Problems on Low-dimensional Topology, 2025

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This is a list of open problems on low-dimensional topology with expositions of their history, background, significance, or importance. This list was made by editing manuscripts written by contributors of open problems to the problem session of the conference "Intelligence of Low-dimensional Topology" held at Research Institute for Mathematical Sciences, Kyoto University in May 26–28, 2025.

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1 Generalizing properties from the curve complex in the framework of Artin groups

(María Cumplido)

The braid group on n strands, introduced by Artin at the beginning of the 20th century, can be identified with the mapping class group of the n-punctured disk. Algebraically, the braid group is an Artin group that admits the following presentation:

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i, & \text{for } |i-j| > 1 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, & \text{for } |i-j| = 1 \end{array} \right\rangle.$$

Over the past few decades, both topological and combinatorial approaches have played central roles in understanding the structure of braid groups. On the one hand, the combinatorial viewpoint, particularly through Garside theory, has provided powerful algebraic tools. On the other hand, the topological perspective via the action of the braid group on the *curve complex* of the punctured disk has offered deep geometric insights.

The *curve complex* of a surface is a simplicial complex whose vertices correspond to isotopy classes of essential simple closed curves on the surface, with higher-dimensional simplices representing collections of such curves that can be realized disjointly. The mapping class group of the surface acts naturally on this complex by sending curves to their images under homeomorphisms.

This action underlies the Nielsen–Thurston classification, which classifies elements of the mapping class group into three types. An element g is called:

- *periodic* if some power of g is the identity,
- *reducible* if some power of g preserves a simplex (*i.e.*, a collection of disjoint curves) in the curve complex,
- *pseudo-Anosov* if neither of the above conditions holds. In the pseudo-Anosov case, there exist a pair of transverse measured foliations on the surface that are invariant under g, scaled respectively by a factor $\lambda > 1$ and $1/\lambda$ each time g acts.

In the 1980s, Birman, Lubotzky and McCarthy [4] established that for every reducible mapping class g, one can canonically choose a collection of invariant curves. These curves decompose the surface into subsurfaces on which g acts either periodically or pseudo-Anosov, yielding a powerful structural understanding of reducible elements.

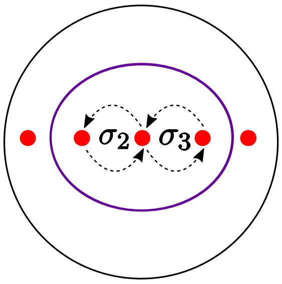
In general, an Artin group is defined by a finite set of generators and a symmetric matrix $\left(m_{s,t}\right)_{s,t\in S}$, where $m_{s,s} = 1$ and $m_{s,t} \in \{2, 3, \ldots, \infty\}$ for $s \neq t$. Its presentation is:

$$A_S = \langle S \mid \underbrace{sts\cdots}_{m_{s,t}} = \underbrace{tst\cdots}_{m_{s,t}},\; m_{s,t} \neq \infty \rangle.$$

It is easy to see that the braid group fits this framework. Moreover, braid groups are examples of *spherical-type* Artin groups, meaning that their associated Coxeter groups (obtained by adding the relations $s^2 = 1$) are finite. All spherical-type Artin groups admit a *Garside structure*.

In the case of braid groups, the vertices of the curve complex of the n-punctured disk are in bijection with certain subgroups called *irreducible parabolic subgroups*. A *standard parabolic subgroup* of an Artin group A_S is a subgroup A_X generated by a subset $X \subseteq S$.

For braid groups, if the generators in X are consecutive (*i.e.*, if A_X is irreducible), then A_X corresponds to a circle C enclosing the punctures associated with the generators in X (see the figure below). More generally, a *parabolic subgroup* is any conjugate of a standard parabolic subgroup. In braid groups, a conjugate $\alpha A_X \alpha^{-1}$ corresponds to the image of C under the action of α.



We can define an analogous flag simplicial complex for Artin groups, mimicking the curve complex. In this complex, two irreducible parabolic subgroups P and Q span an edge if one is contained in the other, or if their intersection is trivial and $pq = qp$ for all $(p, q) \in P \times Q$.

In spherical-type Artin groups, the Garside structure simplifies checking this condition: it suffices to verify whether the elements z_P and z_Q that generate the centres of P and Q commute [10].

