A GENERALIZED EXTENSION OF AVERAGED MAPPINGS DEFINED IN A HILBERT SPACE TO BANACH SPACES

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Abstract. It is well known that there is an averaged mapping as one of useful nonlinear mappings for many iterative algorithms. In a Hilbert space we can obtain a theorem that an averaged mapping is determined by some inequality, since we can use the important tool of relation between a norm and an inner product. However, this determinism can not have a power in Banach spaces. In this paper, we introduce a generalized extension of an averaged mapping determined in a Hilbert space to Banach spaces.

1. An averaged mapping and its properties

At first we give an introduction of an averaged mapping defined in a Hilbert space.

Definition 1.1. Let H be a Hilbert space, and let $\langle \cdot, \cdot \rangle$ be an inner product, and let $\|\cdot\|$ be a norm on H. Let T be a mapping from H to H. T is called an averaged mapping if there exists $\alpha \in [0,1]$ and a nonexpansive mapping $N : H \to H$ such that

$$T = (1-\alpha)I + \alpha N,$$

where I is an identity mapping. More explicitly, T is called an α-averaged mapping, or an averaged mapping with α.

We obtained some properties of averaged mappings in a Hilbert space as follows([1]).

Proposition 1.1. *In a Hilbert space H, the following hold:*
(a) A mapping $T : H \to H$ is firmly nonexpansive if and only if T is a $(\frac{1}{2})$-averged mapping.
(b) Let T_1 and T_2 be firmly nonexpansive on H. Then, T_2T_1 is $(\frac{2}{3})$-averaged.
(c) Let T_i be α_i-averaged for every $i \in I = \{1, 2, 3, \cdot, \cdot, m\}$. Then, $T_mT_{m-1} \cdot \cdot T_2T_1$ is α-averaged with

$$\alpha = \frac{\sum_{i\in I}(\frac{\alpha_i}{1-\alpha_i})}{1 + \sum_{i\in I}(\frac{\alpha_i}{1-\alpha_i})}.$$

In a Hilbert space, we obtain the following theorem by using a good relation between a norm and an inner product (cf. [1]).

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Theorem 1.1. *Let $T : H \to H$, and suppose $\alpha \in (0,1)$. Then (a) and (b) are equivalence.*
(a) T is an α-averaged mapping.
(b) T satisfies the following inequality (1) for all $x, y \in E$

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - (\frac{1-\alpha}{\alpha}) \|(I-T)x - (I-T)y\|^2 . \tag{1}$$

2. Bregman distance in Banach spaces

We also introduce Bregman distance defined in Banach spaces (cf. [6]).

Definition 2.1. Let E be a Banach space. Let $f : E \to \mathbb{R}$ a *Gâteaux* differentiable and convex function and $f'(x)$ stands for the derivative of f at $x \in E$.
Then, Bregman distance D_f is defined and denoted as follows:

$$D_f(x, y) = f(x) - f(y) - \langle x - y, f'(y) \rangle ,$$

for all $x, y \in E$, where $\langle \cdot, \cdot \rangle$ stands for the duality pair

Moreover, we give a definition of $V(\cdot, \cdot)$, as a special version of Bregman distance with $f = \|\cdot\|^2$.

Definition 2.2. Let E be a smooth Banach space with a norm $\|\cdot\|$. We denote a function V on a $E \times E$ as follows:

$$V(x, y) = \|x\|^2 + \|y\|^2 - 2 \langle x, Jy \rangle$$

for all $x, y \in E$, where $\langle \cdot, \cdot \rangle$ stands for the duality pair and J is the normalized duality mapping, i.e.

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 , \quad \|x^*\| = \|x\|\}.$$

It is well known that the value of the normalized duality mapping J is singleton in a smooth Banach space. It is trivial $V(x, y) \geq 0$ for all $x, y \in E$. In a Hilbert space H, as a special case of a Banach space, it is shown that $V(x, y) = \|x - y\|^2$ for all $x, y \in H$.

