Plat presentation of surface-links and their invariants

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1 Introduction

This note is a summary of the results given in [26, 27, 29]. A *surface-link* is a closed surface smoothly embedded in $\mathbb{R}^4$, and a *surface-knot* is a connected surface-link. Two surface-links are said to be *equivalent* if they are related by a smooth isotopy of $\mathbb{R}^4$. Throughout this paper, we work in either the smooth category or the PL category. Surfaces embedded in $\mathbb{R}^4$ are assumed to be locally flat in the PL category.

For a braid β of degree $2m$ ($m \geq 1$), the *plat closure* of β, denoted by $\tilde{\beta}$, is the link obtained by attaching arcs to β as shown in Figure 1. We call $\tilde{\beta}$ a *plat presentation* of a link L if $\tilde{\beta}$ is ambiently isotopic to L. Every link has a plat presentation.

Figure 1: The plat closure of a braid β

In [17, 18], Rudolph introduced *braided surfaces* as higher-dimensional analogues of classical braids, and Viro [23] introduced *2-dimensional braids*, which are special cases of braided surfaces. These surfaces are defined in Section 2. A 2-dimensional braid gives rise to an orientable surface-link by taking its closure. It was shown in [6, 23] that every orientable surface-link is equivalent to the closure of a 2-dimensional braid. Therefore, orientable surface-links can be studied via 2-dimensional braids. On the other hand, non-orientable surface-links cannot be studied in this way.

The aims of this note are to introduce plat presentations of surface-links and to study invariants of surface-links via this presentation. In Section 2, we define the plat closure $\tilde{S}$ of an adequate braided surface S. We call $\tilde{S}$ a *plat presentation* of a surface-link F if it is equivalent to F. Every (possibly non-orientable) surface-link has a plat presentation (Theorem 2.1). Therefore, this framework allows us to study surface-links using braided surfaces.

The *plat index* and *genuine plat index* are invariants of a surface-link defined by using plat presentations. In Section 3, we review several results on these indices. Also, we discuss the additivity of the plat index.

The knot group of a surface-link is the fundamental group of its exterior. As an application of plat presentations, we provide a necessary and sufficient condition for a group to be the knot group of a surface-link (Theorem 4.7). A *symmetric quandle* is a useful algebraic system to study surface-links. The *knot symmetric quandle* is defined for a surface-link, which is an invariant of a surface-link. Then, we provide similar condition for a symmetric quandle to be the knot symmetric quandle of a surface-link (Theorem 5.5).

2 Plat closures of braided surfaces

In this section, we review braids and braided surfaces, and define the plat closure of a braided surface.

2.1 Braids and braided surfaces

Let D^2 be a 2-disk in $\mathbb{R}^2$ and X_m be an m-point subset of $\mathrm{int}(D^2)$, where m is a positive integer. Let $I = [0, 1]$, and let $p_2 : D^2 \times I \to I$ be the projection map onto the second factor. A *braid* of degree m is a union β of m curves in $D^2 \times I$ such that the restriction $\pi_\beta = p_2|_\beta : \beta \to I$ is a covering map of degree m and $\partial\beta = X_m \times \partial I$. Two braids of degree m are said to be *equivalent* if they are related by an isotopy of $D^2 \times I$ fixing $D^2 \times \partial I$ pointwise. The *braid group* B_m of degree m is the group of equivalence classes of braids of degree m. B_m has the following presentation due to Artin [1]:

$$\langle \sigma_1, \ldots, \sigma_{m-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ \sigma_i\sigma_j = \sigma_j\sigma_i\ (|i-j| > 1)\rangle .$$

Let B^2 be a 2-disk in $\mathbb{R}^2$, and let $\mathrm{pr}_2 : D^2 \times B^2 \to B^2$ be the projection onto the second factor. A *braided surface* [17, 18] is a compact surface S embedded in $D^2 \times B^2$ such that the restriction $\pi_S = \mathrm{pr}_2|_S : S \to B^2$ is a simple branched covering map[1]. The *degree* of a braided surface S is the degree of the covering map π_S. Two braided surfaces S_0 and S_1 are said to be *equivalent* (*in the strong sense*) if there exists an isotopy $\{H_t\}_{t\in I}$ of $D^2 \times B^2$ fixing $D^2 \times \partial B^2$ pointwise such that

- $H_0 = \mathrm{id}$, $H_1(S_0) = S_1$, and
- $H_t(S_0)$ is a braided surface for all $t \in I$.

