Modular Vector Fields in Non-commutative Geometry

Toyo TANIGUCHI
Graduate School of Mathematical Sciences,
The University of Tokyo

Abstract

This is a summary of the paper [Tan24] and the presentation in the conference "Intelligence in Low-dimensional Topology 2025". We construct a non-commutative geometry analogue of the modular vector field in Poisson geometry based on a (flat) connection and show that, in the case of an oriented surface, it coincides with Turaev's loop operation μ with a suitable choice of the connection.

1. Introduction

In [Gol86], Goldman defined a symplectic structure on the $\mathrm{GL}(n)$-character variety X of the fundamental group of a connected, closed and oriented surface Σ based on previous works by Atiyah and Bott. In the same paper, he found the topological counterpart of the Lie algebra structure on the ring $\mathcal{O}(X)$ of regular functions on X induced from the symplectic structure, which is now known as the *Goldman Lie algebra*, denoted by $\mathfrak{g}(\Sigma)$ in this summary. More specifically, $\mathfrak{g}(\Sigma)$ is the $\mathbb{R}$-vector space spanned by homotopy classes of free loops on the surface Σ, and we have the surjective map

$$\mathfrak{g}(\Sigma) \to \mathcal{O}(X)$$

of Lie algebras. The above construction is extended by Fock and Rosly [FR99] for the case of non-empty boundary, in which case X only has a Poisson structure. In contrast, the definition of the Lie bracket, the Goldman bracket, works as is since it is written in terms of the intersection of loops; hence, it is called a *loop operation*.

The Goldman Lie algebra $\mathfrak{g}(\Sigma)$ can be thought of as the non-commutative geometry counterpart of the symplectic structure on X. For simplicity, we only deal with the case of non-empty boundary and an approach involving double brackets. First of all, let $\mathbb{R}\pi$ be the group algebra of the fundamental group $\pi = \pi_1(\Sigma)$. Then, the vector space $\mathfrak{g}(\Sigma)$ is identified as the *trace space*

$$|\mathbb{R}\pi| := \mathrm{HH}_0(\mathbb{R}\pi) \cong \mathbb{R}\pi/[\mathbb{R}\pi, \mathbb{R}\pi]$$

of $\mathbb{R}\pi$, where the most-right-hand side is the quotient by the *vector space* spanned by commutators and not the abelianisation. In [vdB08], van den Bergh introduced a non-commutative geometry analogue of a (Poisson) bivector: a *double bracket* Π on an associative but not-necessarily-commutative algebra A is a linear map

$$\Pi\colon A \otimes A \to A \otimes A$$

satisfying an appropriate version of the Leibniz rule. More details are covered in the next section. In the case of $\mathbb{R}\pi$, we can define the double bracket κ in terms of the intersection of based loops, which induces the Goldman bracket mentioned above.

As in the title, the purpose of my paper [Tan24] is to formulate modular vector fields in non-commutative geometry. With a Poisson manifold P together with a volume form α, we can associate the vector field $\mathbf{m}_\alpha$ as the composition

$$\mathbf{m}_\alpha \colon C^\infty(P) \xrightarrow{\text{Ham}} \text{Der}(C^\infty(P)) \xrightarrow{\text{div}_\alpha} C^\infty(P).$$

Here, Ham is the Hamiltonian flow and div_α is the divergence with respect to α. In this summary of my paper [Tan24], we start with formulating these two maps in the setting of non-commutative geometry and show, in the case of the surface $\mathbb{R}\pi$, that it coincides with another kind of loop operation μ introduced by Turaev.

2. Double Brackets, Connections and the Triple Divergence

Let $\mathbb{K}$ be a field and A a unital associative $\mathbb{K}$-algebra. The two-fold tensor product $A \otimes A$ has two commuting A-bimodule structures: for $a, b, x, y \in A$, the *outer structure* is given by

$$a \cdot (x \otimes y) \cdot b = ax \otimes yb,$$

while the *inner structure* is given by

$$a * (x \otimes y) * b = xb \otimes ay.$$

Following [AKKN23], we make the following:

Definition 2.1. A *double braket* on A is a $\mathbb{K}$-linear map $\Pi \colon A \otimes A \to A \otimes A$ satisfying

$$\Pi(a, bc) = \Pi(a, b) \cdot c + b \cdot \Pi(a, c) \text{ and}$$
$$\Pi(ab, c) = \Pi(a, c) * b + a * \Pi(b, c)$$

for $a, b, c \in A$.

Remark 2.2. Van den Bergh defined double brackets in [vdB08] with the additional condition of skew-symmetry. Our definition is based on [AKKN23], and the skew-symmetry is not imposed to deal with the important example below.

