MGR coloring invariants of Seifert surfaces

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1 Introduction

A *spatial surface* is a compact surface embedded in the 3-sphere $S^3 = \mathbb{R}^3 \sqcup \{\infty\}$. In this note, we assume that (1) spatial surfaces are oriented and that (2) connected components of spatial surfaces are neither 2-disks nor closed surfaces. The aim of this note is to introduce the concept of a *groupoid rack*, an algebraic system for constructing invariants of spatial surfaces.

2 Handlebody-knots and spatial surfaces

A *spatial trivalent graph* is a finite trivalent graph embedded in S^3. In this note, we allow trivalent graphs to have loops, multiple edges, and S^1*-components*, i.e., edges without vertices. We regard knots as spatial trivalent graphs without vertices. Diagrams of spatial trivalent graphs are defined as usual in knot theory. An *edge* of a diagram D of a spatial trivalent graph G is a sub-diagram of D that presents an edge of G. In particular, an edge of D of G that presents an S^1-component of G is called an S^1-component of D. A *handlebody-knot* [3] is a handlebody embedded in S^3. Two handlebody-knots H_1 and H_2 are said to be *equivalent* ($H_1 \cong H_2$) if they are ambiently isotopic in S^3. Every handlebody-knot is obtained as a regular neighborhood of a spatial trivalent graph. A *diagram* of a handlebody-knot H is a diagram of a spatial trivalent graph G such that the regular neighborhood of G is equivalent to H. We denote by $H(D)$ the handlebody-knot whose diagram is D. In [3], a Reidemeister-type theorem for handlebody-knots was introduced.

Theorem 2.1 ([3])**.** *Two handlebody-knots are equivalent if and only if their diagrams are related by a finite sequence of* R1–R6 *moves, depicted in Fig. 1, and isoopies in* S^2.

A *spatial surface* is a compact surface embedded in S^3. Two spatial surfaces F_1 and F_2 are said to be *equivalent* ($F_1 \cong F_2$) if they are ambiently isotopic in S^3. Thoroughout this note, we assume that (1) a spatial surface is oriented and that (2) each component of a spatial surface is neither a closed disk nor a closed surface. Under the assumptions, spatial surfaces are Seifert surfaces for their boundaries. As a remark, if two spatial surfaces with the same boundary are not equivalent, then they are not equivalent as Seifert surfaces for the boundary. Let D be a diagram of a spatial trivalent graph. The spatial surface $F(D)$ is obtained from D as illustrated in Fig. 2.

