Properties of vector-valued cone-dc functions and its applications

Syuuji Yamada*and Tamaki Tanaka
Faculty of Science, Niigata University

1 Introduction

An optimization problem including dc functions (difference of two convex functions) is called dc programming. Dc programming is one of the important subjects in global optimization and has been studied by Rosen [8], Avriel and Williams [1], Meyer [7], Ueing [12], Hillestad and Jacobsen [3], and Tuy [10]. It is known that many global optimization problems can be transformed into or approximated by dc programming problems. For dc programming problems, a necessary and sufficient optimality condition has been proposed by Hiriart [4].

The concept of vector-valued cone-dc function has been proposed by Hojo, Tanaka and Yamada [5]. Moreover, several properties of cone-dc function have been analysed by Yamada, Tanaka and Tanino [13]. In particular, it has been shown that every locally vector-valued cone-dc function having a compact convex domain can be rewritten as a vector-valued cone-dc function with the same domain. From such results, we notice that many vector optimization problems can be transformed into or approximated by vector-valued cone-dc vector optimization problems.

In this paper, we propose necessary and sufficient optimality conditions for three types of solutions of an unconstrained vector-valued cone-dc vector optimization problem by applying Hiriart's optimality condition in the case where the ordering cone is obtained as a convex polyhedral cone. By utilizing a penalty function method, the proposed optimality conditions adapt to a constrained vector-valued cone-dc vector optimization problem. Moreover, we define a dual problem for a cone-dc vector optimization problem. Furthermore, we propose an algorithm for finding a weakly efficient point of a cone-dc vector optimization problem.

The organization of this paper is as follows: In Section 2, we introduce some notation and mathematical preliminaries in convex analysis. In Section 3, we introduce some properties of real-valued dc functions. In Section 4, we introduce some properties of vector-valued cone-dc functions. In Section 5, we propose optimality conditions for a cone-dc vector optimization problem. In Section 6, we define a dual problem for a cone-dc vector optimization problem based on quasi-conjugation. In Section 7, we propose an algorithm for finding a weakly efficient point of a cone-dc vector optimization problem.

*Corresponding author

2 Mathematical Preliminaries

Throughout this paper, we use the following notation: $\mathbb{R}$ denotes the set of all real numbers. For a natural number m, $\mathbb{R}^m$ denotes an m-dimensional Euclidean space. The origin of $\mathbb{R}^m$ is denoted by $\mathbf{0}_m$. Let $\mathbb{R}^m_+ := \{\boldsymbol{x} \in \mathbb{R}^m : \boldsymbol{x} \geq \mathbf{0}_m\}$. Given a vector $\boldsymbol{x} \in \mathbb{R}^m$, $\boldsymbol{x}^\top$ denotes the transposed vector of $\boldsymbol{x}$. For a subset $X \subset \mathbb{R}^m$, $\operatorname{int} X$ denotes the interior of X. Given a nonempty cone $D \subset \mathbb{R}^m$, D^+ denotes the positive polar cone of D, that is, $D^+ := \{\boldsymbol{u} \in \mathbb{R}^m : \boldsymbol{u}^\top \boldsymbol{x} \geq 0, \text{ for all } \boldsymbol{x} \in D\}$. Given a real-valued function $\alpha : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ and a positive real number ε, $\partial_\varepsilon \alpha(\boldsymbol{y})$ denotes the ε-subdifferential of α at $\boldsymbol{y}$, that is,

$$\partial_\varepsilon \alpha(\boldsymbol{y}) := \{\boldsymbol{a} \in \mathbb{R}^m : \boldsymbol{a}^\top(\boldsymbol{x} - \boldsymbol{y}) + \alpha(\boldsymbol{y}) - \varepsilon \leq \alpha(\boldsymbol{x}) \text{ for all } \boldsymbol{x} \in \mathbb{R}^m\}.$$

