FIXED POINT THEOREMS FOR MAPPINGS DETERMINED BY SEVERAL PARAMETERS IN METRIC SPACES

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1. Fixed point theorems in metric spaces: type 1 [5]

Let (X, d) be a metric space. A mapping T from X into itself is said to be widely more generalized hybrid if there exist real numbers $\alpha, \beta, \gamma, \delta, \varepsilon$ and ζ such that

$$\begin{aligned} &\alpha d(Tx, Ty)^2 + \beta d(x, Ty)^2 + \gamma d(Tx, y)^2 + \delta d(x, y)^2 \\ &\qquad + \varepsilon d(x, Tx)^2 + \zeta d(y, Ty)^2 \\ &\leq 0 \end{aligned}$$

for any $x, y \in X$. Such a mapping is called an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-widely more generalized hybrid mapping.

Lemma 1.1. *Let (X, d) be a metric space, let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-widely more generalized hybrid mapping from X into itself, and $\lambda \in [0, 1]$ Then T is an $(\alpha, (1 - \lambda)\beta + \lambda\gamma, \lambda\beta + (1 - \lambda)\gamma, \delta, (1 - \lambda)\varepsilon + \lambda\zeta, \lambda\varepsilon + (1 - \lambda)\zeta)$-widely more generalized hybrid mapping from X into itself.*

Lemma 1.2. *Let X be a metric space and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-widely more generalized hybrid mapping from X into itself. Suppose that there exists $\lambda \in [0, 1]$ such that*

(1) $\alpha + (1 - \lambda)\varepsilon + \lambda\zeta + 2 \min\{\lambda\beta + (1 - \lambda)\gamma, 0\} \geq 0$;
(2) $\alpha + \delta + \varepsilon + \zeta + 4 \min\{\lambda\beta + (1 - \lambda)\gamma, 0\} > 0$.

Then $\{T^n x \mid n \in \mathbb{N} \cup \{0\}\}$ is a Cauchy sequence for any $x \in X$.

By Lemma lemma:Cauchy-m we obtain directly the following theorem.

Theorem 1.1. *Let X be a complete metric space and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-widely more generalized hybrid mapping from X into itself. Suppose that there exists $\lambda \in [0, 1]$ such that*

(1) $\alpha + (1 - \lambda)\varepsilon + \lambda\zeta + 2 \min\{\lambda\beta + (1 - \lambda)\gamma, 0\} \geq 0$;
(2) $\alpha + \delta + \varepsilon + \zeta + 4 \min\{\lambda\beta + (1 - \lambda)\gamma, 0\} > 0$.

Then for any $x \in X$ there exists $\lim_{n\to\infty} T^n x$.

By Theorem 1.1, we obtain the following theorem.

Theorem 1.2. *Let X be a complete metric space and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-widely more generalized hybrid mapping from X into itself. Suppose that therre exists $\lambda \in [0, 1]$ such that*

(1) $\alpha + (1 - \lambda)\varepsilon + \lambda\zeta + 2 \min\{\lambda\beta + (1 - \lambda)\gamma, 0\} \geq 0$;
(2) $\alpha + \delta + \varepsilon + \zeta + 4 \min\{\lambda\beta + (1 - \lambda)\gamma, 0\} > 0$;
(3) $\alpha + (1 - \lambda)(\beta + \zeta) + \lambda(\gamma + \varepsilon) > 0$.

Then T has a fixed point u, where $u = \lim_{n\to\infty} T^n x$ for any $x \in X$. Additionally, if $\alpha+\beta+\gamma+\delta > 0$, then T has a unique fixed point.

Proof. By Theorem 1.1 there exists $u \in X$ such that $u = \lim_{n\to\infty} T^n x$. Replacing x and y by $T^n x$ and u, respectively, we obtain

$$\begin{aligned}&\alpha d(T^{n+1}x, Tu)^2 + ((1-\lambda)\beta + \lambda\gamma) d(T^n x, Tu)^2 \\ &\quad + (\lambda\beta + (1-\lambda)\gamma) d(T^{n+1}x, u)^2 + \delta d(T^n x, u)^2 \\ &\quad + ((1-\lambda)\varepsilon + \lambda\zeta) d(T^n x, T^{n+1}x)^2 + (\lambda\varepsilon + (1-\lambda)\zeta) d(u, Tu)^2 \\ &\le 0.\end{aligned}$$

Since $u = \lim_{n\to\infty} T^n x$, we obtain

$$(\alpha + (1-\lambda)(\beta+\zeta) + \lambda(\gamma+\varepsilon)) d(u, Tu)^2 \le 0.$$

Since $\alpha + (1-\lambda)(\beta+\zeta) + \lambda(\gamma+\varepsilon) > 0$, we obtain $d(u,Tu)^2 \le 0$ and hence u is a fixed point of T.

