliques, Ann. Inst. Fourier 48 (1998), 1441–1453.

[O] D. Osin. Acylindrically hyperbolic groups. Trans. Amer. Math. Soc. 368 (2016), no. 2, 851-888.

[RW] C. Reinfeldt, R. Weidmann, Makanin-Razborov diagrams for hyperbolic groups. Ann. Math. Blaise Pascal 26 (2019), no. 2, 119-208.

[S] Z. Sela, Diophantine geometry over groups I: Makanin-Razborov diagrams, Publication de la IHES 93 (2001), 31-105.

[W] J. S. Wilson, On exponential growth and uniformly exponential growth for groups, Inventiones math. 155 (2004), 287-303.

Department of Mathematics
Graduate School of Science
Kyoto University
Kyoto 606-8502
JAPAN
E-mail address: kfujiwara@math.kyoto-u.ac.jp

京都大学大学院・理学研究科・数学専攻　藤原　耕二