By definition, the boundary ∂S is a closed braid of degree m in $D^2 \times \partial B^2$. We fix a base point $y_0 \in \partial B^2$. A braided surface S of degree m is called *pointed* if $\partial S \cap (D^2 \times \{y_0\}) = X_m \times \{y_0\}$. In this note, a braided surface is assumed to be pointed unless otherwise stated. A *2-dimensional braid* [5, 23] of degree m is a braided surface S such that $\partial S = X_m \times \partial B^2$.

2.2 The plat closure of braided surfaces

A *wicket* [2] is a semicircle properly embedded in $\mathrm{int}(D^2) \times I$ that meets $D^2 \times \{0\}$ orthogonally at its endpoints. A wicket is uniquely determined by its boundary; If two wickets share the same boundary, then they are identical. A union of m disjoint wickets is called a *configuration of m wickets* ($m \geq 1$).

[1]A branched covering map $f : X \to Y$ of degree m is called *simple* if $|f^{-1}(y)| \in \{m, m-1\}$ holds for $y \in Y$.

Suppose that $X_{2m} = \{x_1, x_2, \dots, x_{2m}\}$ is a set of $2m$ points lying in this order on a line in D^2. Let w_m be the configuration of m wickets such that $\partial w_m = X_{2m}$ and, for each $i = 1, 2, \dots, m$, the pair $\{x_{2i-1}, x_{2i}\}$ is the set of endpoints of a wicket in w_m.

The configuration space of m wickets is denoted by $\mathcal{W}_m$. For a loop $f : (I, \partial I) \to (\mathcal{W}_m, w_m)$, we define the braid β_f of degree $2m$ by

$$\beta_f = \bigcup_{t \in I} p_1(\partial f(t)) \times \{t\} \subset D^2 \times I,$$

where $p_1 : D^2 \times I \to D^2$ is the projection onto the first factor. Then, the group homomorphism $\Phi : \pi_1(\mathcal{W}_m, w_m) \to B_{2m}$ is defined as the map that sends $[f] \in \pi_1(\mathcal{W}_m, w_m)$ to $[\beta_f] \in B_{2m}$. In [2], Brendle and Hatcher proved that this homomorphism Φ is injective, and the image of Φ is the subgroup of B_{2m} defined by

$$\langle \sigma_1, \sigma_2\sigma_1\sigma_3\sigma_2, \sigma_{2i}\sigma_{2i-1}\sigma_{2i+1}^{-1}\sigma_{2i}^{-1} \mid i = 1, 2, \dots, m-1 \rangle.$$

We call it the *Hilden subgroup* of B_{2m}, denoted by K_{2m}.

Next, we define the plat closure of a braided surface. Let S be a braided surface of degree $2m$. We assume that B^2 is the unit disk in $\mathbb{C}$. Then, we define the braid β_S of degree $2m$ by

$$\beta_S = \bigcup_{t \in I} \mathrm{pr}_1(\partial S \cap D^2 \times \{e^{2\pi i t}\}) \times \{t\} \subset D^2 \times I,$$

where $\mathrm{pr}_1 : D^2 \times B^2 \to D^2$ is the projection onto the first factor. Then, a braided surface S is called *adequate* if $\beta_S = \beta_f$ holds for some loop $f : (I, \partial I) \to (\mathcal{W}_m, w_m)$. By the rigidity of a wicket, such a loop f is unique for S, so we denote it by f_S. The degree of an adequate braided surface is even, and every 2-dimensional braid of even degree is adequate.

We assume that S is adequate. For $t \in I$, let $J_t = \{re^{2\pi i t} \mid 1 \le r \le 2\}$ and $N = \bigcup_{t \in I} J_t$, where $J_0 = J_1$. Let w_t be the configuration of $2m$ wickets $f_S(t)$ in $D^2 \times J_t$ such that $\partial w_t \subset D^2 \times \{e^{2\pi i t}\}$. We define A_S as the union of w_t for $t \in I$. Then, A_S is the surface embedded in $D^2 \times N$ such that $A_S \cap S = \partial A_S = \partial S$. Hence, the union $A_S \cup S$ is a surface-link in $\mathbb{R}^4$. We call it the *plat closure* of S, denoted by $\tilde{S}$. We call $\tilde{S}$ a *plat presentation* of a surface-link F if it is equivalent to F.