3 Properties of real-valued dc function

In this section, we introduce some properties of real-valued dc function

Definition 3.1 (**Real-valued dc function**) Let X be a nonempty convex subset on $\mathbb{R}^m$. A function $\alpha : X \to \mathbb{R}$ is said to be *dc* on X if there exist two real-valued convex functions $\beta, \gamma : X \to \mathbb{R}$ such that

$$\alpha(\boldsymbol{x}) = \beta(\boldsymbol{x}) - \gamma(\boldsymbol{x}) \quad \text{for all } \boldsymbol{x} \in X. \tag{1}$$

The representation (1) is called a *dc decomposition* of α on X. When $X = \mathbb{R}^m$, F is simply called a *dc* function and the representation (1) is simply called a *dc decomposition.*

Proposition 3.2 (See, e.g., Tuy [11], Proposition 2.4) *Let $X \subset \mathbb{R}^m$ be a nonempty convex subset and let F be a real-valued function on X. If for every $\boldsymbol{x} \in X$ there exists a convex open neighborhood $\mathcal{M}_x$ of $\boldsymbol{x}$ such that F is convex on $\mathcal{M}_x \cap X$, then F is convex on X.*

Theorem 3.3 (See, e.g., Hartman [2]) *Let $X \subset \mathbb{R}^m$ be nonempty and convex. A real-valued function $F : X \to \mathbb{R}$ is dc on X if and only if F is locally dc on X.*

The mathematical programming problem where the objective function and all constraint functions are given by dc functions is called a *dc programming problem.* From Theorem 3.3, the following statements hold.

I. Every continuously twice differentiable function is dc.

II. Every continuous function on a compact convex set can be formulated as the limit of a sequence of dc functions.

From the above statements, we note that many global optimization problems can be transformed into or approximated by dc programming problems.

For real-valued dc programming problems, an optimal condition is obtained by the following proposition.

Proposition 3.4 (See Hiriart [4]) *Let $\beta : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ be an arbitrary function and $\gamma : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ a convex proper l.s.c function. A point $\boldsymbol{x} \in (\text{dom } \beta) \cap (\text{dom } \gamma)$ is a global minimizer of $\beta(\boldsymbol{x}) - \gamma(\boldsymbol{x})$ of $\mathbb{R}^m$ if and only if*

$$\partial_\varepsilon \gamma(\boldsymbol{x}) \subset \partial_\varepsilon \beta(\boldsymbol{x}) \quad \text{for each } \varepsilon > 0.$$

4 Properties of vector-valued cone-dc function

Let $C \subset \mathbb{R}^n$ be a convex polyhedral cone defined as

$$C := \left\{ \boldsymbol{x} \in \mathbb{R}^n : \left(\boldsymbol{c}^i\right)^\top \boldsymbol{x} \geq 0,\ i = 1, \dots, l \right\}.$$

We assume that $\operatorname{int} C \neq \emptyset$ and that $\dim\{\boldsymbol{c}^1, \dots, \boldsymbol{c}^l\} = n$. Then, we note that C is pointed.

Now, we define the order $\leq_C$ as follows:

$$\boldsymbol{x} \leq_C \boldsymbol{y} \text{ if } \boldsymbol{y} - \boldsymbol{x} \in C \text{ for } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n.$$

Moreover, we review the following concept for vector-valued functions.

Definition 4.1 (**C-convexity**, see, e.g., Luc [6], Definition 6.1) Let $X \subset \mathbb{R}^m$ be nonempty and convex. A vector-valued function $g : X \to \mathbb{R}^n$ is said to be *C-convex* on X if for each $\boldsymbol{x}^1, \boldsymbol{x}^2 \in X$ and $0 \leq \lambda \leq 1$, g satisfies that

$$g((1-\lambda)\boldsymbol{x}^1 + \lambda \boldsymbol{x}^2) \leq_C (1-\lambda) g(\boldsymbol{x}^1) + \lambda g(\boldsymbol{x}^2).$$

