Convex minimization problems with fixed point constraint on Hadamard spaces
アダマール空間上の不動点制約をもつ凸最小化問題

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Abstract

This work studies a convex minimization problem with a fixed point constraint on Hadamard spaces. We propose a new method to generate an iterative sequence with an anchor point and prove its strong convergence to the solution closest to the anchor point.

1 Introduction

Let X be a metric space having a convexity structure and $f\colon X \to]-\infty, \infty]$ a convex function. For a nonempty subset C of X, we consider the problem to find a point $x_0 \in C$ satisfying that

$$f(x_0) = \inf_{y \in C} f(y).$$

This problem is called the convex minimization problem with a constraint set C, and the solution is called a minimizer of f. The constraint set C is given in various ways. In this work, we will mainly focus on the case that C is the set of fixed points of a given mapping.

The following result shows the Δ-convergence of an iterative scheme to a solution to a convex minimization problem with a fixed point constraint.

Theorem 1.1 (Kimura [5]). *Let X be a Hadamard space such that a subset $\{z \in X \mid d(z,y) \le d(z,x)\}$ is convex for any $x, y \in X$. Let $f\colon X \to]-\infty, \infty]$ be a proper lower semicontinuous convex function and $T\colon X \to X$ a nonexpansive mapping. Let $\{\lambda_n\} \subset]0, \infty[$ be a positive real sequence such that $\inf_{n \in \mathbb{N}} \lambda_n > 0$. Suppose that*

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$\operatorname{argmin}_X f \cap \operatorname{Fix} T \neq \varnothing$. *Generate a sequence* $\{x_n\} \subset X$ *as follows:* $x_1 \in X$, $f_1 = f$, *and*

$$\begin{aligned} X_n &= \{z \in X \mid d(z, Tx_n) \leq d(z, x_n)\}, \\ f_{n+1} &= f_n + i_{X_n}, \\ x_{n+1} &= R_{\lambda_{n+1} f_{n+1}} x_n \end{aligned}$$

for $n \in \mathbb{N}$. *Then* $\{x_n\}$ *is* Δ*-convergent to* $x_0 \in \operatorname{argmin}_X f \cap \operatorname{Fix} T$.

In this work, we study a convex minimization problem with a fixed point constraint on a Hadamard space and consider a technique similar to the result above to generate an approximating sequence to the solution. We propose a new method to generate an iterative sequence with an anchor point and prove its strong convergence to the solution closest to the anchor point.

2 Preliminaries

Let X be a metric space with a metric d. A geodesic connecting two points $x, y \in X$ is a mapping $\gamma_{xy} \colon [0,1] \to X$ such that $\gamma_{xy}(0) = x$, $\gamma_{xy}(1) = y$, and $d(\gamma_{xy}(s), \gamma_{xy}(t)) = |s-t|\, d(x,y)$ for $s, t \in [0,1]$. If for every pair of points in X, there exists a geodesic connecting them, we call X a geodesic space.

A Hadamard space is defined as a complete CAT(0) space. The definition of CAT(0) space is usually stated using geodesic triangles and comparison triangles on the geodesic space and its model space, respectively. In this work, we define it by the equivalent condition described by a particular inequality as follows: We say a geodesic space X is a CAT(0) space if the inequality

$$d(\gamma_{xy}(t), z)^2 \leq (1-t) d(x,z)^2 + t d(y,z)^2 - t(1-t) d(x,y)^2$$

holds for any $x, y, z \in X$ and $t \in [0,1]$.

A geodesic space X is said to be uniquely geodesic if for each pair of points $x, y \in X$, there is a unique geodesic γ_{xy} connecting them. It is easy to see that every CAT(0) space is uniquely geodesic. In this case, we can define a convex combination $(1-t)x \oplus ty \in X$ of $x, y \in X$ with a parameter $t \in [0,1]$ by $(1-t)x \oplus ty = \gamma_{xy}(t)$. Moreover, we define a geodesic segment $[x,y]$ between x and y by

$$[x,y] = \gamma_{xy}([0,1]) = \{(1-t)x \oplus ty \in X \mid 0 \leq t \leq 1\}.$$

Using this notion, we define the convexity of sets in a natural way; a subset C of X is said to be convex if $[x,y] \subset C$ for any $x, y \in C$.