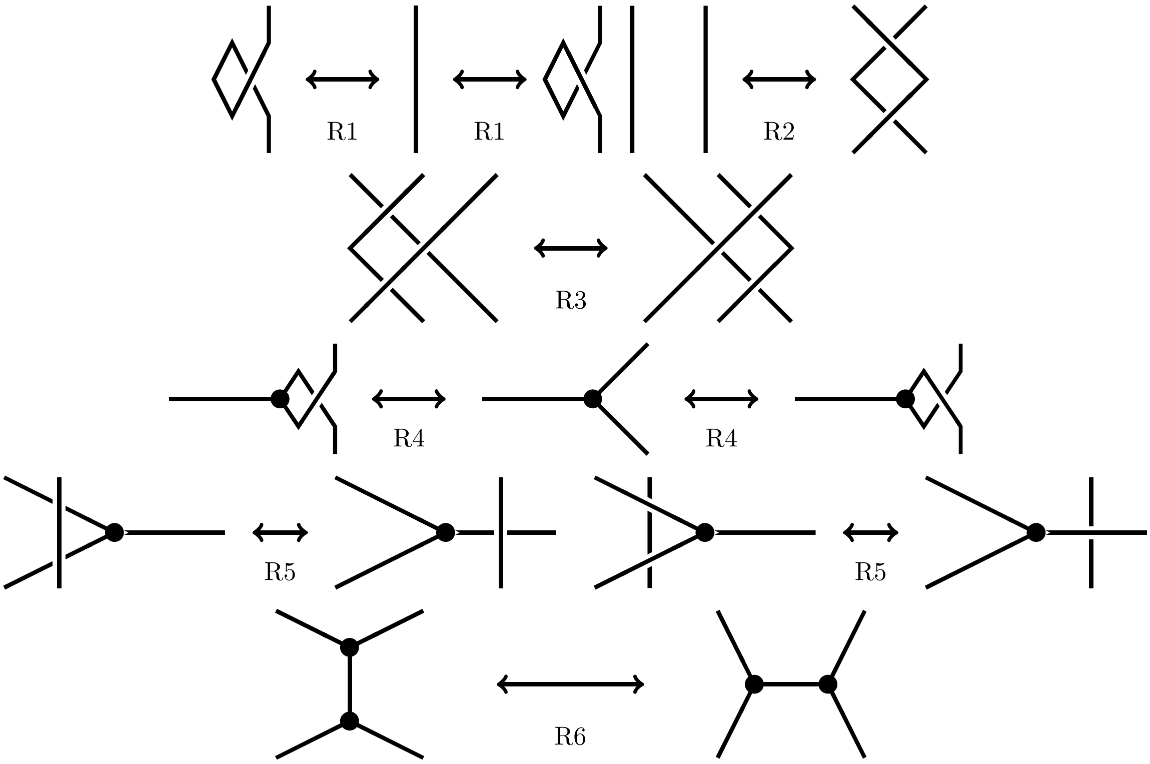


Figure 1: Local moves on diagrams of spatial trivalent graphs

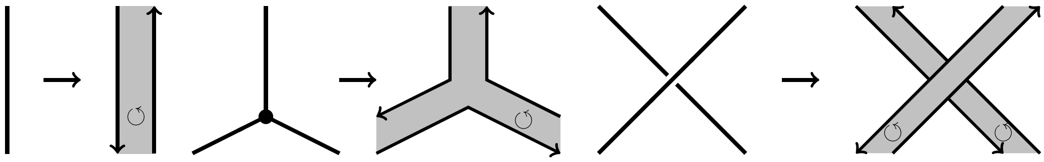


Figure 2: A construction of $F(D)$

For any spatial surface F, there exists a diagram D such that $F \cong F(D)$, [6, 9]. A *diagram* of a spatial surface F is a diagram D such that $F(D)$ is equivalent to F. As in the case of handlebody-knots, a Reidemeister-type theorem holds for spatial surfaces, [9].

Theorem 2.2 ([9]). *Two spatial surfaces are equivalent if and only if their diagrams are related by a finite sequence of* R2, R3, R5, *and* R6 *moves, depicted in Fig. 1, and isotopies in* S^2.

3 Groupoid racks

Definition 3.1 ([7, 10]). A *rack* is a pair $X = (X, \triangleleft)$ of a set X and a binary operation $\triangleleft : X \times X \to X$ satisfying the following two conditions:

- For any $y \in X$, the map $S_y : X \to X$, defined by $S_y(x) = x \triangleleft y$, is bijective.

- For any $x, y, z \in X$, $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$.

A rack $X = (X, \triangleleft)$ is called a *quandle* if it satisfies the following condition:

- For any $x \in X$, $x \triangleleft x = x$.

We give some examples of racks and quandles. Let $n \in \mathbb{Z}_{>0}$ be a positive integer. The cyclic group $\mathbb{Z}_n$ with the binary operation $\triangleleft$, defined by $x \triangleleft y = 2y - x$, is a quandle, called the *dihedral quandle*. The cyclic group $\mathbb{Z}_n$ with the binary operation $\triangleleft$, defined by $x \triangleleft y = x + 1$, is a rack, called *cyclic rack*. Let R_1 and R_2 be racks. Then $R_1 \times R_2$ is a rack with the binary operation $\triangleleft$ defined by $(x_1, x_2) \triangleleft (y_1, y_2) = (x_1 \triangleleft_1 y_1, x_2 \triangleleft_2 y_2)$, where for each $i \in \{1, 2\}$, $\triangleleft_i : R_i \times R_i \to R_i$ is a rack operation on R_i.