Proposition 4.2 (See, e.g., Luc [6], Proposition 6.2) *Let $X \subset \mathbb{R}^m$ be a nonempty convex subset. Then, a vector-valued function $g : X \to \mathbb{R}^n$ is C-convex on X if and only if $\boldsymbol{c}^\top g(\,\cdot\,)$ is a real-valued convex function on X for each $\boldsymbol{c} \in C^+$.*

Proposition 4.3 (See, e.g., Lucc [6], Proposition 6.7) *Let $X \subset \mathbb{R}^m$ be a nonempty convex subset and let vector-valued functions $g, h : X \to \mathbb{R}^n$ be C-convex on X. Then, the following assertions hold.*

(i) *μg is C-convex on X for each $\mu \in \mathbb{R}$ $(\mu > 0)$.*

(ii) *$g + h$ is C-convex on X.*

From Propositions 3.2, 4.2 and 4.3, the following proposition holds.

Proposition 4.4 *Let $X \subset \mathbb{R}^m$ be a nonempty convex subset and let a vector valued function $g : X \to \mathbb{R}^n$ be locally C-convex on X. Then, g is C-convex on X.*

Definition 4.5 (**C-dc function**, see Hojo, Tanaka and Yamada [5]) Let $X \subset \mathbb{R}^m$ be nonempty and convex. A vector-valued function $f : X \to \mathbb{R}^n$ is said to be *C-dc* on X if there exist two C-convex functions g and h on X such that

$$f(\boldsymbol{x}) = g(\boldsymbol{x}) - h(\boldsymbol{x}) \tag{2}$$

for all $\boldsymbol{x} \in X$. Moreover, formulation (2) is called C-dc decomposition of f over X.

The following theorem is straightforwardly satisfied from Definition 4.5.

Theorem 4.6 *Let $X \subset \mathbb{R}^m$ be a convex subset satisfying $\operatorname{int} X \neq \emptyset$. If a vector-valued function $f : X \to \mathbb{R}^n$ is C-dc on X, then f is locally C-dc on X.*

Theorem 4.7 (Yamada, Tanaka and Tanino [13]) *Let $X \subset \mathbb{R}^m$ be a convex subset and S a compact convex subset satisfying $\operatorname{int} X \supset S$ and $\operatorname{int} S \neq \emptyset$. If a vector-valued function $f : X \to \mathbb{R}^n$ is locally C-dc on X, then f is C-dc on S.*

Theorem 4.8 (Yamada, Tanaka and Tanino [13]) *Let $X \subset \mathbb{R}^m$ be a convex subset, $S \subset \operatorname{int} X$ a compact convex subset satisfying $\operatorname{int} S \neq \emptyset$, $f : X \to \mathbb{R}^n$, and let $f_i : X \to \mathbb{R}$ $(i = 1, \ldots, m)$ satisfy $f(\boldsymbol{x}) := [f_1(\boldsymbol{x}), \ldots, f_n(\boldsymbol{x})]^\top$. If every f_i $(i = 1, \ldots, n)$ is continuously twice differentiable on X, then f is C-dc on S.*

Theorem 4.9 (Yamada, Tanaka and Tanino [13]) *Let $X \subset \mathbb{R}^m$ be a convex subset and S a compact convex subset satisfying $\operatorname{int} X \supset S$ and $\operatorname{int} S \neq \emptyset$. If a vector-valued function $f(\boldsymbol{x}) := [f_1(\boldsymbol{x}), \ldots, f_n(\boldsymbol{x})]^\top$ $(f_i : Y \to \mathbb{R},\ i = 1, \ldots, n)$ is continuous on X, then f is the limit of a sequence of C-dc functions on S which converges uniformly in S.*

From Theorems 4.8 and 4.9, we notice that many vector optimization problems can be transformed into or approximated by C-dc vector optimization problems.