For more details of Hadamard space, see [1, 2] for instance.

Let $T \colon X \to X$ be a mapping on a metric space X. We say T is nonexpansive if $d(Tx, Ty) \leq d(x,y)$ for any $x, y \in X$. A point $z \in X$ is called a fixed point of T if it satisfies $z = Tz$. The set of all fixed points of T is denoted by $\operatorname{Fix} T$. We know that $\operatorname{Fix} T$ is closed and convex if T is nonexpansive.

Let $f \colon X \to \,]-\infty, \infty]$ be a function on a Hadamard space X. Then,

- f is said to be proper if $f(x) \in \mathbb{R}$ for some $x \in X$;
- f is said to be lower semincontinuous if $f(x) \le \liminf_{n\to\infty} f(x_n)$ for any $x \in X$ and $\{x_n\} \subset X$ with $x_n \to x$;
- f is said to be convex if $f(tx \oplus (1-t)y) \le tf(x) + (1-t)(y)$ for any $x, y \in X$ and $t \in]0,1[$.

We denote the set of all minimizers of f on X by $\operatorname{argmin}_X f$.

For a subset C of X, we define the indicator function $i_C\colon X \to]-\infty,\infty]$ of C by

$$i_C(x) = \begin{cases} 0 & (x \in C), \\ \infty & (x \notin C) \end{cases}$$

for $x \in X$. It is easy to see that C is nonempty, closed, and convex, if and only if i_C is proper, lower semicontinuous, and convex, respectively.

If f is a proper lower semicontinuous convex function, then so is λf for $\lambda \in]0,\infty[$. Moreover, for such λf and $x \in X$, there exists a unique minimizer $y_x \in X$ of the function $\lambda f + d(x,\cdot)^2/2$. Using this point we define the resolvent $R_{\lambda f}\colon X \to X$ of λf by

$$R_{\lambda f}x = y_x = \operatorname*{argmin}_X \left(\lambda f + \frac{1}{2} d(x,\cdot)^2 \right)$$

for each $x \in X$. In particular, if $f = i_C$ for a nonempty closed convex subset C of X, the resolvent operator $R_f = R_{i_C}$ is the mapping making $x \in X$ correspond to the unique closest point $y_x \in C$ to x. We call it a metric projection onto C and denote it by P_C.

Let $\{x_n\}$ be a bounded sequence in a metric space X. Define a function $g\colon X \to [0,\infty]$ by $g(x) = \limsup_{n\to\infty} d(x_n, x)$ for $x \in X$. The subset $\operatorname{argmin}_X g$ of X is called the asymptotic center of $\{x_n\}$. If X is a Hadamard space, the asymptotic center is a singleton for every bounded sequence [3]. We say $\{x_n\}$ is Δ-convergent [7] to $x_0 \in X$ if any subsequence $\{x_{n_i}\}$ of $\{x_n\}$ has a unique asymptotic center x_0. In this case, we call x_0 the Δ-limit of $\{x_n\}$.

We know that any closed convex subset C of a Hadamard space is Δ-closed in the sense that any Δ-convergent sequence $\{x_n\}$ in C has its Δ-limit in C. Further, any proper lower semincontinuous convex function $f\colon X \to]-\infty,\infty]$ is Δ-lower semicontinuous in the sense that

$$f(x_0) \le \liminf_{n\to\infty} f(x_n)$$

holds for any sequence $\{x_n\} \subset X$ which is Δ-convergent to $x_0 \in X$.