Question 1.1 (M. Cumplido, V. Gebhardt, J. González-Meneses, B. Wiest)**.** *For which Artin groups is the complex of irreducible parabolic subgroups Gromov-hyperbolic?*

This question is only known for braid groups and three additional specific cases [6].

We can also define a Nielsen–Thurston-type classification based on whether elements preserve parabolic subgroups.

Question 1.2 (M. Cumplido, J. González-Meneses)**.** *Is there a notion of canonical reduction system for a reducible element of an Artin group? If so, can it be computed?*

This question builds on more basic ones, such as the following:

Question 1.3 (folklore)**.** *Is the intersection of parabolic subgroups again a parabolic subgroup?*

This is known for spherical-type [10], certain 2-dimensional [11, 5], some FC-type [30, 29, 1], and some Euclidean-type cases [21].

Question 1.4 (L. Paris, E. Godelle). *Can the normalizer of a parabolic subgroup be described?*

This has been answered for spherical-type [33, 17], FC-type [18], 2-dimensional [19], and some Euclidean-type groups [21].

Question 1.5 (M. Cumplido, J. González-Meneses). *Is the intersection of all irreducible maximal parabolic subgroups (with respect to inclusion) preserved by a reducible element of an Artin group non-empty?*

Question 1.6 (M. Cumplido, J. González-Meneses). *Does there exist a combinatorial algorithm to determine whether an element of an Artin group is periodic, reducible, or pseudo-Anosov?*

Question 1.7 (M. Cumplido, J. González-Meneses). *When cutting along the canonical reduction system of a braid, we obtain a description of how the braid acts on each part of the surface. How does this translate into the context of Artin groups?*

2 (Pure) cactus groups

(Kazuhiro Ichihara[2] and Takatoshi Hama[3])

The *cactus group*, an analogue of the braid group, was introduced in [22], motivated by the study of quantum groups.[4] More precisely, for any integer $n \geq 2$, the *cactus group of degree* n, denoted by J_n, is defined by a presentation with generators $s_{p,q}$ for $1 \leq p < q \leq n$ and the following relations:

- $s_{p,q}^2 = e$ for all $1 \leq p < q \leq n$,
- $s_{p,q}s_{m,r} = s_{m,r}s_{p,q}$ if $[p,q] \cap [m,r] = \emptyset$,
- $s_{p,q}s_{m,r} = s_{p+q-r,,p+q-m}s_{p,q}$ if $[m,r] \subset [p,q]$,

where e denotes the identity element, and $[p,q]$ denotes the set $\{p, p+1, \ldots, q\}$ of integers for $p < q$.

As with the braid group, elements of the cactus group can be represented by planar diagrams consisting of vertical strands. Examples of such diagrams for J_4 are shown in Figure 1.

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[4]It is worth noting that the group itself had been studied earlier in different contexts; see [12, 13] for examples.