Definition 3.2 ([5, 6])**.** Let $\{G_\lambda\}_{\lambda \in \Lambda}$ be a family of groups and e_λ the identity element of the group G_λ for each $\lambda \in \Lambda$. A *multiple group rack (MGR)* $X = (X, \triangleleft)$ is a pair of the disjoint union $X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$ of groups and a binary operation $\triangleleft : X \times X \to X$ satisfying the following three conditions:

- For any $x \in X$, for any $\lambda \in \Lambda$, and for any $a, b \in G_\lambda$, $x \triangleleft (ab) = (x \triangleleft a) \triangleleft b$ and $x \triangleleft e_\lambda = x$.
- For any $x, y, z \in X$, $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$.
- For any $x \in X$ and for any $\lambda \in \Lambda$, there exists $\mu \in \lambda$ such that for any $a, b \in G_\lambda$, $a \triangleleft x, b \triangleleft x \in G_\mu$ and $(ab) \triangleleft x = (a \triangleleft x)(b \triangleleft x)$.

An MGR $X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$ is called a *multiple conjugation quandle (MCQ)* if it satisfies the following condition:

- For any $\lambda \in \Lambda$ and for any $a, b \in G_\lambda$, $a \triangleleft b = b^{-1}ab$.

An example of an MGR is given in the proof of Theorem 4.3.

Definition 3.3 ([11])**.** Let R be a rack with a rack operation $* : R \times R \to R$. Then $R \times R$ is a rack with the binary operation $\triangleleft : (R \times R) \times (R \times R) \to R \times R$ defined by $(x, y) \triangleleft (z, w) = ((x *^{-1} z) * w, (y *^{-1} z) * w)$. The *heap rack* $R \times R$ is the rack $R \times R$ with the partial operation $(x, y)(y, z) = (x, z)$.

A *groupoid* is a category in which all morphisms are invertible. In this note, we denote the composition of morphisms f and g with $\mathrm{cod}(f) = \mathrm{dom}(g)$ in a category by fg.

Definition 3.4 ([1])**.** Let $\mathcal{C}$ be a groupoid. A *groupoid rack* $X = (X, \triangleleft)$ associated with $\mathcal{C}$ is a pair of the set X of all morphisms of $\mathcal{C}$ and a binary operation $\triangleleft : X \times X \to X$ satisfying the following three conditions:

- For any $x, f, g \in X$ with $\mathrm{cod}(f) = \mathrm{dom}(g)$, $x \triangleleft (fg) = (x \triangleleft f) \triangleleft g$ and $x \triangleleft \mathrm{id}_\lambda = x$, where id_λ is the identity of the object λ.
- For any $x, y, z \in X$, $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$.
- For any $x, f, g \in X$ with $\mathrm{cod}(f) = \mathrm{dom}(g)$,

An MGR $X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$ can be regarded as the groupoid rack associated with the following groupoid $\mathcal{C}$:

- $\mathrm{Ob}(\mathcal{C}) = \Lambda$.
- $\mathrm{Mor}(\mathcal{C}) = \begin{cases} G_\lambda & \text{if} \lambda = \mu, \\ \emptyset & \text{otherwise.} \end{cases}$
- Composition: $G_\lambda \times G_\lambda \to G_\lambda, (a, b) \mapsto ab$.
- The identity morphism of $\lambda \in \Lambda$ is the identity element of the group G_λ.
- The inverse morphism of a morphism $x \in G_\lambda$ is $x^{-1} \in G_\lambda$.