Let $\{C_n\}$ be a sequence of nonempty closed convex subsets in a Hadamard space X. We define two subsets $d\text{-}\mathrm{Li}_n C_n$ and $\Delta\text{-}\mathrm{Ls}_n C_n$ of X as follows: $x \in d\text{-}\mathrm{Li}_n C_n$ if and only if there exists $\{x_n\} \subset X$ such that $x_n \in C_n$ for all $n \in \mathbb{N}$ and $d(x_n, x) \to 0$. On the other hand, $y \in \Delta\text{-}\mathrm{Ls}_n C_n$ if and only if there exist a sequence $\{y_i\} \subset X$ and a subsequence $\{n_i\}$ of $\mathbb{N}$ such that $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$ and the asymptotic center of $\{y_i\}$ is $\{y\}$. We say that $\{C_n\}$ converges to C_0 in the sense of Δ-Mosco [4] if a subset

C_0 of X satisfies that

$$C_0 = d\text{-}\operatorname*{Li}_n C_n = \Delta\text{-}\operatorname*{Ls}_n C_n.$$

In this case, C_0 is called a Δ-Mosco limit of $\{C_n\}$ and we denote it by $\Delta\text{M-lim}_{n\to\infty} C_n$.

The following result shows the relation between a sequence of closed convex sets in a Hadamard space and the corresponding sequence of metric projections onto them.

Theorem 2.1 (Kimura [4]). *Let X be a Hadamard space and C_0 a nonempty closed convex subset of X. Then, for a sequence $\{C_n\}$ of nonempty closed convex subsets in X, the following are equivalent:*

(a) $C_0 = \Delta\text{M-lim}_{n\to\infty} C_n$;
(b) $\{P_{C_n} x\}$ *converges to* $P_{C_0} x \in X$ *for every* $x \in X$.

As a simple and important example of Δ-Mosco convergence, the following fact is mentioned in [4]: If $\{C_n\}$ is a nonempty closed convex subsets in X such that

$$C_1 \supset C_2 \supset \cdots \supset C_n \supset C_{n+1} \supset \cdots$$

and $\bigcap_{k=1}^{\infty} C_k \neq \varnothing$, then $\bigcap_{k=1}^{\infty} C_k = \Delta\text{M-lim}_{n\to\infty} C_n$.

3 Main result

In this section, we prove our main result, for generating an approximation sequence convergent to a solution to a convex minimization problem with a fixed point constraint of a nonexpansive mapping.

The following result will serve an essential part of the proof for the main theorem.

Theorem 3.1 (Kimura–Shindo [6]). *Let X be a Hadamard space and let $f_n \colon X \to]-\infty, \infty]$ be a proper lower semicontinuous convex function having a minimizer on X for $n \in \mathbb{N} \cup \{\infty\}$. Let $\{\lambda_n\} \subset]0, \infty[$ be an increasing sequence diverging to ∞. Suppose the following conditions:*

(i) $\Delta\text{M-lim}_{n\to\infty} \operatorname{argmin}_X f_n = \operatorname{argmin}_X f_\infty$;
(ii) *for any $b \in X$, there exists $\{b_n\} \subset X$ such that $b_n \to b$ and $\limsup_{n\to\infty} f_n(b_n) \leq f_\infty(b)$;*
(iii) *for any subsequence $\{f_{n_i}\}$ of $\{f_n\}$ and a Δ-convergent sequence $\{c_i\} \subset X$ whose Δ-limit is $c \in X$, it holds that $f_\infty(c) \leq \liminf_{i\to\infty} f_{n_i}(c_i)$.*

Then,

$$\lim_{n\to\infty} R_{\lambda_n f_n} u = P_{\operatorname{argmin}_X f_\infty} u$$

for any $u \in X$.

We also need the following lemma.