5 Optimality conditions for a cone-dc vector optimization problem

Let us consider the following vector optimization problem.

$$(C\text{-DC}) \begin{cases} C\text{-min} & f(\boldsymbol{x}) := g(\boldsymbol{x}) - h(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in X := \{\boldsymbol{x} \in \mathbb{R}^m : p(\boldsymbol{x}) \leq 0\} \end{cases}$$

where $g, h : \mathbb{R}^m \to \mathbb{R}^n$ are vector-valued C-convex functions, $p : \mathbb{R}^m \to \mathbb{R}$ is a convex function and C-min denotes minimizing with respect to the ordering cone $C \subset \mathbb{R}^n$. For problem (C-DC), we assume the following conditions.

Assumptions:

(A1) X is a compact convex set.

(A2) $p(\mathbf{0}) < 0$

By utilizing a penalty function methd, problem (C-DC) can be transformed into the following problem.

$$(C\text{-}\overline{\text{DC}}) \begin{cases} C\text{-min} & \bar{f}(\boldsymbol{x}) := \bar{g}(\boldsymbol{x}) - h(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \mathbb{R}^m, \end{cases}$$

where $\bar{g}(\boldsymbol{x}) := g(\boldsymbol{x}) + \tau \max\{0, p(\boldsymbol{x})\}\boldsymbol{v}$, $\boldsymbol{v} \in \operatorname{int} C$, τ is a positive real number satisfying

$$\arg\min\{\boldsymbol{c}^\top \bar{g}(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{R}^m\} \subset X$$

for each $\boldsymbol{c} \in C^+ \backslash \{\mathbf{0}_m\}$.

Definition 5.1 (**Efficiency**, see Luc [6], Definition 2.1) Let $X \subset \mathbb{R}^m$ be nonempty and let $f : X \to \mathbb{R}^n$ be a vector-valued function. We say that

(i) $\boldsymbol{y} \in X$ is an ideal efficient point of f over X with respect to C if $f(\boldsymbol{y}) \leq_C f(\boldsymbol{x})$ for all $\boldsymbol{x} \in X$.

(ii) $\boldsymbol{y} \in X$ is an efficient point of f over X with respect to C if $f(\boldsymbol{x}) \leq_C f(\boldsymbol{y})$ for some $\boldsymbol{x} \in X$, then $f(\boldsymbol{y}) \leq_C f(\boldsymbol{x})$.

(iii) $\boldsymbol{y} \in X$ is a weakly efficient point of f over X with respect to C if $f(\boldsymbol{x}) \leq_{\{0_n\} \cup \mathrm{int}\ C} f(\boldsymbol{y})$ for some $\boldsymbol{x} \in X$, then $f(\boldsymbol{y}) \leq_{\{0_n\} \cup \mathrm{int}\ C} f(\boldsymbol{x})$.

By applying Proposition 3.4 to (C-DC), we obtain the following theorems.

Theorem 5.2 *A vector $\boldsymbol{y} \in \mathbb{R}^m$ is an ideal efficient point of (C-DC) if and only if*

$$\partial_\varepsilon \left(\boldsymbol{c}^i\right)^\top h(\boldsymbol{y}) \subset \partial_\varepsilon \left(\boldsymbol{c}^i\right)^\top g(\boldsymbol{y}) \quad \text{for each } i \in \{1, \ldots, l\} \text{ and } \varepsilon \in \mathbb{R}_+.$$

Theorem 5.3 *A vector $\boldsymbol{y} \in \mathbb{R}^m$ is an efficient point of (C-DC) if and only if $\boldsymbol{y}$ satisfies*