Furthermore, suppose that $\alpha+\beta+\gamma+\delta > 0$ holds. Let u and v be fixed points of T. Then we obtain

$$\begin{aligned}&\alpha d(Tu,Tv)^2 + \beta d(u,Tv)^2 + \gamma d(Tu,v)^2 + \delta d(u,v)^2 \\ &\quad + \varepsilon d(u,Tu)^2 + \zeta d(v,Tv)^2 \\ &= (\alpha+\beta+\gamma+\delta) d(u,v)^2 \le 0.\end{aligned}$$

Since $\alpha+\beta+\gamma+\delta > 0$, we obtain $d(u,v)^2 \le 0$ and hence $u = v$. □

2. Fixed point theorems in metric spaces: type 2 [7]

Lemma 2.1. *Let (X,d) be a metric space and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$-widely more generalized hybrid mapping from X into itself. Suppose that therre exists $\lambda \in [0,1]$ such that*

(1) $\alpha + (1-\lambda)\varepsilon + \lambda\zeta + 2\min\{\lambda\beta + (1-\lambda)\gamma, 0\} > 0.$

Then

$$d(T^2x, Tx) \le \sqrt{A}\, d(Tx, x)$$

holds for any $x \in X$, where

$$A = \max\left\{-\frac{\delta + \lambda\varepsilon + (1-\lambda)\zeta + 2\min\{\lambda\beta + (1-\lambda)\gamma, 0\}}{\alpha + (1-\lambda)\varepsilon + \lambda\zeta + 2\min\{\lambda\beta + (1-\lambda)\gamma, 0\}}, 0\right\}.$$

Lemma 2.2. *Let (X,d) be a metric space and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$-widely more generalized hybrid mapping from X into itself. Suppose that therre exists $\lambda \in [0,1]$ such that*

(1) $\alpha + (1-\lambda)\varepsilon + \lambda\zeta + 2\min\{\lambda\beta + (1-\lambda)\gamma, 0\} > 0$;

(2) $\alpha + 2\min\{\lambda\beta + (1-\lambda)\gamma, 0\} > 0.$

Then

$$d(T^3x, Tx) \le \sqrt{B}\, d(Tx, x)$$

holds for any $x \in X$, where

$$\begin{aligned}B &= \max\left\{\max\left\{-\frac{(1-\lambda)\varepsilon + \lambda\zeta}{\alpha + 2\min\{\lambda\beta + (1-\lambda)\gamma, 0\}}, 0\right\} A^2\right. \\ &\quad + \max\left\{-\frac{(1-\lambda)\beta + \lambda\gamma + 2\min\{\delta, 0\}}{\alpha + 2\min\{\lambda\beta + (1-\lambda)\gamma, 0\}}, 0\right\} A\end{aligned}$$

$$-\frac{\lambda\varepsilon+(1-\lambda)\zeta+2\min\{\lambda\beta+(1-\lambda)\gamma,0\}+2\min\{\delta,0\}}{\alpha+2\min\{\lambda\beta+(1-\lambda)\gamma,0\}},0\Bigg\}.$$

Lemma 2.3. *Let (X,d) be a metric space and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$-widely more generalized hybrid mapping from X into itself. Suppose that therre exists $\lambda\in[0,1]$ such that*

(1) $\alpha+(1-\lambda)\varepsilon+\lambda\zeta+2\min\{\lambda\beta+(1-\lambda)\gamma,0\}>0$;
(2) $\alpha+2\min\{\lambda\beta+(1-\lambda)\gamma,0\}>0$.