Example 2.3. Let Σ be a connected, oriented and compact surface with non-empty boundary. Take a base point $*$ on $\partial\Sigma$, and let $\pi = \pi_1(\Sigma, *)$ be the fundamental group of Σ. Then, there is a double bracket on the group algebra $\mathbb{K}\pi$, denoted by κ below, which is independently introduced by Massuyeau and Turaev in [MT14] and by Kawazumi and Kuno in [KK15], respectively. It is given as follows: first, fix an embedded short positive arc $\nu \colon [0, 1] \to \partial\Sigma$ with $\nu(1) = *$, and put $\nu(0) = \bullet$, by which $\pi_1(\Sigma, \bullet)$ is identified with $\pi_1(\Sigma, *)$. For generic loops α based at $\bullet$, and β at $*$, respectively, we define

$$\kappa(\alpha, \beta) = \sum_{p \in \alpha \cap \beta} \text{sign}(\alpha, \beta; p) \beta_{*p} \alpha_{p\bullet} \nu \otimes \nu^{-1} \alpha_{\bullet p} \beta_{p*}\,,$$

where $\text{sign}(\alpha, \beta; p) \in \{\pm 1\}$ is the local intersection number with respect to the orientation of Σ, and β_{*p} is the path along β from $*$ to p and so on.

Definition 2.4. A *double deriavtion* on A is a $\mathbb{K}$-linear map $\theta\colon A \to A \otimes A$ satisfying

$$\theta(ab) = \theta(a) \cdot b + a \cdot \theta(b)$$

for $a, b \in A$. We denote the set of all double derivations on A by $\mathrm{DDer}(A)$.

Given a double bracket Π, we have $\Pi(a, \cdot\,) \in \mathrm{DDer}(A)$ by the first condition in Definition 2.1. We abusively put

$$\Pi\colon A \to \mathrm{DDer}(A) : a \mapsto \Pi(a, \cdot\,)\,,$$

which is an analogue of a Hamiltonian flow.

Next, we move on to the divergence map. As mentioned above, the divergence map is defined upon the choice of a volume form. More specifically, for a Poisson manifold P together with a volume form α, the divergence $\mathrm{div}_\alpha(\xi)$ of a vector field ξ on P is the smooth function uniquely specified by

$$\mathrm{div}_\alpha(\xi)\alpha = L_\xi(\alpha) \ \text{ in } \Omega^{\mathrm{top}}(P),$$

where L_ξ is the Lie derivative by ξ. Recall that $\Omega^{\mathrm{top}}(P)$ is the exterior product of the space of 1-forms $\Omega^1(P)$ over the $\mathbb{R}$-algebra $C^\infty(P)$ of smooth functions.

For an arbitrary associative algebra A, the exterior product of modules over A is not well-defined. However, notice the following: taking a local frame $(e_1, \dots, e_r)$ of T^*P such that

$$e_1 \wedge \cdots \wedge e_r = \mu$$

and putting $L_\xi(e_i) = f_i^j e_j$, we have

$$L_\xi(\mu) = \sum_i e_1 \wedge \cdots \wedge f_i^i e_i \wedge \cdots \wedge e_r = \sum_i f_i^i \mu\,.$$

Then, define a (local) flat connection ∇ on T^*X by $\nabla(e_i) = 0$ and the divergence associated with the connection by

$$\mathrm{Div}^\nabla(\xi) = \mathrm{Tr}(L_\xi - \nabla_\xi)\,. \tag{1}$$

This formulation only uses a connection, the Lie derivative and the trace map, all of which can be defined within non-commutative geometry. In this virtue, we regard a flat connection ∇ as a substitute for a volume form.

With this seen, we recall some preliminary tools from non-commutative geometry. Let $A^e = A \otimes A^{\mathrm{op}}$ be the universal enveloping algebra of A.

Definition 2.5.

- The space of 1-*forms* in defined by

$$\Omega^1 A = \mathrm{Ker}(A \otimes A \xrightarrow{\mathrm{mult}} A)$$

  This is naturally an A-bimodule. We put $da = 1 \otimes a - a \otimes 1$. With this notation, we have the Leibniz rule for d:

$$d(ab) = (da)b + a(db) \text{ for all } a, b \in A.$$

  A double derivation $\theta \in \mathrm{DDer}(A)$ is equivalent to an A-bimodule map $i_\theta\colon \Omega^1 A \to A \otimes A$, and they are related by the formula $i_\theta(da) = \theta(a)$.

