On metrics for quandles

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1 Introduction

This manuscript is based on [6]. A *quandle* is an algebraic system which can be regarded as a generalization of the conjugation of a group. Recently, quandles are studied in many branches of mathematics, for example, in knot theory as an invariant of knots, symmetric space theory, and so on. The axioms of quandles correspond to the fundamental transformations for knot diagrams called the Reidemeister moves (see [8] for details), and correspond to the properties of point symmetries of symmetric spaces (see [9] for details). In these fields, one often focuses on finite quandles because they give explicit and computable knot invariants, and are regarded as a discretization of compact symmetric spaces. On the other hand, there are many interesting examples of countable quandles. For example, the knot quandle of a non-trivial knot in the 3-sphere is countable. In addition, discrete subquandles in non-compact symmetric spaces, like the Euclidean space, and the hyperbolic plane, are countable quandles in general. We focus on such countable quandles.

Similar to group theory, it is more difficult to study infinite quandles than finite ones. Here, we recall techniques for geometric group theory. Finitely generated groups are naturally equipped with a different structure from the group structure: let G be a finitely generated group and let S be its finite generating set. Then they give a graph structure for G called the Cayley graph. The graph structure induces a metric on G by the path metric. The metric depends on the choice of the generating set S, but the quasi-isometry class is determined independently of that. In other words, the quasi-isometry invariant for the metric space can be regarded as an invariant of the group.

The notion of the Cayley graph for group is generalized to the Schreier graph for a set with a group action, whose set of vertices is the set, and whose edges are defined by a group action on the set (see Subsection 2.2). Connected components of this graph correspond to the orbits of the action, and become metric spaces by the path metric induced by the graph structure. The metrics depend on the choice of the generating set, but quasi-isometry classes of metric spaces are uniquely determined up to the choice of that.

A quandle structure naturally defines two groups acting on the quandle. One of these is called the *inner automorphism group*, which is a group generated by the point symmetries. The other is called the *displacement group*, which roughly corresponds to the identity component of the inner automorphism group. We introduce graph structures on a quandle by the Schreier graphs with respect to these natural actions. In particular, the graph structure corresponding to the action of the inner automorphism group is a generalization of the notion of a diagram of a quandle, which was defined by Winker [14], and has been studied by several researchers [5, 1, 11]. By rephrasing the properties of the Schreier graph in terms of our graph of quandles, we immediately obtain that the metrics are determined up to the choice of finite generating sets (see Theorem 3.4). Therefore, we can now investigate the geometry of quandles. We note that for a finitely generated quandle, the inner automorphism group is finitely generated, but the displacement group may not be finitely generated. If both the inner automorphism group and the displacement group are finitely generated, then we have two quasi-isometry classes for the quandle. In general, they are not quasi-isometric by Proposition 3.5.

Finally, we provide some examples whose connected component is quasi-isometric to typical metric spaces. Similar to the free group, the free quandle with the inner metric is quasi-isometric to the the tree. In particular, we focus on the *generalized Alexander quandles* with the displacement metric. This type of quandle is a group equipped with a quandle structure given by a group automorphism, which is studied in detail in [4, 3]. It is important in quandle theory, for example, every homogeneous quandle is presented as a quotient of a generalized Alexander quandle. Any connected component of a generalized Alexander quandle with the displacement metric is rephrased to the word metric of the displacement group (Theorem 4.2). Using generalized Alexander quandles, we give examples of quandles whose connected component is quasi-isometric to the Euclidean spaces, the hyperbolic plane, and some 3-dimensional homogeneous spaces.

2 Preliminaries

2.1 Quandle

In this subsection, we review some notions of quandles and properties related to group actions on quandles. The following definitions were originally given by Joyce [7].