$$\left(\bigcap_{\substack{i=1 \\ i \neq j}}^{l} \bigcup_{a^i \in \partial_{\varepsilon_i} (c^i)^\top h(y)} \left\{ \boldsymbol{x} \in \mathbb{R}^m : \begin{array}{l} \left(\boldsymbol{a}^i\right)^\top (\boldsymbol{x} - \boldsymbol{y}) - \varepsilon_i \\ \geq \left(\boldsymbol{c}^i\right)^\top (g(\boldsymbol{x}) - g(\boldsymbol{y})) \end{array} \right\} \right)$$
$$\bigcap \left(\bigcup_{a^j \in \partial_{\varepsilon_j} (c^j)^\top h(y)} \left\{ \boldsymbol{x} \in \mathbb{R}^m : \begin{array}{l} \left(\boldsymbol{a}^j\right)^\top (\boldsymbol{x} - \boldsymbol{y}) - \varepsilon_j \\ > \left(\boldsymbol{c}^j\right)^\top (g(\boldsymbol{x}) - g(\boldsymbol{y})) \end{array} \right\} \right) = \emptyset$$

for each $j \in \{1, \ldots, l\}$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)^\top \in \mathbb{R}^n_+$.

Theorem 5.4 *A vector $\boldsymbol{y} \in \mathbb{R}^m$ is a weakly efficient point of (C-DC) if and only if $\boldsymbol{y}$ satisfies*

$$\bigcap_{i=1}^{l} \bigcup_{a^i \in \partial_{\varepsilon_i} (c^i)^\top h(y)} \left\{ \boldsymbol{x} \in \mathbb{R}^m : \begin{array}{l} \left(\boldsymbol{a}^i\right)^\top (\boldsymbol{x} - \boldsymbol{y}) - \varepsilon_i \\ > \left(\boldsymbol{c}^i\right)^\top (g(\boldsymbol{x}) - g(\boldsymbol{y})) \end{array} \right\} = \emptyset$$

for each $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)^\top \in \mathbb{R}^n_+$.

6 Duality based on a quasi-conjugation

For each $\boldsymbol{c} \in C^+ \backslash \{\boldsymbol{0}_m\}$, we consider the following problem:

$$(\mathrm{MP}(\boldsymbol{c})) \begin{cases} \text{maximize} & h_c(\boldsymbol{x}, \alpha) := \boldsymbol{c}^\top \bar{h}(\boldsymbol{x}) - \alpha \\ \text{subject to} & (\boldsymbol{x}, \alpha) \in S(\boldsymbol{c}), \end{cases}$$

where $S(\boldsymbol{c}) := \{(\boldsymbol{x}, \alpha) : \boldsymbol{c}^\top g(\boldsymbol{x}) - \alpha \leq 0\}$.

From assumptions (A1) and (A2), without loss of generality, we can assume that $(\boldsymbol{0}_m, 0) \in \mathrm{int}\, S(\boldsymbol{c})$ for each $\boldsymbol{c} \in S^+ \backslash \{\boldsymbol{0}_n\}$. Since h is C-convex, $S(\boldsymbol{c})$ is a closed convex set.

Definition 6.1 (Quasi-conjugate function, Thach [9]) *Let $G : \mathbb{R}^n \to \bar{\mathbb{R}}$ and let $G^H : \mathbb{R}^n \to \bar{\mathbb{R}}$ be a function defined as follows:*

$$G^H(\boldsymbol{u}) := \begin{cases} -\sup\{G(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{R}^n\} & \text{if } \boldsymbol{u} = \boldsymbol{0}_n, \\ -\inf\{G(\boldsymbol{x}) : \boldsymbol{u}^\top \boldsymbol{x} \geq 1\} & \text{if } \boldsymbol{u} \neq \boldsymbol{0}_n. \end{cases}$$

The function G^H is called the quasi-conjugate function of G.

For each $\boldsymbol{c} \in C^{+}\backslash\{\mathbf{0}_m\}$, we consider the following problem:

$$(\mathrm{DP}(\boldsymbol{c}))\left\{\begin{array}{ll}\text{minimize} & h_c(\boldsymbol{u},\beta)^H \\ \text{subject to} & (\boldsymbol{u},\beta) \in \mathbb{R}^{m+1}\backslash S(\boldsymbol{c})^{\circ}.\end{array}\right.$$