Proposition 3.5. *Let X be a groupoid rack associated with a groupoid $\mathcal{C}$. If $\mathcal{C}$ satisfies that for any $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$, $\mathrm{Mor}(\lambda, \mu) = \emptyset$, then X is an MGR.*

Proposition 3.6. *Let $X = (X, \triangleleft)$ be a groupoid rack associated with a groupoid $\mathcal{C}$. If $\mathcal{C}$ satisfies the following, then X is an MCQ.*

1. *For any $\lambda, \mu \in \mathrm{Ob}(\mathcal{C})$ with $\lambda \neq \mu$, $\mathrm{Mor}(\lambda, \mu) = \emptyset$.*
2. *For any $\lambda \in \mathrm{Ob}(\mathcal{C})$ and for any $a, b \in \mathrm{Mor}(\lambda, \lambda)$, $a \triangleleft b = b^{-1}ab$.*

Remark 3.7. Heap racks can be also regarded as groupoid racks.

A *Y-orientation* of a diagram of a spatial trivalent graph is an assignment of orientations to all edges of D such that no vertices are sinks or sources, as shown in Fig. 3.

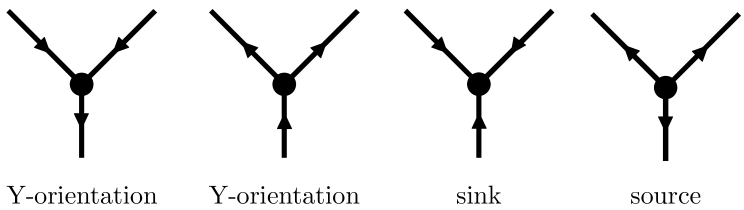


Figure 3: Orientations around vertices

We remark that every diagram admits a Y-orientation, [5, 8]. A *Y-oriented diagram* of a spatial trivalent graph is a diagram with a Y-orientation.

Let D be a Y-oriented diagram. We denote the set of all arcs of D by $\mathcal{A}(D)$. Let X be a set, $\triangleleft : X \times X \to X$ a binary operation, $P \subset X \times X$ a subset, and $\mu : P \to X$ a partial operation on X. An *X-coloring* of D or a *coloring* of D by X is a map $C : \mathcal{A}(D) \to X$ satifying the conditions depicted in Fig. 4.

We denote the set of all X-colorings by $\mathrm{Col}_X(D)$.

Theorem 3.8 ([1])**.** *Let D be a Y-oriented diagram, X a set, $\triangleleft : X \times X \to X$ a binary operation, $P \subset X \times X$ a subset, and $\mu : P \to X$ a partial operation on X.*

1. *If μ is a composition of a groupoid $\mathcal{C}$ and $(X, \triangleleft)$ is a groupoid rack associated with $\mathcal{C}$, then $|\mathrm{Col}_X(D)|$ is an invariant of the spatial surface $F(D)$.*
2. *If $|\mathrm{Col}_X(D)|$ is an invariant of the spatial surface $F(D)$, then $\bigcup_{(x,y) \in P} \{x, y\}$ is a groupoid rack.*

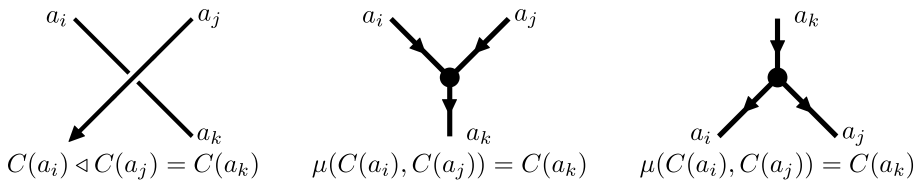


Figure 4: Coloring conditions at crossings or vertices

Remark 3.9. In Theorem 3.8, if D has a vertex, for any X-coloring C of D, we have $C(\mathcal{A}(D)) \subset \bigcup_{(x,y)\in P}\{x,y\}$.

In what follows, when we consider X-colorings of a diagram by a set X, we assume the assumption of 1 in Theorem 3.8.