- The *Lie derivative* L_θ on $\Omega^1 A$ by a double derivation $\theta \in \mathrm{DDer}(A)$ is defined by

$$L_\theta(adb) = \theta(a)' \otimes \theta(a)'' db + ad\theta(b)' \otimes \theta(b)'' + a\theta(b)' \otimes d\theta(b)''.$$

  This is an element of $(\Omega^1 A \otimes A) \oplus (A \otimes \Omega^1 A)$.

- A *connection* on an A-bimodule E is a $\mathbb{K}$-linear map

$$\nabla\colon E \to \Omega^1 A \otimes_A E \oplus E \otimes_A \Omega^1 A$$

  satisfying the Leibniz rule

$$\nabla(aeb) = da \otimes eb + a\nabla(e)b + ae \otimes db$$

  for $a, b \in A$ and $e \in E$.

- For a connection ∇ on E and a double derivation θ, we put

$$\begin{aligned} \nabla_\theta\colon E &\xrightarrow{\nabla} \Omega^1 A \otimes_A E \oplus E \otimes_A \Omega^1 A \\ &\xrightarrow{i_\theta \otimes \mathrm{id} + \mathrm{id} \otimes i_\theta} (A \otimes A) \otimes_A E \oplus E \otimes_A (A \otimes A) \cong (A \otimes E) \oplus (E \otimes A)\,. \end{aligned}$$

  Note that, in the case of $E = \Omega^1 A$, the domains and codomains of the maps ∇_θ and L_θ are the same.

We want to imitate the formula (1), so we need the trace map.

Definition 2.6. Let B be an associative $\mathbb{K}$-algebra, E a left B-module and W a B-bimodule. Suppose that E is finitely generated and projective. Then, the trace map Tr is defined by the composition

$$\begin{aligned} \mathrm{Tr}\colon \mathrm{Hom}_B(E, W \otimes_B E) \cong E^* \otimes_B W \otimes E &\to |W| := \mathrm{HH}_0(B, W) \cong W/[B, W]\,, \\ \varepsilon \otimes w \otimes e &\mapsto \varepsilon(e)w \end{aligned}$$

where $E^* = \mathrm{Hom}_B(E, B)$ is the dual module, which is a *right* B-module.

Now set $B = A^e$, $E = \Omega^1 A$ and $W = (A \otimes A \otimes A^{\mathrm{op}}) \oplus (A \otimes A^{\mathrm{op}} \oplus A^{\mathrm{op}})$. Then, we have the following canonical isomorphism:

$$(A \otimes E) \oplus (E \otimes A) \cong W \otimes_B E.$$

Definition 2.7. The *triple divergence* associated with a connection ∇ on $\Omega^1 A$ is defined by

$$\begin{aligned} \mathrm{TDiv}^\nabla\colon \mathrm{DDer}(A) &\to |W| \cong A \otimes |A| \oplus |A| \otimes A\,, \\ \theta &\mapsto \mathrm{Tr}(L_\theta - \nabla_\theta)\,. \end{aligned}$$

Combining a double bracket Π with the triple divergence above, we have the map

$$\phi_{\Pi,\nabla}\colon A \xrightarrow{\Pi} \mathrm{DDer}(A) \xrightarrow{\mathrm{TDiv}^\nabla} A \otimes |A| \oplus |A| \otimes A\,,$$

which is an analogue of a modular vector field.

3. Turaev's Loop Operation μ

We begin this section by recalling another loop operation μ introduced by Turaev [Tur79]. Fix a framing fr (i.e., a non-vanishing vector field) on Σ. The base points $\bullet$, $*$ and an arc ν are taken as in Example 2.3.

Definition 3.1. The $\mathbb{K}$-linear map $\mu_r\colon \mathbb{K}\pi \to |\mathbb{K}\pi| \otimes \mathbb{K}\pi$ is defined as follows: for $\alpha \in \pi$ represented by a generically immersed path based at $*$, first deform α into a path from $\bullet$ to $*$ by sliding the endpoint along the arc ν, and insert positive or negative monogons so that $\mathrm{rot}^{\mathsf{fr}}(\alpha) = -1/2$. Then,

$$\mu_r(\alpha) = \sum_{p\in \mathrm{Self}(\alpha)} \mathrm{sign}(p; \alpha_{\mathrm{first}}, \alpha_{\mathrm{second}})|\alpha_{pp}| \otimes \alpha_{\bullet p *}.$$

Here $\mathrm{Self}(\alpha)$ is the set of self-intersections of α, and α_{first} is the velocity vector of α passing p for the first time. α_{second} is analogously defined.