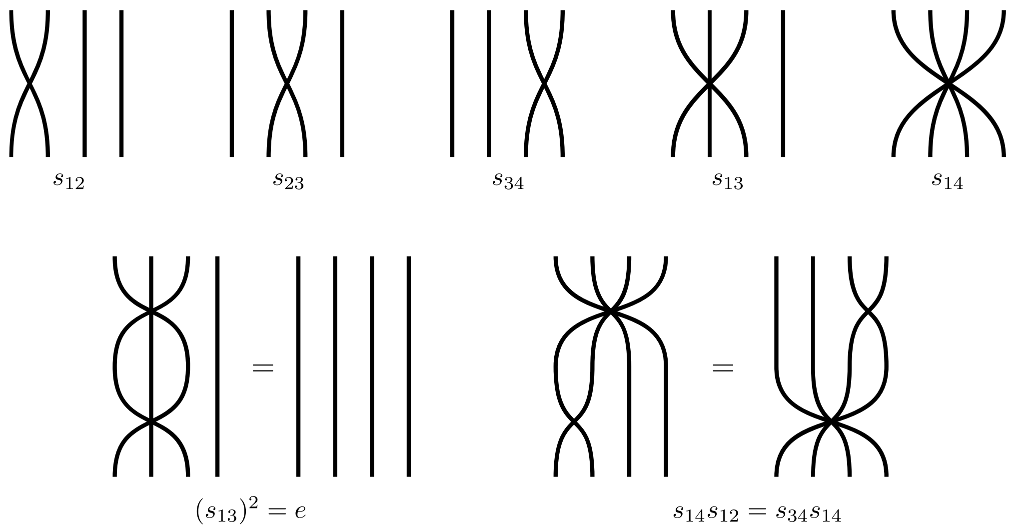

Figure 1: Diagrams for some elements of J_4

Owing to this diagrammatic representation, the cactus group J_n admits a natural projection $\pi : J_n \to S_n$ onto the symmetric group S_n of degree n. The kernel of this projection is called the *pure cactus group* of degree n, denoted by PJ_n. For further details, see [22, Subsection 3.1] or [15, Section 1].

It is shown in [15, Proposition 6.1] that the cactus group J_n is not hyperbolic for any $n \geq 6$. For small values of n, the hyperbolicity of J_n is well understood:

- $J_2 \cong \mathbb{Z}/2\mathbb{Z}$,
- J_3 is virtually the infinite dihedral group which is isomorphic to $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$,
- J_4 is virtually a hyperbolic surface group,

and all of these are hyperbolic. (See, for instance, [15, 3].)

Thus, it is natural to consider the following question.

Question 2.1 ([15, Question 8.7])**.** *Are the cactus group J_5 and the pure cactus group PJ_5 hyperbolic?*

In [22, Theorem 9], Henriques and Kamnitzer showed that PJ_n is isomorphic to the fundamental group of the Deligne–Mumford compactification $\overline{M_{0,n+1}}(\mathbb{R})$ of the moduli space of real genus-zero curves with $n+1$ marked points. Therefore, the following question is closely related to the geometric structure of $\overline{M_{0,6}}(\mathbb{R})$, which is known to be a non-orientable closed smooth 3-manifold.

Question 2.2 (K. Ichihara, T. Hama)**.** *Is the 3-manifold whose fundamental group is isomorphic to PJ_5 a hyperbolic 3-manifold?*

On the other hand, since non-elementary (*i.e.*, infinite and not virtually cyclic) hyperbolic groups have always exponential growth rate [16], the following question also arises naturally in connection with the above question.

Problem 2.3 (K. Ichihara, T. Hama)**.** *Determine the growth rate of the cactus group, in particular* J_5*.*

It is also known that, for any finitely generated hyperbolic group, the generating function encoding its growth function is rational (see, for instance, [16]). This leads to another natural question:

Question 2.4 (K. Ichihara, T. Hama)**.** *Can we describe the generating function that represents the growth function of the cactus group* J_5*?*

3 Extension of skein modules to TQFTs, and skein modules of mapping tori

(Patrick Kinnear)

The *skein module* $\mathrm{Sk}(M)$ of a 3-manifold M is the $\mathbb{Q}(q^{1/2})$-vector space spanned by isotopy classes of framed links in M, subject to the Kauffman bracket skein relations.

In [27], the following was shown.

Theorem ([27])**.** *Let* $\gamma \in MCG(\mathbb{T}^2) \cong \mathrm{SL}_2(\mathbb{Z})$*. Write* $\sim$ *for the relation of conjugacy in* $\mathrm{SL}_2(\mathbb{Z})$*,* T *for the generator corresponding to a Dehn twist, and* M_γ *for the mapping torus of* γ*. Then for* $|\mathrm{tr}(\gamma)| \leq 2$*,*

$$\dim_{\mathbb{Q}(q^{1/2})}\mathrm{Sk}(M_\gamma) = \begin{cases} 6 & |\mathrm{tr}(\gamma)| = 0 \\ 4 & |\mathrm{tr}(\gamma)| = 1 \\ 9+k & \gamma \sim T^{2k} \\ 6+k & \gamma \sim T^{2k+1} \end{cases}$$

and for $|\mathrm{tr}(\gamma)| > 2$ *we have*

$$\dim_{\mathbb{Q}(q^{1/2})}\mathrm{Sk}(M_\gamma) = |\mathrm{tr}(\gamma)| + 2^{p+1}$$

where p *is the number of* $\{d(\gamma), \mathrm{tr}(\gamma)\}$ *which are even, with* $d(\gamma)$ *the GCD of the entries of* $I_2 - \gamma$*.*

Here are some problems related to this result.