Theorem 3.10 ([5]). *Let D be a Y-oriented diagram and X groupoid rack.*

1. *If X is an MCQ, then $|\mathrm{Col}_X(D)|$ is an invariant of the handlebody-knot $H(D)$.*
2. *If $|\mathrm{Col}_X(D)|$ is an invariant of the handlebody-knot $H(D)$, then X is an MCQ.*

4 An infinite family of pairs of Seifert surfaces

A square matrix P is called a *unimodular matrix* if all its entries are integers and $\det P = \pm 1$. Two square matrix V_1 and V_2 with integer entries are said to be *unimodular-congruent* if there exists a unimodular matrix P such that $V_2 = P^T V_1 P$, where P^T is the transpose of P. Although a Seifert matrix of a spatial surface depends on the choice of a basis for the first homology group of the spatial surface, the following result holds for Seifert matrices for spatial surfaces.

Proposition 4.1. *If two spatial surfaces F_1 and F_2 are equivalent, then their Seifert matrices are unimodular-congruent.*

According to Proposition 4.1, a Seifert matrix of a spatial surface is an invariant of the spatial surface up to unimodular-congruent.

Corollary 4.2. *Let V_1 and V_2 be Seifert matrices of spatial surfaces F_1 and F_2, respectively. If $F_1 \cong F_2$, then $\gcd\{k \times k\text{-minors of } V_1\} = \gcd\{k \times k\text{-minors of } V_2\}$ for any $k \in \mathbb{Z}_{>0}$.*

Theorem 4.3. *For any oriented link L, there exists a family $\{(F_n, F'_n)\}_{n\in\mathbb{Z}_{\geq 0}}$ of pairs of Seifert surfaces for L satisfying the following:*

1. *For any $n \in \mathbb{Z}_{\geq 0}$, the regular neighborhoods of F_n and F'_n are equivalent as handlebody-knots.*
2. *For any $n \in \mathbb{Z}_{\geq 0}$, Seifert matrices of F_n and F'_n are unimodular congruent.*
3. *For any $n \in \mathbb{Z}_{\geq 0}$, F_n and F'_n are not equivalent as spatial surfaces.*

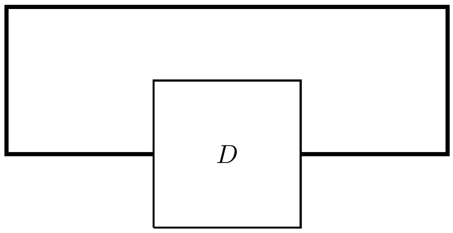


Figure 5: A diagram D of F

Sketch of proof. We construct a desired family. We take a Seifert surface F for L such that F is not equivalent to the closed 2-disk. Let D be a diagram of F, as shown in Fig. 5.

For each $n \in \mathbb{Z}_{\geq 0}$, let D_n and D'_n denote the diagrams obtained from by replacing the outer edge of D in Fig. 5 with the edges shown in Fig. 6.