Theorem 2.1 ([29]). *Every surface-link has a plat presentation.*

We call $\tilde{S}$ a *genuine plat presentation* of a surface-link F if S is a 2-dimensional braid whose plat closure is equivalent to F.

Theorem 2.2 ([29]). *Every orientable surface-link has a genuine plat presentation.*

Let $e(F)$ be the normal Euler number of a surface-knot F ([9, 15]). For a 2-dimensional braid S of even degree, each component of $\tilde{S}$ has the normal Euler number zero. On the other hand, the converse remains open:

Conjecture 2.3. *Every surface-link, each of whose components F satisfies $e(F) = 0$, has a genuine plat presentation.*

3 Plat index of surface-links

In this section, we define the plat index and genuine plat index of a surface-link, and give several results on these indices.

Definition 3.1. The *plat index* of a surface-link F, denoted by $\mathrm{Plat}(F)$, is defined as the half of minimum number of degrees of adequate braided surfaces whose plat closures are equivalent to F.

Definition 3.2. The *genuine plat index* of a surface-link F, denoted by $\mathrm{g.Plat}(F)$, is defined as either the half of minimum number of degrees of 2-dimensional braids whose plat closures are equivalent to F, or $+\infty$ if there are no such 2-dimensional braids.

The plat index of a classical link is defined in a similar way. It coincides with the bridge index of a classical link. Hence, the plat index and genuine plat index of a surface-link are higher-dimensional analogues of the bridge index of a link.

A *2-knot* is a surface-knot homeomorphic to the 2-sphere S^2. Surface-links with the plat index (or the genuine plat index) equal to one are determined as follows:

Proposition 3.3 ([29])**.** *Let F be a surface-link.*

- *The plat index of F is one if and only if F is either a trivial 2-knot or a trivial non-orientable surface-knot.*
- *The genuine plat index of F is one if and only if F is either a trivial 2-knot or a trivial non-orientable surface-knot with $e(F) = 0$.*

We remark that for a trivial orientable surface-knot F of positive genus, both the plat index and the genuine plat index of F are equal to two.

A *2-link* is a union of disjoint 2-knots. A surface-link is called *ribbon* [12, 24] if it is obtained from a trivial 2-link F_0 by surgery along 1-handles attached to F_0.

Proposition 3.4 ([28, 29])**.** *Let F be an m-component 2-link, where m is a positive integer. Then, F is ribbon if $\mathrm{g.Plat}(F) = m + 1$.*

By definition, the inequality $\mathrm{Plat}(F) \leq \mathrm{g.Plat}(F)$ holds for any surface-link F. Furthermore, we have the following result.

Proposition 3.5 ([27, 29])**.** *Let F be a surface-link. Let $\mathrm{Braid}(F)$ denote the braid index [5] of F, and let $\mathrm{mrk}(F)$ denote the meridional rank[2] of F. Here, we define $\mathrm{Braid}(F) = +\infty$ if F is non-orientable. Then, we have the following inequalities:*

$$\mathrm{mrk}(F) \leq \mathrm{Plat}(F) \leq \mathrm{g.Plat}(F) \leq \mathrm{Braid}(F).$$

Remark 3.6. For each inequality in Proposition 3.5, there exists a surface-link satisfying the inequality strictly.

- Let F be a trivial orientable surface-knot with positive genus. Then, we have $\mathrm{Plat}(F) = 2$. On the other hand, we have $\mathrm{mrk}(F) = 1$ since F is trivial. The author does not know such examples of 2-knots.

[2]The *meridional rank* of F is the minimum number of meridians of F needed to generate the knot group $\pi_1(\mathbb{R}^4 \setminus F)$.

- Let F be the 2-twist spun 2-knot of the trefoil [30]. It was shown in [29] that $\mathrm{Plat}(F) = 2$. On the other hand, F is not ribbon. Hence, we have $2 < \mathrm{g.Plat}(F)$ by Proposition 3.4.
- Let F be the non-trivial 2-knot denoted by 2_2 in the table in [12]. It was shown in [29] that $\mathrm{g.Plat}(F) = 2$. Since any surface-link with the braid index less than three is trivial ([5]), we have $\mathrm{Braid}(F) \geq 3$.