Definition 2.1 ([7, 10])**.** A non-empty set X equipped with a binary operation $\triangleleft$ is called a *quandle* if the following conditions hold:

1. $x \triangleleft x = x$ holds for any $x \in X$.
2. The map $s_y \colon X \to X$ defined by $s_y(x) := x \triangleleft y$ is a bijection for any $y \in X$.
3. $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$ holds for any $x, y, z \in X$.

The bijection $s_y \colon X \to X$ is called the *point symmetry* at y. We denote $x \triangleleft^{-1} y := s_y^{-1}(x)$.

Let X and Y be quandles. A map $f \colon X \to Y$ is called a *quandle homomorphism* if it satisfies $f(x \triangleleft x') = f(x) \triangleleft f(x')$ for any elements $x, x' \in X$. A bijective quandle homomorphism $f \colon X \to Y$ is called a *quandle isomorphism*. The automorphism group $\mathrm{Aut}(X)$ is a group consisting of all quandle isomorphisms from X to itself. The binary operation of $\mathrm{Aut}(X)$ is given by $fg := g \circ f$, and the group acts on the quandle from the right. A quandle X is said to be *homogeneous* if $\mathrm{Aut}(X)$ acts transitively on X. Note that any point symmetry s_y is a quandle isomorphism.

Definition 2.2. The *inner automorphism group* of X is a subgroup in $\mathrm{Aut}(X)$ generated by $\{s_y \mid y \in X\}$, and is denoted by $\mathrm{Inn}(X)$. A *connected component* of X is an orbit under the action of $\mathrm{Inn}(X)$. The set of all connected components of X is denoted by $\pi_0(X) = X/\mathrm{Inn}(X)$. A quandle X is said to be *connected* if $\mathrm{Inn}(X)$ acts transitively on X.

The following group was defined by Joyce [7] as the transvection group. Roughly speaking, this group is the unit component of inner automorphism group.

Definition 2.3. The *displacement group* of a quandle X is the subgroup of $\mathrm{Aut}(X)$ generated by the set $\{s_x s_y^{-1} \mid x, y \in X\}$, and is denoted by $\mathrm{Dis}(X)$.

We end this subsection with some examples of quandles.

Example 2.4. We define a quandle structure $\triangleleft$ on $\mathbb{Z}$ by

$$x \triangleleft y := 2y - x \qquad \text{for all } x, y \in \mathbb{Z}.$$

This quandle is called the *infinite dihedral quandle*, and is denoted by R_∞. This quandle can be regarded as a discrete subquandle of the symmetric space $\mathbb{R}$.

Example 2.5. Let G be a group, and let $X \subseteq G$ be a nonempty subset that is closed under conjugation. Then, the set X is a quandle equipped with a binary operation $\triangleleft$ defined by

$$x \triangleleft y := y^{-1}xy, \qquad \text{for all } x, y \in X,$$

and is called the *conjugation quandle*.

2.2 Schreier graph

In this subsection, we introduce the notion of Schreier graphs. This graph is regarded as a generalization of the Cayley graph.

Definition 2.6. Let X be a nonempty set equipped with a right action of a group G, and let $S \subset G$ be a generating set. The *Schreier graph* of the right action of G with respect to S is the undirected graph $\mathrm{Sch}(X, G, S)$ which is defined as follows:

1. The set of vertices is X.
2. Two vertices x and y are connected by an edge if it satisfies that $y = x \cdot s$ for some $s \in S \cup S^{-1}$, where $S^{-1} := \{s^{-1} \mid s \in S\}$. Then the edge is labeled by s.

Remark 2.7. The Cayley graph of a group G can be regarded as a special case of the Schreier graph. In fact, the Schreier graph of the natural right action of G on the set G defined by $g \cdot h = gh$ for $g, h \in G$ is the Cayley graph with a certain generating set.

It is easy to show that connected components of the Schreier graph correspond to orbits of the action. The graph structure of $\mathrm{Sch}(X, G, S)$ provides a metric d_S^{Sch} on each connected component, in other words, each G-orbit, as the path metric. This metric depends on the choice of the generating set S. However, if G is finitely generated, then quasi-isometry class of the metric is determined up to the choice of S as the following proposition.