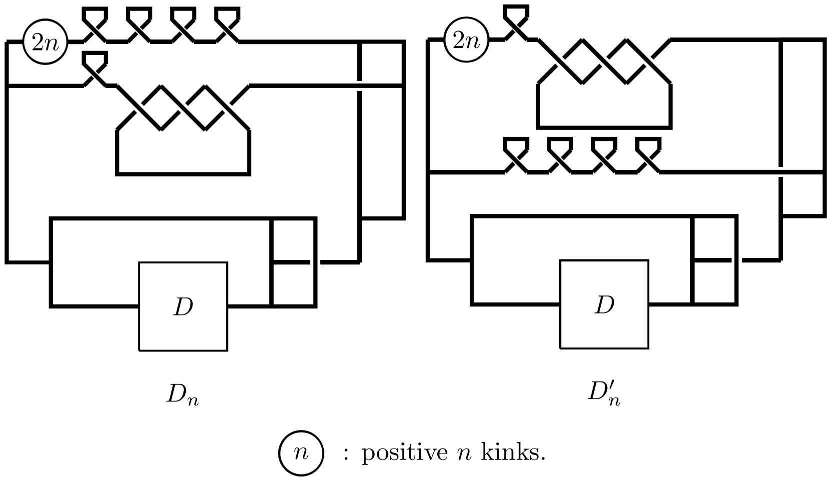


Figure 6: Diagrams D_n and D'_n $(n \in \mathbb{Z}_{\geq 0})$

Then it holds $\partial F(D_n) \cong \partial F(D'_n) \cong \partial F(D) \cong L$. For each $n \in \mathbb{Z}_{\geq 0}$, we set $F_n = F(D_n)$ and $F'_n = F(D'_n)$. Then the family $\{(F_n, F'_n)\}_{n \in \mathbb{Z}_{\geq 0}}$ is the desired family.

Next, we show that the family $\{(F_n, F'_n)\}_{n \in \mathbb{Z}_{\geq 0}}$ consists of infinitely many distinct pairs.

Let V be a Seifert matrix of the Seifert surface $F(D)$ for L. For each $k \in \mathbb{Z}_{\geq 0}$, F_k has the Seifert matrix $V_k = \begin{pmatrix} 0 & 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 1 & 0 & \mathbf{0} \\ -1 & 0 & 4 & 0 & \mathbf{0} \\ -1 & 0 & 0 & 4+2k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & V \end{pmatrix}$. We set $s \in \mathbb{Z}_{\geq 0}$ to be $\max\{i \in \mathbb{Z}_{>0} \mid \text{non-zero } (i \times i)\text{-minors of } V\}$. If such number does not exist, we define $s = 0$. For each $k \in \mathbb{Z}_{\geq 0}$, we define $E_{k,3+s} := \gcd\{(3+s) \times (3+s)\text{-minors of } V_k\}$. Then, for any $m, n \in \mathbb{Z}_{\geq 0}$ with $m \neq n$, $E_{m,3+s} \neq E_{n,3+s}$. Using Corollary 4.2, it follows $F_m \ncong F_n$.

Finally, we prove that the claims 1–3.

1. For any $n \in \mathbb{Z}_{\geq 0}$, the regular neighborhood $N(F_n)$ of F_n is equivalent to $H(D_n)$ and

the regular neighborhood $N(F_n')$ of F_n' is equivalent to $H(D_n')$. The diagrams D_n and D_n' are related by a finite sequence of R1–R6 moves and isotopies in S^2. Using Theorem 2.1, $H(D_n) \cong H(D_n')$. Therefore, $N(F_n) \cong N(F_n')$.

2. For any $n \in \mathbb{Z}_{\geq 0}$, F_n and F_n' has the same Seifert matrix $V_n = \begin{pmatrix} 0 & 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 1 & 0 & \mathbf{0} \\ -1 & 0 & 4 & 0 & \mathbf{0} \\ -1 & 0 & 0 & 4+2n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & V \end{pmatrix}$.

3. We take the MGR $X = \bigsqcup_{(x,i)\in R_3\times C_2} (\{(x,i)\} \times \mathbb{Z}_2)$ defined by

$$((x,i),a) \triangleleft ((y,j),b) = \begin{cases} ((x,i),a) & \text{if } b = 0, \\ ((2y-x, i+1), a) & \text{if } b = 1 \end{cases}, \quad ((x,i),a)((x,i),b) = ((x,i),a+b).$$

Give Y-orientations to the diagrams D_n and D_n'. Any X-coloring of D_n is given in the Fig. 7.