Theorem 3.7 ([27]). *For any integers $g \geq 0$ and $m \geq 1$, there exist infinitely many orientable surface-knots F of genus g with* $\mathrm{Plat}(F) = \mathrm{g.Plat}(F) = m$.

In the rest of this section, we consider the additivity of the plat index for connected sums. For the bridge index of a knot, the following is well-known:

Theorem 3.8 ([20]). *The bridge index is additive for the connected sum of knots:*

$$b(K_1 \sharp K_2) = b(K_1) + b(K_2) - 1.$$

For the plat index of 2-knots, the following remains open:

Question 3.9. Is the plat index additive for the connected sum of 2-knots:

$$\mathrm{Plat}(F_1 \sharp F_2) = \mathrm{Plat}(F_1) + \mathrm{Plat}(F_2) - 1?$$

In general, the plat index is not always additive for the connected sum of surface-knots, as shown by the following examples:

- Let F_1 and F_2 be trivial orientable surface-knots with positive genus. Since $F_1 \sharp F_2$ is also trivial, we have $\mathrm{Plat}(F_1 \sharp F_2) = 2 < \mathrm{Plat}(F_1) + \mathrm{Plat}(F_2) - 1 = 3$.
- In [22], Viro showed that there exist non-trivial ribbon 2-knots F such that $F \sharp P^2$ is equivalent to P^2, where P^2 is a trivial $\mathbb{R}\mathrm{P}^2$-knot. Therefore, we have $\mathrm{Plat}(F \sharp P^2) = 1 < 2 + 1 - 1 \leq \mathrm{Plat}(F) + \mathrm{Plat}(P^2) - 1$. Similar examples are shown in [14, 19].

In [11], Kamada, Satoh, and Takabayashi proved that the braid index is not additive for the connected sum of non-trivial 2-knots.

4 Characterizations of knot groups of surface-links

For a link/surface-link, the *knot group* is defined as the fundamental group of its complement. In this section, we review a necessary and sufficient condition for a group to be the knot group of a link/surface-link. Using the plat closure of a braided surface, we obtain a characterization of the knot groups of surface-links (Theorem 4.7).

Let $F_m = \langle x_1, x_2, \ldots, x_m \rangle$ be the free group, where m is a positive integer. We define the left action of the braid group B_m on F_m as follows: For $\sigma_i \in B_m$ and $x_j \in F_m$, we define $\sigma_i \cdot x_j$ and $\sigma_i^{-1} \cdot x_j$ by

$$\sigma_i \cdot x_j = \begin{cases} x_i \, x_{i+1} \, x_i^{-1} & (j = i), \\ x_i & (j = i+1), \\ x_j & (\text{otherwise}), \end{cases} \qquad \sigma_i^{-1} \cdot x_j = \begin{cases} x_{i+1} & (j = i), \\ x_{i+1}^{-1} \, x_i \, x_{i+1} & (j = i+1), \\ x_j & (\text{otherwise}). \end{cases}$$

Then, this induces the action of B_m on F_m.

A *Wirtinger presentation* $\langle A \mid R \rangle$ of a group is a finite presentation such that each relation in R is of the form $w^{-1}aw = b$ for some $a, b \in S$ and $w \in \langle A \rangle$.

Theorem 4.1 ([1])**.** *A group G is the knot group of a classical link if and only if there exist a positive integer m and a braid $b \in B_m$ such that G has a Wirtinger presentation*

$$\langle x_1, x_2, \ldots, x_m \mid b \cdot x_i = x_i (i = 1, 2, \ldots, m) \rangle .$$

Let $n \geq 1$ be an integer. An *n-knot* is a smoothly embedded n-sphere S^n in S^{n+2}. For higher-dimensional knots, Kervaire gave the following characterization of knot groups.

Theorem 4.2 ([13])**.** *For $n \geq 3$, a group G is the knot group of an n-knot if and only if G satisfies the following conditions:*

(1) The weight of G is one; $G \cong \langle\langle x \rangle\rangle$.

(2) $H_2(G) = 0$.

(3) $G/[G, G]$ is isomorphic to $\mathbb{Z}$.

We consider the knot group of 2-knots and surface-links. In the case of ribbon 2-knots, a criterion involving Wirtinger presentations is known.