Skein TQFT in (3+1)-dimension

Theorem leads to the following.

Corollary. *There does not exist a symmetric monoidal functor*

$$\mathrm{Bord}^{\mathrm{or}}_{(4,3)} \to \mathrm{Vect}$$

sending a 3-manifold to its skein module.

We recall that $\mathrm{Bord}^{\mathrm{or}}_{(4,3)}$ is the category of 3-manifolds and 4-dimensional bordisms between them, and such functors are called (3+1)-dimensional TQFTs. The argument, due to R. Detcherry, is given in [27, Remark 1.6]. Essentially, a TQFT would allow us to calculate traces of mapping class group actions on 3-manifolds, and in particular the dimensions of Theorem will appear as traces of the mapping class group of the 3-torus. The mapping class group of a 3-manifold acts in a natural way on the skein module. But in the case of the 3-torus this action is already well-understood due to work of Carrega [7], and the dimensions of Theorem do not appear as traces.

Question 3.1 (P. Kinnear)**.** *On which 4-bordisms can skein theory be extended to a* (3+1)*-dimensional TQFT with the natural mapping class group action?*

That is, we are looking for a symmetric monoidal sub-category $\mathcal{B}$ of $\mathrm{Bord}^{\mathrm{or}}_{(4,3)}$ on which we can define a symmmetric monoidal functor $\mathcal{B} \to \mathrm{Vect}$. Ideally we should find a $\mathcal{B}$ with the same objects as $\mathrm{Bord}^{\mathrm{or}}_{(4,3)}$ but different morphisms.

Higher genus mapping tori

It would be interesting to understand skein modules of mapping tori of higher genus surfaces. To prove Theorem, we used the following work of [20], who investigated G-skein algebras for $G = \mathrm{SL}_2$ (the case above) as well as $G = \mathrm{SL}_N, \mathrm{GL}_N$.

Lemma ([20, Cor. 1.7])**.** *Let the quantum parameter be generic. Let M be an oriented 3-manifold and fix $\mathbb{T}^2 \subset M$. Then the SL_N-skein module is spanned by skeins intersecting $\mathbb{T}^2$ at most $N-1$ times. The GL_N-skein module is spanned by skeins not intersecting $\mathbb{T}^2$.*

We therefore pose:

Question 3.2 (P. Kinnear)**.** *Can a similar result to Lemma be obtained for surfaces of higher genus?*

The idea of the proof is to define algebras $\mathrm{SkAlg}_{G,n}$ of tangles in a thickened torus, modulo skein relations, together with a surjection $\phi : \mathbb{H}^G_n \to \mathrm{SkAlg}_{G,n}$ where $\mathbb{H}^G_n$ is a (possibly modified) version of the Double Affine Hecke Algebra (DAHA). The result then follows by identifying an idempotent $e \in \mathbb{H}^G_n$ such that $\mathbb{H}^G_n e \mathbb{H}^G_n \cong \mathbb{H}^G_n$, and such that $\phi(e)$ is a tangle meeting a horizontal torus in $\mathbb{T}^2 \times [0,1]$ at $n-N$ points (for SL_N) or zero points (for GL_N). The case of SL_2 admits a diagrammatic proof, while the proof for other groups is in terms of representation theory.

The DAHA is closely related to the topology of $\mathbb{T}^2$. Its spherical subalgebra $e\mathbb{H}e$ is isomorphic to the skein algebra of $\mathbb{T}^2$ [20, Thm. 1.13]. There is a proposal for a "genus 2 spherical DAHA" and a result relating it to the skein algebra [2, 9]. Answering Question 3.2 using a similar strategy to [20] will involve finding an appropriate higher genus Hecke algebra (some proposals exist in [14, 24, 2, 23]), understanding its relationship to skein theory and understanding its representation theory.

Answering Question 3.2 would allow us to approach:

Problem 3.3 (P. Kinnear). *Give a version of Theorem for mapping tori of surfaces of higher genus.*

4 Finite generating set giving the minimal growth rate

(Koji Fujiwara)

Let G be a finitely generated group, and S a finite generating set. Let S^n be the set of elements in G which are obtained as products of at most n elements, or their inverses, in S. The growth rate of (G, S) is defined by $e(G, S) = \lim_n |S^n|^{1/n}$. The minimal growth rate of G is defined by $e(G) = \inf_S e(G, S)$, where the infimum is taken over all finite generating sets S of G. Let Σ_g be the closed orientable surface of genus $g \geq 2$. It is known that $e(\pi_1(\Sigma_g))$ is achieved by some S, but it is unknown which S attains the infimum.