Proposition 2.8. *Let X be a nonempty set equipped with a right action of a finitely generated group G. If finite subsets S and T generate the group G, then for any G-orbit O in X, the identity map* $\mathrm{id}\colon (O, d_S^{\mathrm{Sch}}) \to (O, d_T^{\mathrm{Sch}})$ *is a quasi-isometry.*

3 Metrics for quandle

In this section, we define graph structures for a quandle X by using the framework of the Schreier graphs.

Definition 3.1. Let X be a quandle.

1. The Schreier graph associated with the action of the inner automorphism group and its generating set A is denoted by $\Gamma_A^{\mathrm{Inn}}(X)$.
2. The Schreier graph associated with the action of the displacement group and its generating set U is denoted by $\Gamma_U^{\mathrm{Dis}}(X)$.

The graph $\Gamma_A^{\mathrm{Inn}}(X)$ is a generalization of a *diagram of a quandle* introduced by Winker [14]. In fact, if T is a generating set of a quandle X, then $A := s(T) = \{s_a \mid a \in T\}$ generates the inner automorphism group, and the graph $\Gamma_A^{\mathrm{Inn}}(X)$ is isomorphic to the diagram of the quandle.

The reason that we focus on the groups $\mathrm{Inn}(X)$ and $\mathrm{Dis}(X)$ comes from the next proposition.

Proposition 3.2. *Let X be a quandle. Then, there exists one to one correspondence among the set of connected components of X, the set of connected components of $\Gamma_A^{\mathrm{Inn}}(X)$ and the set of connected components of $\Gamma_U^{\mathrm{Dis}}(X)$.*

Hence, we can define metrics on each connected component of a quandle by the path metrics induced by the graphs.

Definition 3.3. Let X be a quandle. The metric on a connected component induced by $\Gamma_A^{\mathrm{Inn}}(X)$ is called the *inner metric* with respect to a generating set $A \subset \mathrm{Inn}(X)$, and denoted by d_A^{Inn}. The metric on a connected component induced by $\Gamma_U^{\mathrm{Dis}}(X)$ is called the *displacement metric* with respect to a generating set $U \subset \mathrm{Dis}(X)$, and denoted by d_U^{Dis}.

As in the case of a word metric for a group, quasi-isometry classes of these metrics are determined up to the choice of finite generating sets.

Theorem 3.4. *Let O be a connected component of a quandle X.*

1. *For finite generating sets $A, B \subset \mathrm{Inn}(X)$, the metric spaces (O, d_A^{Inn}) and (O, d_B^{Inn}) are quasi-isometric.*
2. *For finite generating sets $U, V \subset \mathrm{Dis}(X)$, the metric spaces (O, d_U^{Dis}) and (O, d_V^{Dis}) are quasi-isometric.*

The inner automorphism group of a finitely generated quandle is finitely generated. However, there exists a finitely generated quandle whose displacement group is not finitely generated. In fact, the knot quandle of a non-fibered knot holds this property.

By the above theorem, if both the inner automorphism group and the displacement group are finitely generated, then we obtain two quasi-isometry classes of metrics on a connected component. In general, they are not quasi-isometric.

Proposition 3.5. *There exist a quandle X and its connected component O which satisfy the following properties:*

1. *The groups* $\mathrm{Inn}(X)$ *and* $\mathrm{Dis}(X)$ *are finitely generated.*
2. *For any finite generating set A of* $\mathrm{Inn}(X)$ *and U of* $\mathrm{Dis}(X)$*, the metric spaces* (O, d_A^{Inn}) *and* (O, d_U^{Dis}) *are not quasi-isometric.*

In fact, the infinite dihedral quandle R_∞ is an example of the above proposition. At the last of this section, we give propositions for special cases.