Theorem 4.3 ([25])**.** *A group G is the knot group of a ribbon 2-knot if and only if G satisfies the following conditions:*

(1) G has a Wirtinger presentation with deficiency one.

(2) $G/[G, G]$ is isomorphic to $\mathbb{Z}$.

González-Acuña [3] and Kamada [6] independently proved a characterization of the knot groups of 2-knots and orientable surface-links. We recall the result due to Kamada as follows: Let $b_1, b_2, \ldots, b_n \in B_m$, where n is a non-negative integer. Then, an *(m, n)-presentation* (*associated with* $b_1, b_2, \ldots, b_n$) of a group is a presentation

$$\langle x_1, x_2, \ldots, x_m \mid b_i \cdot x_1 = b_i \cdot x_2 \ (i = 1, 2, \ldots, n) \rangle .$$

We say that an (m, n)-presentation (associated with $b_1, b_2, \ldots, b_m$) satisfies the *∂-condition* if there exist n signs $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \{1, -1\}$ such that

$$\prod_{i=1}^{n} b_i^{-1} \sigma_1^{\varepsilon_i} b_i \ = \ 1_m,$$

where $1_m \in B_m$ is the unit element.

Proposition 4.4 ([6, 17])**.** *Let S be a braided surface of degree m with n branch points. There exist n braids $b_1, b_2, \ldots, b_n \in B_m$ and n signs $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \{1, -1\}$ such that the knot group $G(S) = \pi_1(D^2 \times B^2 \setminus S)$ of S has an (m, n)-presentation associated with $b_1, b_2, \ldots, b_n$ satisfying*

$$\prod_{i=1}^{n} b_i^{-1} \sigma_1^{\varepsilon_i} b_i \ = \ [\beta_S].$$

In particular, the knot group of a 2-dimensional braid of degree m with n branch points has an (m, n)-presentation satisfying the ∂-condition.

Theorem 4.5 ([6])**.** *A group G is the knot group of a c-component orientable surface-link with the Euler characteristic χ if and only if G satisfies the following conditions for some integers $m \geq 1$ and $n \geq 0$ such that $\chi = 2m - n$:*

(1) G has an (m, n)-presentation satisfying the ∂-condition.

(2) $G/[G, G]$ is isomorphic to $\mathbb{Z}^c$.

Corollary 4.6. *A group G is the knot group of a 2-knot if and only if G satisfies the following conditions for some integer $m \geq 1$:*

(1) G has an $(m, 2m-2)$-presentation satisfying the ∂-condition.

(2) $G/[G, G]$ is isomorphic to $\mathbb{Z}$.

We remark that a group is the knot group of an orientable surface-link if and only if it has a finite Wirtinger presentation, and such a surface-link can be realized by a ribbon surface-link (cf. [6, 21]).

A $(2m, n)$-presentation *with inverses* (associated with $b_1, b_2, \ldots, b_n$) is a group presentation

$$\left\langle x_1, x_2, \ldots, x_{2m} \,\middle|\, b_i \cdot x_1 = b_i \cdot x_2,\ x_{2j-1} = x_{2j}^{-1} \quad (i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m) \right\rangle.$$

This presentation is obtained from a $(2m, n)$-presentation by adding m new relations $x_{2j-1} = x_{2j}^{-1}$ for $j = 1, 2, \ldots, m$. We say that a $(2m, n)$-presentation satisfies the *weak ∂-condition* if there exist n signs $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \{1, -1\}$ such that

$$\prod_{i=1}^{n} b_i^{-1} \sigma_1^{\varepsilon_i} b_i \in K_{2m}.$$

By Proposition 4.4, the knot group of an adequate braided surface of degree $2m$ with n branch points has a $(2m, n)$-presentation satisfying the weak ∂-condition.

A surface-link F is called *(c, d)-component* if F consists of c orientable surface-knots and d non-orientable ones.

Theorem 4.7. *A group G is the knot group of a (c, d)-component surface-link with the Euler characteristic χ if and only if G satisfies the following conditions for some integers $m \geq 1$ and $n \geq 0$ such that $\chi = 2m - n$:*

(1) G has a $(2m, n)$-presentation with inverses satisfying the weak ∂-condition.

(2) $G/[G, G]$ is isomorphic to $\mathbb{Z}^c \oplus (\mathbb{Z}/2)^d$.