Theorem 6.2 *For each* $\boldsymbol{c} \in C^{+}\backslash\{\mathbf{0}_m\}$,

$$\max(\mathrm{MP}(\boldsymbol{c})) = -\min(\mathrm{DP}(\boldsymbol{c})).$$

Here, $\max(\mathrm{MP}(\boldsymbol{c}))$ *and* $\min(\mathrm{DP}(\boldsymbol{c}))$ *denote the optimal values of* $(\mathrm{MP}(\boldsymbol{c}))$ *and* $(\mathrm{DP}(\boldsymbol{c}))$*, respectively.*

Theorem 6.3 *A vector* $\bar{\boldsymbol{x}} \in \mathbb{R}^m$ *a weakly efficient point of* $(C-\overline{\mathrm{DC}})$ *if and only if there exists* $\bar{\boldsymbol{c}} \in C^{+}\backslash\{\mathbf{0}_n\}$ *such that*

$$\bar{\boldsymbol{c}}^{\top}(g(\bar{\boldsymbol{x}}) - h(\boldsymbol{x})) = \min(\mathrm{DP}(\bar{\boldsymbol{c}})).$$

7 Optimization algorithm

In this section, we propose an algorithm to find a weakly efficient solution of (C-DC) in the case where C is a convex polyhedral cone.

Algorithm FES

Step 0: Select a tolerance $\gamma \geq 0$. Choose an initial provisional solution $\boldsymbol{y}^1 \in X$. Construct $T_1 := \{\boldsymbol{z} \in \mathbb{R}^{m+1} : (-\rho,\ldots,-\rho,0)^{\top} \leq \boldsymbol{z} \leq (\rho,\ldots,\rho,\sqrt{m+1}\rho+\delta)^{\top}\}$ and the vertex set $V(T_1)$ of T_1, where $\delta > 0$. Set $k = 1$ and go to Step 1.

Step 1: If $\max\{\|\boldsymbol{v}\| : \boldsymbol{v} \in V(T_k)\} \leq \sqrt{n}\rho + \delta + \gamma$, then stop; $\boldsymbol{y}^k$ is an approximate point of an efficient solution. Otherwise, go to Step 2.

Step 2: Choose $\boldsymbol{v}^k \in V(T_k)$ satisfying $\|\boldsymbol{v}^k\| = \max\{\|\boldsymbol{v}\| : \boldsymbol{v} \in V(T_k)\}$ and set $T_{k+1} := T_k \cap \{\boldsymbol{z} \in \mathbb{R}^{m+1} : (\boldsymbol{v}^k)^{\top}(\boldsymbol{z}-\boldsymbol{v}^k) = (\sqrt{n}\rho+\delta)^2\}$ and calculate $V(T_{k+1})$. Set $M = V(T_{k+1})\backslash V(T_k)$ and go to Step 3.

Step 3: If $M = \emptyset$, set $\boldsymbol{y}^{k+1} := \boldsymbol{w}$, $k \leftarrow k+1$ and return to Step 1. Otherwise, choose $\boldsymbol{v} \in M \cap \{\boldsymbol{z} \in \mathbb{R}^{m+1} : \|\boldsymbol{z}\| \geq \rho + \delta\}$, and $\boldsymbol{w} := \boldsymbol{y}^k$ and go to Step 4.

Step 4: For each $i \in \{1,\ldots,n\}$, calculate

$$\omega_i := \min\{\left(\boldsymbol{c}^i\right)^{\top}(\bar{g}(\boldsymbol{x}) - \bar{g}(\boldsymbol{w})) - \left(\nabla\left(\boldsymbol{c}^i\right)^{\top} g(\boldsymbol{v})\right)^{\top}(\boldsymbol{x}-\boldsymbol{v}) - \varepsilon_i : \boldsymbol{x} \in \mathbb{R}^m\}$$

where

$$\varepsilon_i := \left(\boldsymbol{c}^i\right)^{\top} h(\boldsymbol{v}) - \left(\boldsymbol{c}^i\right)^{\top} h(\boldsymbol{w}) - \left(\nabla\left(\boldsymbol{c}^i\right)^{\top} g(\boldsymbol{v})\right)^{\top}(\boldsymbol{y}-\boldsymbol{v}).$$