Then

$$d(T^3x,T^2x)\le\sqrt{C}d(Tx,x)$$

holds for any $x\in C$, where

$$C=\max\left\{-\frac{\lambda\beta+(1-\lambda)\gamma}{\alpha+(1-\lambda)\varepsilon+\lambda\zeta},0\right\}B+\max\left\{-\frac{\delta+\lambda\varepsilon+(1-\lambda)\zeta}{\alpha+(1-\lambda)\varepsilon+\lambda\zeta},0\right\}A.$$

By Lemma 2.3, we obtain the following theorem.

Theorem 2.1. *Let (X,d) be a complete metric space and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$-widely more generalized hybrid mapping from X into itself. Suppose that therre exists $\lambda\in[0,1]$ such that*

(1) $\alpha+(1-\lambda)\varepsilon+\lambda\zeta+2\min\{\lambda\beta+(1-\lambda)\gamma,0\}>0$;
(2) $\alpha+2\min\{\lambda\beta+(1-\lambda)\gamma,0\}>0$;
(3) $C<1$.

Then T has a fixed point u, where $u=\lim_{n\to\infty}T^nx$ for any $x\in X$. Additionally, if $\alpha+\beta+\gamma+\delta>0$, then T has a unique fixed point.

Proof. By Lemma 2.3, we obtain

$$d(T^{n+1}x,T^nx)\le\sqrt{C}d(T^{n-1}x,T^{n-2}x).$$

Since $0\le C<1$, we obtain

$$\begin{aligned}
&d(T^mx,T^nx)\\
&\le\sum_{k=n}^{m-1}d(T^{k+1}x,T^kx)\\
&\le\sum_{k=n,k:even}^{m-1}C^{\frac{k}{4}}d(Tx,x)+\sum_{k=n,k:odd}^{m-1}C^{\frac{k-1}{4}}d(T^2x,Tx)\\
&\le\sum_{k=n,k:even}^{\infty}C^{\frac{k}{4}}d(Tx,x)+\sum_{k=n,k:odd}^{\infty}C^{\frac{k-1}{4}}d(T^2x,Tx)\\
&=\sum_{k=\left[\frac{n+1}{2}\right]}^{\infty}C^{\frac{k}{2}}d(Tx,x)+\sum_{k=\left[\frac{n}{2}\right]}^{\infty}C^{\frac{k}{2}}d(T^2x,Tx)\\
&=\frac{C^{\frac{1}{2}\left[\frac{n+1}{2}\right]}d(Tx,x)+C^{\frac{1}{2}\left[\frac{n}{2}\right]}d(T^2x,Tx)}{1-\sqrt{C}}
\end{aligned}$$

for any $m,n\in\mathbb{N}$ with $m\ge n$. Therefore $\{T^nx\mid n\in\mathbb{N}\cup\{0\}\}$ is Cauchy. Since X is complete, there exists $u=\lim_{n\to\infty}T^nx$. Replacing x and y by u and T^nx, respectively, we obtain

$$\alpha d(Tu,T^{n+1}x)^2+((1-\lambda)\beta+\lambda\gamma)d(u,T^{n+1}x)^2$$

$$
\begin{aligned}
&+(\lambda\beta+(1-\lambda)\gamma)d(Tu,T^nx)^2+\delta d(u,T^nx)^2\\
&+((1-\lambda)\varepsilon+\lambda\zeta)d(u,Tu)^2+(\lambda\varepsilon+(1-\lambda)\zeta)d(T^nx,T^{n+1}x)^2\\
\le{}& 0.
\end{aligned}
$$

Putting $n\to\infty$, we obtain

$$(\alpha+\lambda(\beta+\zeta)+(1-\lambda)(\gamma+\varepsilon))d(Tu,u)^2\le 0.$$

Since

$$
\begin{aligned}
&\alpha+\lambda(\beta+\zeta)+(1-\lambda)(\gamma+\varepsilon)\\
&\quad\ge \alpha+(1-\lambda)\varepsilon+\lambda\zeta+\min\{\lambda\beta+(1-\lambda)\gamma,0\}\\
&\quad> -\min\{\lambda\beta+(1-\lambda)\gamma,0\}\\
&\quad\ge 0,
\end{aligned}
$$

u is a fixed point.