The key proposition used to prove Theorem 4.7 is the following:

Proposition 4.8. *Let S be an adequate braided surface of degree $2m$ with n branch points. Then, the knot group of $\tilde{S}$ has a $(2m, n)$-presentation satisfying the weak ∂-condition.*

We remark that a group is the knot group of a surface-link if and only if it has a twisted Wirtinger presentation. Here, a *twisted Wirtinger presentation* of a group is a finite presentation $\langle A \mid R \rangle$ such that each relation in R is of the form $w^{-1} a^{\varepsilon} w = b$, where $a, b \in A$, $w \in \langle A \rangle$, and $\varepsilon \in \{1, -1\}$.

5 Characterizations of knot symmetric quandles of surface-links

In this section, we recall symmetric quandles and provide a necessary and sufficient condition for a group to be the knot symmetric quandle of a surface-link.

A *quandle* [4, 16] is a set Q with a binary operation $* : Q \times Q \to Q$ satisfying the following conditions:

(1) For any $x \in Q$, we have $x * x = x$.

(2) For any $y \in Q$, the map $S_y : Q \to Q$ defined by sending x to $x * y$ is a bijection.

(3) For any $x, y, z \in Q$, we have $(x * y) * z = (x * y) * (x * z)$.

For a quandle Q, we define another operation $\overline{*}$ on Q by $x \overline{*} y = S_y^{-1}(x)$ for $x, y \in Q$. Then, $(Q, \overline{*})$ is a quandle. This operation $\overline{*}$ is called the *dual operation* of Q.

For quandles Q_1 and Q_2, a map $f : Q_1 \to Q_2$ is a *quandle homomorphism* if $f(x * y) = f(x) * f(y)$ holds for any $x, y \in Q_1$. It is known that S_y is a quandle automorphism of Q. Let $\mathrm{Aut}(Q)$ be the group of quandle automorphisms of Q, and let $\mathrm{Inn}(Q)$ be the subgroup of $\mathrm{Aut}(Q)$ generated by S_y for $y \in Q$.

The action of $\mathrm{Inn}(Q)$ on Q is defined by $f \cdot x = f(x)$ for $f \in \mathrm{Inn}(Q)$ and $x \in Q$. A quandle Q is called *connected* if the action of $\mathrm{Inn}(Q)$ on Q is transitive. Similarly, an orbit of the action of $\mathrm{Inn}(Q)$ on Q is called a *connected component* of Q. We notice that each connected component of Q is a subquandle of Q.

A *symmetric quandle* [7, 10] is a pair (Q, ρ) of a quandle Q and an involution map $\rho : Q \to Q$ of Q, called a *good involution*, satisfying the following conditions:

(1) For any $x, y \in Q$, we have $\rho(x * y) = \rho(x) * y$.

(2) For any $x, y \in Q$, we have $x * \rho(y) = x \overline{*} y$.

For symmetric quandles (Q_1, ρ_1) and (Q_2, ρ_2), a map $f : Q_1 \to Q_2$ is a *symmetric quandle homomorphism* if it is a quandle homomorphism satisfying $f \circ \rho_1 = \rho_2 \circ f$.

We here give examples of symmetric quandles.

Example 5.1 (Double of a quandle [7, 10])**.** Let Q be a quandle and $\overline{Q} = \{\overline{a} \mid a \in Q\}$ a copy of Q. We extend the binary operation $*$ on Q into the disjoint union $D(Q) = Q \cup \overline{Q}$:

$$a * \overline{b} := a \overline{*} b, \quad \overline{a} * b := \overline{a * b}, \quad \overline{a} * \overline{b} := \overline{a \overline{*} b}.$$

Then $(D(Q), *)$ is a quandle. We define a good involution ρ on $D(Q)$ by setting $\rho(a) = \overline{a}$. The symmetric quandle $(D(Q), \rho)$ is called the *double* of Q.

Example 5.2 (Knot symmetric quandles [7, 10])**.** Let K be a properly embedded n-submanifold in a connected $(n+2)$-manifold M. We fix a point $p \in E(K)$. An (*oriented*) *noose* of K is a pair (D, α) of an oriented meridional disk D of K and an oriented arc α in $E(K)$ connecting from a point of ∂D to p.