If $\omega_i \leq 0$ for each $i \in \{1,\ldots.n\}$ and $\boldsymbol{w}_j < 0$ for some $j \in \{1,\ldots.n\}$, set

$$\boldsymbol{w} \in \arg\min\{\min_{i=1,\ldots,n}\left(\boldsymbol{c}^i\right)^{\top}(\bar{g}(\boldsymbol{x}) - \bar{g}(\boldsymbol{w})) - \left(\nabla\left(\boldsymbol{c}^i\right)^{\top} g(\boldsymbol{v})\right)^{\top}(\boldsymbol{x}-\boldsymbol{v}) - \varepsilon_i : \boldsymbol{x} \in \mathbb{R}^m\}.$$

Return to Step 3.

Theorem 7.1 *Every accumulation point of* $\{\boldsymbol{y}^k\}$ *generated by executing Algorithm FES is an weakly efficient point of problem* (C-DC).

References

[1] M. Avriel and A.C. Williams, Complementary Geometric Programming, *SIAM Journal of Applied Mathematics* **19(1)** (1970), 125–141.

[2] P. Hartman, On Functions Representable as a Difference of Convex Functions, *Pacific Journal of Mathematics* **9(3)** (1959), 707–713.

[3] R.J. Hillestad and S.E. Jacobsen, Reverse Convex Programming, *Applied Mathematics and Optimization* **6(1)** (1980), 63–78.

[4] J.-B. Hiriart-Urruty, From convex optimization to nonconvex optimization, Part I: Necessary and sufficient conditions for global optimality, Nonsmooth optimization and related topics, Plenum, New York (1989).

[5] M. Hojo, T. Tanaka and S. Yamada, Optimality Condition for Vector-Valued DC Programming Problems, *Journal of Nonlinear and Convex Analysis* **13(1)** (2012), 57–64.

[6] D.T. Luc, Theory of Vector Optimization. In Lecture Notes in Economics and Mathematical Systems **319**, Springer, Berlin (1989).

[7] R. Meyer, The Validity of a Family of Optimization Methods, *SIAM Journal on Control* **8(1)** (1970), 42–54.

[8] J.B. Rosen, Iterative Solution of Nonlinear Optimal Control Problems, *SIAM Journal on Control* **4(1)** (1966), 223–244.

[9] P.T. Thach, Quasiconjugates of functions, duality relationship between quasiconvex minimization under a reverse convex constraint and quasiconvex maximization under a convex constraint, and applications, *Journal of Mathematical Analysis and Applications***159 (1991)**, 299–322.

[10] H. Tuy, Convex Programs with an Additional Reverse Convex Constraint, *Journal of Optimization Theory and Applications* **52(3)** (1987), 463–486.

[11] H. Tuy, Convex Analysis and Global Optimization, Kluwer Academic Publishers, The Netherlands (1998).

[12] U.A. Ueing, A Combinatorial Method to Compute a Global Solution of Certain Nonconvex Optimization Problems, *Numerical Methods for Nonlinear Optimization Edited by F.A. Lootsma, Academic Press, New York* (1972), 223–230.

[13] S. Yamada, T. Tanaka and T. Tanino, Relationships between vector-valued cone-dc and locally cone-dc functions, *Journal of Mathematical Analysis and Applications*, **398(2)** (2013), 588-593.

Syuuji Yamada
Department of Mathematics
Faculty of Science, Niigata University
8050, Ikarashi 2-no-cho, Nishi-ku, Niigata 950-2181, JAPAN
E-mail: yamada@math.sc.niigata-u.ac.jp

Tamaki Tanaka
Department of Mathematics
Faculty of Science, Niigata University
8050, Ikarashi 2-no-cho, Nishi-ku, Niigata 950-2181, JAPAN
E-mail: tamaki@math.sc.niigata-u.ac.jp