Similarly, the $\mathbb{K}$-linear map $\mu_l\colon \mathbb{K}\pi \to \mathbb{K}\pi \otimes |\mathbb{K}\pi|$ is defined as follows: for $\alpha \in \pi$ represented by a generically immersed path based at $*$, first deform α into a path from $*$ to $\bullet$ by sliding the endpoint along ν, and insert positive or negative monogons so that $\mathrm{rot}^{\mathsf{fr}}(\alpha) = 1/2$. Then,

$$\mu_l(\alpha) = -\sum_{p\in \mathrm{Self}(\alpha)} \mathrm{sign}(p; \alpha_{\mathrm{first}}, \alpha_{\mathrm{second}})\alpha_{* p \bullet} \otimes |\alpha_{pp}|.$$

Set $\mu = \mu_r + \mu_l\colon \mathbb{K}\pi \to (|\mathbb{K}\pi| \otimes \mathbb{K}\pi) \oplus (\mathbb{K}\pi \otimes |\mathbb{K}\pi|)$.

Now we apply the construction in the last section in the case of the surface $A = \mathbb{K}\pi$ and see what the map $\phi_{\mathrm{II},\nabla}$ describes. First, note that the fundamental group π of the surface Σ is a free group since we assumed that the boundary is non-empty. We have to check that $\Omega^1 A$ satisfies the assumption in Definition 2.6 to apply the construction of the triple divergence. In fact, we have the following:

Lemma 3.2. *For a surface Σ above, the A-bimodule $\Omega^1 A$ is finitely generated and free. In particular, for any free generating system $\mathcal{C} = \{\gamma_i\}_{1\le i\le r}$ of π, the set $\{d\gamma_i\}_{1\le i\le r}$ is an A^e- free basis of $\Omega^1 A$.*

By the lemma above, we can define the connection $\nabla_{\mathcal{C}}$ associated with $\mathcal{C}$ by

$$\nabla_{\mathcal{C}}(d\gamma_i\,\gamma_i^{-1}) = 0$$

for all i; this uniquely defines the connection due to the Leibniz rule.

The main result is the following:

Theorem 3.3 ([Tan24])**.** *We have $\phi_{\kappa,\nabla_{\mathcal{C}}} = \mu$ for a suitable free-generating system $\mathcal{C}$ and a framing fr such that $\mathrm{rot}^{\mathsf{fr}}(c) = 0$ for any $c \in \mathcal{C}$ represented by a simple curve.*

For the concrete description of the generating system $\mathcal{C}$, see Figure 2 of [Tan24].

References

[AKKN23] Anton Alekseev, Nariya Kawazumi, Yusuke Kuno, and Florian Naef. The Goldman-Turaev Lie bialgebra and the Kashiwara-Vergne problem in higher genera, Version 3. 2023. arXiv:1804.09566v3.

[FR99] V. V. Fock and A. A. Rosly. Poisson structure on moduli of flat connections on Riemann surfaces and the r-matrix. *American Mathematical Society Translations*, 191(2):67–86, 1999. arXiv:math/9802054.

[Gol86] William M. Goldman. Invariant functions on Lie groups and Hamiltonian flows of surface group representations. *Inventiones mathematicae*, 85(2):263–302, 1986. doi:10.1007/BF01389091.

[KK15] Nariya Kawazumi and Yusuke Kuno. Intersection of curves on surfaces and their applications to mapping class groups. *Annales de l'Institut Fourier*, 65(6):2711–2762, 2015. arXiv:1112.3841, doi:10.5802/aif.3001.

[MT14] Gwenael Massuyeau and Vladimir Turaev. Quasi-Poisson structures on representation spaces of surfaces. *International Mathematics Research Notices*, 2014:1:1–64, 2014. arXiv:1205.4898v3.

[Tan24] Toyo Taniguchi. Modular vector fields in non-commutative geometry. 2024. arXiv:2410.24064.

[Tur79] Vladimir G. Turaev. Intersections of loops in two-dimensional manifolds. *Mathematics of the USSR-Sbornik*, 35(2):229–250, 1979. doi:10.1070/sm1979v035n02abeh001471.

[vdB08] Michel van den Bergh. Double Poisson algebras. *Transactions of the American Mathematical Society*, 360(11):5711–5769, 2008. arXiv:math/0410528.

Graduate School of Mathematical Sciences, The University of Tokyo
3-8-1, Komaba, Meguro-ku, Tokyo, 153-8914, Japan
E-mail: toyo@ms.u-tokyo.ac.jp

東京大学大学院数理科学研究科　谷口東曜