The set of homotopy classes $[(D, \alpha)]$ of all oriented nooses of K is denoted by $\widetilde{Q}(M, K, p)$ or $\widetilde{Q}(K, p)$ simply. Note that $[(D, \alpha)]$ and $[(-D, \alpha)]$ represent different homotopy classes if K is orientable, where $-D$ is D with the reversed orientation. The *full knot quandle* of K is $\widetilde{Q}(M, K, p)$ with the binary operation $*$ defined by

$$[(D, \alpha)] * [(D', \alpha')] = [(D, \alpha \cdot \alpha'^{-1} \partial D' \alpha')].$$

See Figure 2 for a topological description of the operation on $\tilde{Q}(M, K, p)$. The isomor-

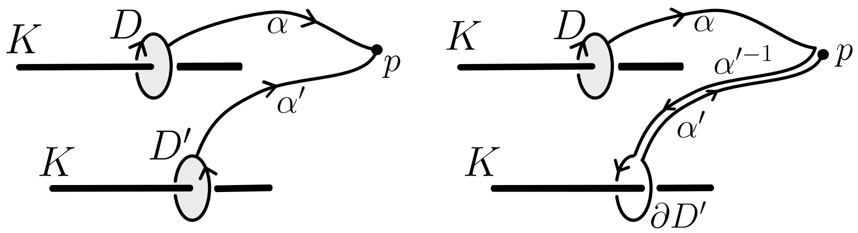

Figure 2: The operation of the full knot quandle $\tilde{Q}(K, p)$

phism class of the full knot quandle does not depend on the base point p. Hence, we denote the full knot quandle by $\widetilde{Q}(M, K)$ or $\tilde{Q}(K)$ simply.

The *knot symmetric quandle* (or *fundamental symmetric quandle*) of K, denoted by $X(M, K)$ (or $X(K)$ simply), is the pair of $\widetilde{Q}(M, K)$ and the good involution ρ_K of $\widetilde{Q}(M, K)$ defined by $\rho_K([(D, \alpha)]) = [(-D, \alpha)]$ for $[(D, \alpha)] \in \tilde{Q}(M, K)$.

Example 5.3 (Free symmetric quandle [8])**.** Let A be a non-empty set and $F(A)$ be the free group on A. We define the binary operation $*$ on $\mathrm{FR}(A) = A \times F(A)$ by

$$(a, w) * (b, u) \ = \ (a, wu^{-1}bu) \quad (a, b \in A,\ w, u \in F(A)).$$

Let $\sim_q$ be the equivalence relation on $\mathrm{FR}(A)$ generated by $(a, w) \sim_q (a, aw)$ for $a \in A$ and $w \in F(A)$. Then, the binary operation on $\mathrm{FR}(A)$ induces a well-defined operation on the quotient set $\mathrm{FQ}(A) = \mathrm{FR}(A)/\sim_q$, which is called the *free quandle* on A.

The *free symmetric quandle* on A, denoted by $\mathrm{FSQ}(A)$, is defined as the double of the free quandle $\mathrm{FQ}(A)$ on A. The underlying set of $\mathrm{FSQ}(A)$ is $(A \cup \overline{A}) \times \mathrm{F}(A)$, where we identify $\overline{a} \in \overline{A}$ with $a^{-1} \in \mathrm{F}(A)$. The good involution of $\mathrm{FSQ}(A)$ sends (a, w) to $(\overline{a}, w)$ for $a \in A$ and $w \in F(A)$.

For $a \in A \cup \overline{A}$ and $w \in F(A)$, we write $(a, 1)$ and (a, w) in $\mathrm{FSQ}(A)$ as a and a^w, respectively, where $1 \in F(A)$ is the unit element.

Next, we introduce a presentation of a symmetric quandle. A free symmetric quandle satisfies the following universal property:

Proposition 5.4 ([27])**.** *Let A be a non-empty set and X a symmetric quandle. Let $\iota_A : A \to \mathrm{FSQ}(A)$ be the map sending $a \in A$ to $a \in \mathrm{FSQ}(A)$. Then, for any map $f : A \to X$, there exists a unique symmetric quandle homomorphism $\overline{f} : \mathrm{FSQ}(A) \to X$ such that $\overline{f} \circ \iota_A = f$.*

Let R be a subset of $\mathrm{FSQ}(A) \times \mathrm{FSQ}(A)$ and X a symmetric quandle. Then, we say that X has a *presentation* $\langle A \mid R \rangle_{\mathrm{sq}}$ if X satisfies the following conditions for some map $\iota_X : A \to X$:

(1) $(\overline{\iota_X} \times \overline{\iota_X})(R) \subset X \times X$ is contained in the diagonal set of $X \times X$, where $\overline{\iota_X}$: $\mathrm{FSQ}(A) \to X$ is the homomorphism obtained from Proposition 5.4.