Proposition 3.6. *Let O and O' be connected components of a homogeneous quandle X.*

1. *If $A \subset \mathrm{Inn}(X)$ is a finite generating set, then the metric spaces (O, d_A^{Inn}) and (O', d_A^{Inn}) are quasi-isometric.*
2. *If $U \subset \mathrm{Dis}(X)$ is a finite generating set, then the metric spaces (O, d_U^{Dis}) and (O', d_U^{Dis}) are quasi-isometric.*

Proposition 3.7. *Let O be a connected component of a quandle X. Let us assume that the displacement group is finitely generated, and acts freely on O. Then, the connected component O with the displacement metric with respect to some finite generating set is quasi-isometric to the group* $\mathrm{Dis}(X)$ *with a word metric with respect to some finite generating set.*

We note that the inner automorphism group cannot act freely on any connected component because of the first axiom of quandles.

4 Example

In this section, we provide some quandles quasi-isometric to certain metric spaces. First, we consider the free quandles (see [8] for details). We recall that the finitely generated free groups with word metrics are quasi-isometric to trees. The free quandles hold a similar property.

Proposition 4.1. *Let A be a finite set with the cardinality more than one, and let $FQ[A]$ be the free quandle generated by A. Then, each connected component of $FQ[A]$ with the inner metric d_A^{Inn} is quasi-isometric to a tree.*

We note that the displacement group of $FQ[A]$ is not finitely generated. Hence, we cannot apply Theorem 3.4 for the displacement metric of $FQ[A]$.

In the following, we consider the displacement metrics. In particular, we focus on a special class of quandles. For a group G and a group automorphism $\sigma \in \mathrm{Aut}(G)$, a quandle structure $\triangleleft$ on G is defined by $x \triangleleft y := \sigma(xy^{-1})y$. Then, the quandle is called the *generalized Alexander quandle*, and denoted by $\mathrm{GAlex}(G, \sigma)$. It is easy to see that

$\mathrm{GAlex}(G, \sigma)$ is homogeneous, and the displacement group of the quandle acts freely on each connected component. The displacement metric of $\mathrm{GAlex}(G, \sigma)$ is determined as follows.

Theorem 4.2. *Let G be a group and let σ be its group automorphism. If the displacement group of the generalized Alexander quandle $X := \mathrm{GAlex}(G, \sigma)$ is finitely generated, then any connected component O of the quandle with a displacement metric is quasi-isometric to the displacement group with a word metric.*

Hence, to determine the quasi-isometric class of a connected component of a generalized Alexander quandle with the displacement metric, it is enough to calculate the quasi-isometric class of the displacement group with a word metric. The displacement group of a generalized Alexander quandle is calculated as follows.

Lemma 4.3 (cf. [4])**.** *Let $X := \mathrm{GAlex}(G, \sigma)$. We denote the connected component of X containing $1 \in G$ by P.*

1. *The subset P is a subgroup of G. Moreover, the group P is isomorphic to the displacement group $\mathrm{Dis}(X)$.*
2. *If the group automorphism σ is given as an inner automorphism with respect to $g \in G$, that is, $\sigma(x) = g^{-1}xg$ for $x \in G$, then P is isomorphic to the commutator subgroup $[\langle\langle g \rangle\rangle_G, \langle\langle g \rangle\rangle_G]$ of $\langle\langle g \rangle\rangle_G$, where $\langle\langle g \rangle\rangle_G$ is the normal closure of $\{g\}$ in G.*

As a conclusion of this section, we provide concrete examples of generalized Alexander quandles quasi-isometric to well-known metric spaces, which is the Euclidean spaces, the hyperbolic plane, and 3-dimensional homogeneous spaces.