Lemma 2.1. (*cf.*[10]) *The V has the following properties.*
(i) For all $x, y, z \in E$,

$$2 \langle x - y, Jy - Jz \rangle \leq V(x, y) + V(x, z).$$

(ii) If a sequence $\{x_n\} \subset E$ satisfies $\lim_{n\to\infty} V(x_n, p) < \infty$ for some $p \in E$, then $\{x_n\}$ is bounded.
(iii) Let E be a smooth and uniformly convex Banach space. There exists a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and for each real real number $r > 0$ and all $x, y \in \{z \in E : \|z\| \leq r\}$

$$0 \leq g(\|x - y\|) \leq V(x, y).$$

Remark 1. This $V(\cdot, \cdot)$ does not mean a distance, since the so-called triangle inequality is not satisfied. In a smooth and uniformly convex Banach spaces, $V(x, y) = 0$ implies $x = y$ by the above lemma 2.1 (*iii*).

3. A nonlinear mapping determined by Bregman distance V

We introduce a definition of a special mapping in Banach spaces and named V-strongly nonexpansive mapping as follows ([10]):

Definition 3.1. Let E be a smooth Banach space. A mapping $T : E \to E$ is said to be V-strongly nonexpansive if there exists a constant $\lambda > 0$ such that

$$V(Tx, Ty) \leq V(x, y) - \lambda V((I - T)x, (I - T)y), \tag{2}$$

for all $x, y \in E$. More explicitly, T is called a V-strongly nonexpansive mapping with λ.

For the sake of showing the relation among other nonlinear mappings, we give definitions of them in a Banach space E and a Hilbert space H.

Definition 3.2. (1) $T : E \to E$ is said to be firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \langle x - y, J(Tx - Ty)\rangle$$

for all $x, y \in E$.
(2) $T : H \to H$ is said to be β-inverse strongly monotone if

$$\beta \|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle$$

for all $x, y \in H$.
(3) $T : H \to H$ is said to be strongly nonexpansive if T is nonexpansive with some fixed point and if it holds that

$$(x_n - y_n) - (Tx - Ty) \to 0$$

for sequences $\{x_n\}$ and $\{y_n\}$ such that $\{x_n - y_n\}$ is bounded and $\lim_{n\to\infty}(\|x_n - y_n\| - \|Tx_n - Ty_n\|) = 0$.

We give a proposition with respect to the relation among the above nonlinear mappings.

Proposition 3.1. ([10]) *In a Hilbert space,*
(*a*) *A firmly nonexpansive mapping is V-strongly nonexpansive with $\lambda = 1$.*
(*b*) *If T is a β-inverse strongly monotone mapping for some $\beta > \frac{1}{2}$, then (I-T) is V-strongly nonexpansive with $\lambda = (2\beta - 1)$.*
(*c*) *A V-strongly nonexpansive mapping with a nonempty subset of asymptotically fixed points is strongly nonexpansive.*

Proposition 3.2. ([10], [12]) *In smooth Banach spaces,*
(*a*) *For any $c \in (-1, 1)$, cI is V-strongly nonexpansive with any $\lambda \in (0, \frac{1+c}{1-c})$. Especially, for $c = 1$, I is V-strongly nonexpansive with any $\lambda > 0$.*
(*b*) *If T is V-strongly nonexpansive with $\lambda > 0$, then for any $\alpha \in [-1, 1]$ with $\alpha \neq 0$, αT is V-strongly nonexpansive with $\alpha^2\lambda$.*
(*c*) *If T is V-strongly nonexpansive with $\lambda \geq 1$, then $(I - T)$ is also V-strongly nonexpansive with λ^{-1}.*
(*d*) *Suppose that T is V-strongly nonexpansive with $\lambda > 0$, and that $\alpha \in [-1, 1]$ satisfies $\alpha^2\lambda \geq 1$. Let $T_\alpha = (I - \alpha T)$. Then T_α is V-strongly nonexpansive with $(\alpha^2\lambda)^{-1}$. Moreover,*

$$V(T_\alpha x, T_\alpha y) \leq V(x, y) - \lambda^{-1} V(Tx, Ty).$$

Remark 2. The V-strongly nonexpansive mapping is not neccesarily nonexpansive. We have an example of a V-strongly nonexpansive mapping which has a nonempty subset of fixed points and which is not even quasi-nonexpansive in some Banach space ([12]).