(2) For any symmetric quandle Y and any map $\iota_Y : A \to Y$ satisfying (1), there exists a unique symmetric quandle homomorphism $f : X \to Y$ such that $\iota_Y = f \circ \iota_X$.

A construction of a symmetric quandle X having a presentation $\langle A \mid R\rangle_{\mathrm{sq}}$ is explained in [8, 27]. We write each relation $(x, y) \in R$ by the form $x = y$.

It is known that $X(D^2, X_m)$ is isomorphic to the free symmetric quandle $\mathrm{FSQ}(X_m)$ on $X_m = \{x_1, x_2, \ldots, x_m\}$. We define the left action of B_m on $\mathrm{FSQ}(X_m)$ as follows: For $\sigma_i \in B_m$ and $x_j \in \mathrm{FSQ}(X_m)$, we define $\sigma_i \cdot x_j$ and $\sigma_i^{-1} \cdot x_j$ by

$$\sigma_i \cdot x_j = \begin{cases} x_{i+1} * \overline{x_i} & (j = i), \\ x_i & (j = i+1), \\ x_j & (\text{otherwise}), \end{cases} \qquad \sigma_i^{-1} \cdot x_j = \begin{cases} x_{i+1} & (j = i), \\ x_i * x_{i+1} & (j = i+1), \\ x_j & (\text{otherwise}). \end{cases}$$

We define $\sigma_i \cdot \overline{x_j} = \overline{\sigma_i \cdot x_j}$, where $\overline{\sigma_i \cdot x_j}$ is the image of $\sigma_i \cdot x_j$ by the good involution of $\mathrm{FSQ}(X_m)$. Then, this induces the action of B_n on $\mathrm{FSQ}(X_m)$.

Let $b_1, b_2, \ldots, b_n \in B_{2m}$, where n is a non-negative integer. A $(2m, n)$*-presentation with inverses* (*associated with* $b_1, b_2, \ldots, b_n$) of a symmetric quandle is a presentation

$$\left\langle x_1, \ldots, x_{2m} \,\middle|\, b_i \cdot x_1 = b_i \cdot x_2,\ x_{2j-1} = \overline{x_{2j}} \quad (i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m) \right\rangle_{\mathrm{sq}}.$$

A $(2m, n)$-presentation with inverses associated with $b_1, b_2, \ldots, b_n$ of a symmetric quandle is said to satisfy the *weak ∂-condition* if there exist $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \{1, -1\}$ such that

$$\prod_{i=1}^{n} b_i^{-1} \sigma_1^{\varepsilon_i} b_i \in K_{2m}.$$

Theorem 5.5 ([26])**.** *A symmetric quandle (Q, ρ) is the knot symmetric quandle of a (c, d)-component surface-link with the Euler characteristic χ if and only if (Q, ρ) satisfies the following conditions for some integers $m \geq 1$ and $n \geq 0$ such that $\chi = 2m - n$:*

(1) (Q, ρ) has a $(2m, n)$-presentation with inverses satisfying the weak ∂-condition.

(2) Q consists of $2c + d$ connected components $X_1, \ldots, X_c$, $Y_1, \ldots, Y_c$, and $Z_1, \ldots, Z_d$ such that $\rho(X_i) = Y_i$ and $\rho(Z_j) = Z_j$ for each $i = 1, 2, \ldots, c$ and $j = 1, 2, \ldots, d$.

We remark that every symmetric quandle admitting a finite presentation is the knot symmetric quandle of some ribbon surface-link. The key proposition used to prove Theorem 5.5 is the following:

Proposition 5.6 ([26])**.** *Let S be an adequate braided surface of degree $2m$ with n branch points. Then, the knot symmetric quandle of $\tilde{S}$ has a $(2m, n)$-presentation with inverses satisfying the weak ∂-condition.*

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