Problem 4.1 (K. Fujiwara). *Find such S for each Σ_g.*

5 Knitted surfaces and their chart description

(Inasa Nakamura[5])

Knitted surfaces are surfaces properly embedded in $D^2 \times B^2$, defined by using "pairings" and "knit structure" [31]. We say that two knitted surfaces are equivalent if they are related by an "isotopy of knitted surfaces". Knitted surfaces can also be given by using deformations of knits. A knitted surface is a generalization of a simple braided surface [8, 25], and a knitted surface has a graphical description called a chart description, which is a generalization of a chart description of a simple braided surface.

Problem 5.1 (I. Nakamura). *Determine a set of chart moves for knitted surfaces, which generates all chart moves. We expect that the set is formed by the moves given in [31].*

Problem 5.2 (I. Nakamura). *Classify the equivalence classes of charts of knitted surfaces under chart move equivalence, in terms of the numbers of vertices of charts.*

Any surface-link, which may be unrientable, is equivalent to the closure of a certain knitted surface [31, 32].

Problem 5.3 (I. Nakamura). *Calculate quandle cocycle invariants, the link/knot group, or other knot invariants of a surface-link by using the chart description of its associated knitted surface.*

Problem 5.4 (I. Nakamura). *Construct a family of an infinite number of surface-links by using knitted surfaces, each of which is distinguished by some invariants associated with the knitted surface.*

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6 Plat presentations for surface-links

(Jumpei Yasuda)

A *surface-link* is an (unoriented) closed surface smoothly embedded in $\mathbb{R}^4$. A *2-knot* is a surface-link that is homeomorphic to S^2. A braided surface [34] is a compact surface embedded in $D^2 \times B^2$ that is a branched covering surface over B^2, and a 2-dimensional braid [25] is a special kind of braided surface. In [37], the plat closure method was introduced to construct a surface-link from a braided surface. If the plat closure of a braided surface is isotopic to a surface-link F, it is called a *plat presentation* of F. Every surface-link can be represented by a plat presentation using a braided surface.

Let $e(F)$ be the normal Euler number of a surface-link F. The plat closure of a 2-dimensional braid is a surface-link each of whose component F satisfies $e(F) = 0$. However, the converse remains open:

Question 6.1 (J. Yasuda)**.** *Is every surface-link, each of whose component F satisfies $e(F) = 0$, equivalent to the plat closure of a 2-dimensional braid?*

The *plat index* of a surface-link F, denoted by $\mathrm{Plat}(F)$, is defined as the half of the minimum degree of braided surfaces whose plat closures are isotopic to F. This is a generalization of the bridge number $b(L)$ of a link L. It is known that surface-links with the plat index one are trivial.

It is known that the bridge number is additive for the connected sum of knots [36]: for two knots K_1 and K_2, we have $b(K_1 \sharp K_2) = b(K_1) + b(K_2) - 1$. The plat index satisfies the following inequality for surface-links F_1 and F_2:

$$\mathrm{Plat}(F_1 \sharp F_2) \;\leq\; \mathrm{Plat}(F_1) + \mathrm{Plat}(F_2) - 1.$$

In [28, 35], it was shown that the connected sum of a 3-twist-spun of a knot [38] and a trivial non-orientable surface-knot P of genus 3 with $e(P) = \pm 2$ is isotopic to P itself. This implies that the inequality above is strict in this case. A similar result is known for the braid index [25]: it is not additive for the connected sum of non-trivial 2-knots [26].

Question 6.2 (J. Yasuda)**.** *Is the plat index additive for the connected sum of 2-knots (or orientable surface-knots)?*

Let F be a 2-twist-spun 2-knot of a twist knot. Then $\mathrm{Plat}(F) = 2$.

Question 6.3 (J. Yasuda)**.** *Is the plat index of a 2-twist-spun 2-knot of a 2-bridge knot equal to two?*

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