4. Comparison between the averaged mapping and the V-strongly nonexpansive mapping in Banach spaces

We recall Theorem 1.1 that a mapping T is α-averaged mapping if and only if T satisfies the following inequality for all $x, y \in E$

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - (\frac{1-\alpha}{\alpha}) \|(I-T)x - (I-T)y\|^2 ,$$

where $\alpha \in (0,1)$. Exchanging $\frac{1-\alpha}{\alpha}$ for λ, we get

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \lambda \|(I-T)x - (I-T)y\|^2 ,$$

as $\lambda > 0$. In a Hilbert space, this means that T is an averaged mapping if and only if T is a V-strongly nonexpansive mapping, since the following correspondanse holds:

$$\lambda = \frac{1-\alpha}{\alpha} \Longleftrightarrow \alpha = \frac{1}{\lambda + 1}.$$

In a Banach space, we could also give a definition of an averaged mapping as the same as one in a Hilbert space;

Definition 4.1. Let E be a Banach space with a norm $\|\cdot\|$. Let T be a mapping from E to E. T is called an averaged mapping if there exists $\alpha \in (0,1)$ and a nonexpansive mapping $N : E \to E$ such that

$$T = (1-\alpha)I + \alpha N.$$

Then T is a nonexpansive mapping since

$$\begin{aligned}
\|Tx - Ty\| &= \|(1-\alpha)x + \alpha Nx - (1-\alpha)y - \alpha Ny\| \\
&= \|(1-\alpha)(x-y) + \alpha(Nx - Ny)\| \\
&\leq (1-\alpha)\|x-y\| + \alpha \|Nx - Ny\| \\
&\leq (1-\alpha)\|x-y\| + \alpha \|x-y\| = \|x-y\| .
\end{aligned}$$

This definition means that an averaged mapping is a nonexpansive mapping in an arbitrary Banach space. However, with respect to a V-strongly nonexpansive mapping, we have obtained an example of V-strongly nonexpansive mappings which has a nonempty set of fixed points and which is not even quasi nonexpansive in a Banach space l^p with $p = \frac{3}{2}$ (see [12]).

The properties of averaged mappings are useful for the convergence theorems for fixed points with an iterative scheme, especially in a Hilbert space. However in Banach spaces, instead of of those V-strongly nonexpansive mappings ought to play important role.

At last we should introduce a theorem with respect to this nonlinear mapping in a Banach space.

Theorem 4.1. ([10]) *Let E be a reflexive, smooth and strictly convex Banach space. Let C be a nonempty, closed, and convex subset of E, and $R_C : E \to C$ sunny and generalized nonexpansive retraction. Let $B : E^* \to 2^E$ be a maximal monotone operator and let $J_{r_n} = (I + r_n BJ)^{-1}$ be a generalized resolvent of B for a sequence $\{r_n\}$, where the duality mapping J of E is weakly sequential continuous, and $\{r_n\} \subset (0, \infty)$. Suppose $A : C \to E$ is a V strongly nonexpansive mapping with $\lambda \geq 1$ such that $C_0 = A^{-1} \cap (BJ)^{-1}(0) \neq 0$. For an $\alpha \in [-1, 1]$ such that $\alpha^2 \lambda \geq 1$, let a sequence $\{x_n\} \subset C$ be defined as follows: For any $x = x_1 \in C$ and every $n \in \mathbb{N}$,*

$$y_n = R_C(I - \alpha A)x_n,$$
$$x_{n+1} = R_C(\beta_n x + (1 - \beta_n)J_{r_n} y_n),$$

where $\{\beta_n\} \subset [0, 1]$ and $\{r_n\}(0, \infty)$ satisfy that

$$\sum_{n \geq 1} \beta_n < \infty, \quad \liminf_{n \to \infty} r_n > 0.$$

Then there exists an element $u \in C_0$ such that

$$x_n \rightharpoonup u, \quad R_{C_0}(x_n) \to u.$$

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