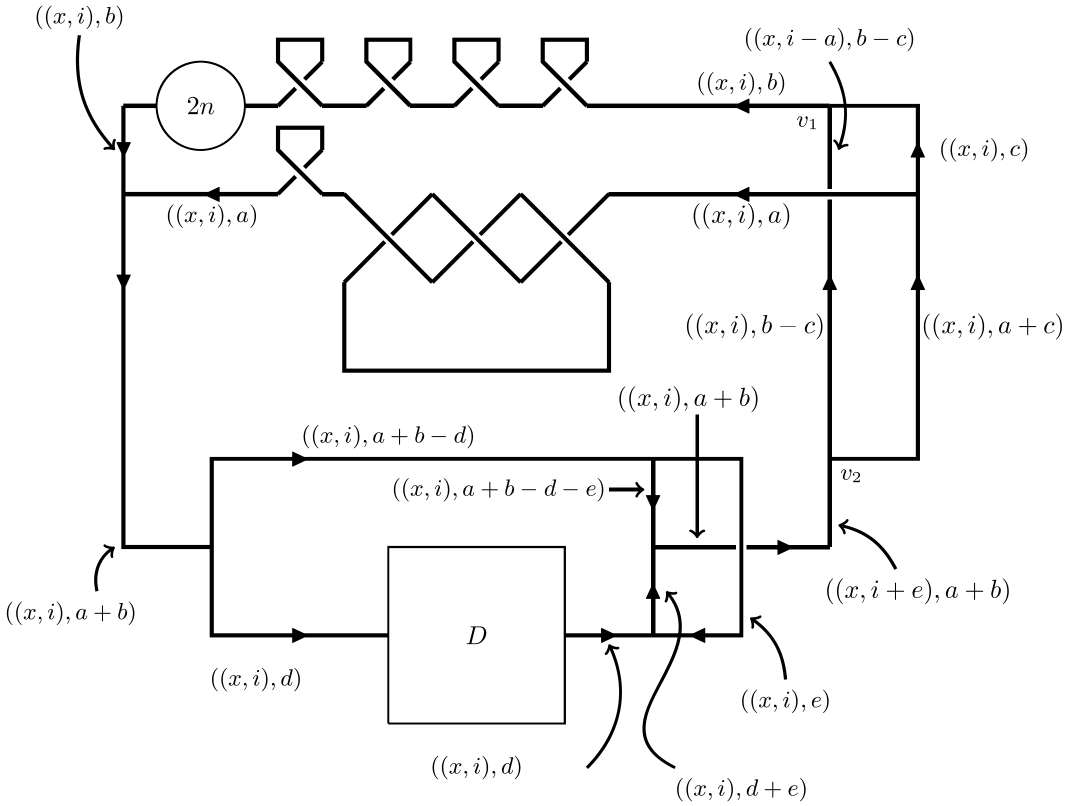
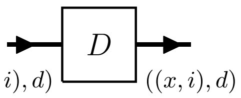


Figure 7: X-colored diagram D_n

The relations from coloring conditions at the vertices v_1 and v_2 is the following:

$$\begin{cases} i - a = i \pmod 2, \\ i + c = i \pmod 2. \end{cases}$$

Thus we have $a = 0$ and $e = 0$.

We set $\mathrm{Col}_X(D; ((x,i),d)) := \mathrm{Col}_X \Bigg($ 

$\Bigg)$.

Then,

$$|\mathrm{Col}_X(D_n)| = \bigsqcup_{((x,i),(b,c,d)) \in (R_3 \times C_2) \times \mathbb{Z}_2^3} |\mathrm{Col}_X(D; ((x,i),d))|$$
$$= 4 \left(\bigsqcup_{((x,i),d) \in X} |\mathrm{Col}_X(D; ((x,i),d))| \right).$$

We remark that $|\mathrm{Col}_X(D_n)| > 0$ because $|\mathrm{Col}_X(D; ((x,i),0))| > 0$ for any $(x,i) \in R_3 \times C_2$.

On the other hand, Any X-coloring of D'_n is given in the Fig. 8.