Furthermore, suppose that $\alpha+\beta+\gamma+\delta>0$ holds. Let u and v be fixed points of T. Then we obtain

$$
\begin{aligned}
&\alpha d(Tu,Tv)^2+\beta d(u,Tv)^2+\gamma d(Tu,v)^2+\delta d(u,v)^2\\
&\qquad+\varepsilon d(u,Tu)^2+\zeta d(v,Tv)^2\\
&\quad=(\alpha+\beta+\gamma+\delta)d(u,v)^2\le 0.
\end{aligned}
$$

Since $\alpha+\beta+\gamma+\delta>0$, we obtain $d(u,v)^2\le 0$ and hence $u=v$. □

3. Regarding parameter conditions

In this section, we consider families of mappings determined by parameter conditions in the case of metric spaces.

Let

$$
\begin{aligned}
&\mathcal{T}(\alpha,\beta,\gamma,\delta,\eta,\zeta)\\
&\overset{\text{def}}{=}\left\{T:X\longrightarrow X\;\middle|\;\begin{array}{l}\forall x,y\in X,\\ \alpha d(Tx,Ty)^2+\beta d(x,Ty)^2+\gamma d(Tx,y)^2\\ +\delta d(x,y)^2+\varepsilon d(x,Tx)^2+\zeta d(y,Ty)^2\\ \le 0\end{array}\right\}.
\end{aligned}
$$

If $(\alpha,\beta,\gamma,\delta,\eta,\zeta)\in[0,\infty)^6\setminus\{(0,0,0,0,0,0)\}$, then $\mathcal{T}(\alpha,\beta,\gamma,\delta,\eta,\zeta)=\emptyset$ clearly.

If $(\alpha,\beta,\gamma,\delta,\eta,\zeta)\in(-\infty,0]^6$, then $\mathcal{T}(\alpha,\beta,\gamma,\delta,\eta,\zeta)$ contains all mappings clearly.

$\alpha+\beta+\gamma+\delta\le 0 \iff \mathcal{T}(\alpha,\beta,\gamma,\delta,\eta,\zeta)$ contains the identity mapping.

If $\beta+\varepsilon\le 0$, $\gamma+\zeta\le 0$, and $\delta\le 0$, then: let $Tx=c$ (where c is a constant). Since

$$
\begin{aligned}
&\alpha d(Tx,Ty)^2+\beta d(x,Ty)^2+\gamma d(Tx,y)^2+\delta d(x,y)^2\\
&\quad+\varepsilon d(x,Tx)^2+\zeta d(y,Ty)^2\\
&=(\beta+\varepsilon)d(x,c)^2+(\gamma+\zeta)d(y,c)^2+\delta d(x,y)^2\\
&\le 0,
\end{aligned}
$$

Therefore, T is included in $\mathcal{T}(\alpha,\beta,\gamma,\delta,\eta,\zeta)$.

Other than the above, we would like to find conditions under which $\mathcal{T}(\alpha,\beta,\gamma,\delta,\eta,\zeta)$ will not be empty.

Program Results

$(\alpha,\beta,\gamma,\delta,\eta,\zeta)\in[-5,5]^6$ (step size: 1)

$(\alpha, \beta, \gamma, \delta, \eta, \zeta) \notin (-\infty, 0]^6 \cup [0, \infty)^6$
$\alpha + \beta + \gamma + \delta > 0$
$\beta + \varepsilon > 0$, $\gamma + \zeta > 0$, or $\delta > 0$

Total number of parameter combinations: 750,987.

Search only for mappings like the following:

$X = [0, 1]$ (step size: 0.1)

$Tx = a + bx$: if the value exceeds the range, it folds up or down to fit within the range. ($a \in [0, 1]$, $b \in [-5, 5]$, step size: 0.1).

Let

$$\mathcal{T}_1(\alpha, \beta, \gamma, \delta, \eta, \zeta) \stackrel{\text{def}}{=} \left\{ \begin{array}{l|l} T : x \longmapsto a + bx & \forall a \in [0,1], \forall b \in [-5,5], \forall x, \forall y \in X, \\ \text{folds up or down} & \alpha d(Tx,Ty)^2 + \beta d(x,Ty)^2 + \gamma d(Tx,y)^2 \\ \text{if necessary} & +\delta d(x,y)^2 + \varepsilon d(x,Tx)^2 + \zeta d(y,Ty)^2 \\ & \leq 0 \end{array} \right\}.$$

Total number of parameter combinations such that $\mathcal{T}_1(\alpha, \beta, \gamma, \delta, \eta, \zeta) \neq \emptyset$ holds: 75,860.

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