Proposition 4.4. *Let t be a group automorphism of $\mathbb{Z}^n$, and let $X = \mathrm{GAlex}(\mathbb{Z}^n, t)$. Then, any connected component of X with a displacement metric is quasi-isometric to the k-dimensional Euclidean space, where $k = \mathrm{rank}(1 - t^{-1})$.*

Proposition 4.5. *Let $\Delta^+(p, q, r)$ be the index-2 subgroup of the triangle group $\Delta(p, q, r)$, that is,*

$$\Delta^+(p, q, r) = \langle a, b, c \mid a^p = b^q = c^r = abc = 1 \rangle_{\mathrm{Grp}}.$$

Let us define a group automorphism $\sigma \colon \Delta^+(p, q, r) \to \Delta^+(p, q, r)$ by $\sigma(g) := a^{-1}ga$. If $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, then a connected component of $\mathrm{GAlex}(\Delta^+(p, q, r), \sigma)$ with a displacement metric is quasi-isometric to the 2-dimensional hyperbolic space.

We note that there exists a surjective quandle homomorphism from the quandle in the above proposition to a discrete subquandle in the 2-dimensional hyperbolic space with certain quandle structure.

For a knot K in the 3-sphere S^3 and a positive integer n, a 3-orbifold of the base space S^3 with the singular set K whose cone-angle is equal to $\frac{2\pi}{n}$ is denoted by $\mathcal{O}(K, n)$. Let $G(K) = \pi_1(S^3 \setminus K)$ be the knot group of K, and fix a meridian $\mu \in G(K)$. Then, it is known that the orbifold fundamental group $G_n(K) := \pi_1^{\mathrm{orb}}(\mathcal{O}(K, n))$ is isomorphic to $G(K)/\langle\langle \mu^n \rangle\rangle_{G(K)}$, where $\langle\langle \mu^n \rangle\rangle_{G(K)}$ is the normal closure of μ^n in $G(K)$. We denote the image of the meridian in $G_n(K)$ by the same symbol μ.

Proposition 4.6. *Let $\sigma\colon G_n(K) \to G_n(K)$ be a group automorphism defined by $\sigma(g) = \mu^{-1}g\mu$. If the orbifold $\mathcal{O}(K,n)$ is geometric, then any connected component of the quandle* $\mathrm{GAlex}(G_n(K),\sigma)$ *with a displacement metric is quasi-isometric to the universal covering space* $\widetilde{\mathcal{O}(K,n)}$.

Note that there exists a surjective quandle homomorphism from $\mathrm{GAlex}(G_n(K),\sigma)$ to the knot n-quandle $Q_n(K)$. Proposition 4.6 gives many examples of quandles quasi-isometric to 3-dimensional homogeneous spaces.

1. Let K be a hyperbolic knot. If a positive integer n is large enough, then the orbifold $\mathcal{O}(K,n)$ is hyperbolic by the hyperbolic Dehn surgery theorem (for example, see [12]). Hence, a connected component of $\mathrm{GAlex}(G_n(K),\sigma)$ is quasi-isometric to $\mathbb{H}^3$.

2. Let K be a Montesinos knot. It is known that the orbifold $\mathcal{O}(K,2)$ has a Seifert structure by the Montesinos trick. Hence, a connected component of $\mathrm{GAlex}(G_n(K),\sigma)$ is quasi-isometric to one of the following geometries: $S^2\times\mathbb{E}^1$, S^3, $\mathbb{E}^3$, Nil, $\mathbb{H}^2\times\mathbb{E}^1$ and $\widetilde{SL}_2$ ([13, 2]). For example, if K is the $(-2,3,7)$-pretzel knot, then a connected component of $\mathrm{GAlex}(G_n(K),\sigma)$ is quasi-isometric to $\widetilde{SL}_2$.

3. According to the classification of geometric orbifolds given by Dunbar [2], we can construct more examples. A connected component of the quandle $\mathrm{GAlex}(G_3(4_1),\sigma)$ is quasi-isometric to $\mathbb{E}^3$, where 4_1 is the figure-eight knot. A connected component of the quandle $\mathrm{GAlex}(G_6(3_1),\sigma)$ is quasi-isometric to Nil, where 3_1 is the trefoil.

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