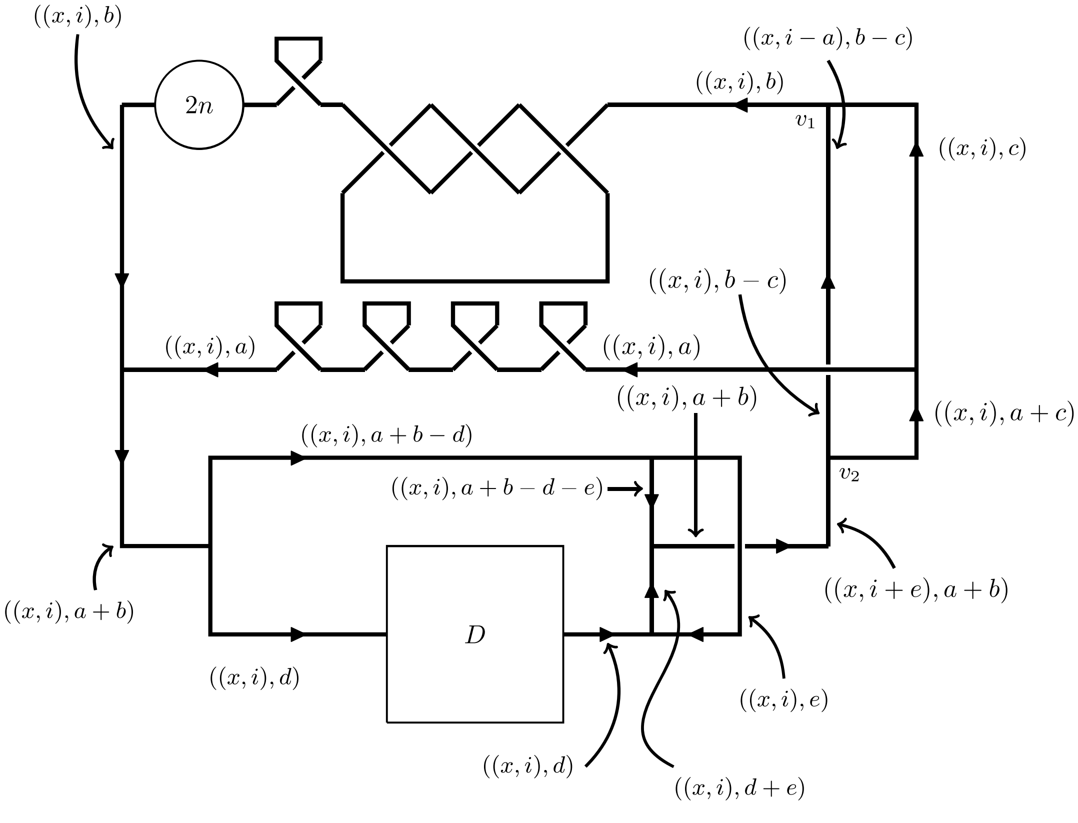


Figure 8: X-colored diagram D'_n

The relations from coloring conditions at the vertices v_1 and v_2 is the following:

$$\begin{cases} i - a = i \pmod 2, \\ i + c = i \pmod 2. \end{cases}$$

Thus we have $a = 0$ and $e = 0$.

Then we have

$$
\begin{aligned}
|\mathrm{Col}_X(D'_n)| &= \bigsqcup_{((x,i),(b,c,d))\in(R_3\times C_2)\times\mathbb{Z}_2^3} \Big(|\mathrm{Col}_X(D,((x,i),d))| \cdot \Big|\mathrm{Col}_X(\rightarrow\boxed{3_1}\rightarrow\textcircled{1}\rightarrow\,,((x,i),b))\Big|\Big) \\
&= 8\left(\bigsqcup_{((x,i),d)\in X} |\mathrm{Col}_X(D,((x,i),d))|\right) > |\mathrm{Col}_X(D_n)|\,.
\end{aligned}
$$

Therefore, $F_n \not\cong F'_n$. □

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