Lemma 3.2. *Let $f\colon X \to \,]-\infty, \infty]$ be a proper function on a nonempty set X, and C a nonempty subset of X such that* $\operatorname{argmin}_X f \cap C \neq \varnothing$. *Then*

$$\operatorname*{argmin}_X (f + i_C) = \operatorname*{argmin}_X f \cap C.$$

Proof. Let $w \in \operatorname{argmin}_X f \cap C$. Since f is proper, we have

$$(f + i_C)(w) = f(w) = \inf_{y \in X} f(y) < \infty.$$

Using this fact, we show $\operatorname{argmin}_X (f + i_C) \subset \operatorname{argmin}_X f \cap C$. Let $z \in \operatorname{argmin}_X (f + i_C)$. Then we have

$$(f + i_C)(z) \leq (f + i_C)(w) < \infty.$$

It implies that $i_C(z) \neq \infty$ and thus $i_C(z) = 0$, or equivalently, $z \in C$. Further, we have

$$f(w) \leq f(z) = (f + i_C)(z) \leq (f + i_C)(w) = f(w),$$

which deduces $f(w) = f(z)$. Hence $z \in \operatorname{argmin}_X f$, and thus $\operatorname{argmin}_X (f + i_C) \subset \operatorname{argmin}_X f \cap C$.

For the opposite inclusion, let $w \in \operatorname{argmin}_X f \cap C$. Then, for all $y \in X$, we have

$$(f + i_C)(w) = f(w) \leq f(y) \leq f(y) + i_C(y) = (f + i_C)(y).$$

It implies that $w \in \operatorname{argmin}_X (f + i_C)$, and hence $\operatorname{argmin}_X (f + i_C) \supset \operatorname{argmin}_X f \cap C$, the desired result. □

We now show the main result of this work.

Theorem 3.3. *Let X be a Hadamard space such that a subset $\{z \in X \mid d(u, z) \leq d(v, z)\}$ is convex for all $u, v \in X$. Let $f\colon X \to \,]-\infty, \infty]$ be a proper lower semicontinuous convex function and $T\colon X \to X$ a nonexpansive mapping. Suppose that $S = \operatorname{argmin}_X f \cap \operatorname{Fix} T \neq \varnothing$. Let $\{\lambda_n\} \subset \,]0, \infty[$ be an increasing real sequence diverging to ∞. For $u \in X$, generate a sequence $\{x_n\} \subset X$ as follows: $x_1 \in X$, $f_1 = f$, and*

$$\begin{aligned} X_n &= \{z \in X \mid d(Tx_n, z) \leq d(x_n, z)\}, \\ f_{n+1} &= f_n + i_{X_n}, \\ x_{n+1} &= R_{\lambda_{n+1} f_{n+1}} u \end{aligned}$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ converges to $P_S u$.

Proof. We first notice that the sequence $\{x_n\}$ is well defined; since $d(Tx_n, z) \leq d(x_n, z)$ for every $z \in \operatorname{Fix} T$, we have

$$\varnothing \neq S \subset \operatorname{Fix} T \subset X_n$$

for all $n \in \mathbb{N}$. It shows that every f_n is proper. Moreover, since X_n is closed and convex, its indicator function i_{X_n} is lower semicontinuous and convex, and so is f_n. Therefore we can define the resolvent $R_{\lambda_n f_n} : X \to X$ and thus $\{x_n\}$ is well defined.

Let $C_n = \bigcap_{k=1}^n X_k$ for $n \in \mathbb{N}$, $C_\infty = \bigcap_{k=1}^\infty X_k$, and $f_\infty = f + i_{C_\infty}$. Then, we have

$$\begin{aligned} f_{n+1} &= f_n + i_{X_n} = f_{n-1} + i_{X_{n-1}} + i_{X_n} = \cdots \\ &= f_1 + \sum_{k=1}^n i_{X_k} = f_1 + i_{\bigcap_{k=1}^n X_k} = f + i_{C_n} \end{aligned}$$

for $n \in \mathbb{N}$. We show that $\{f_n\}$ and f_∞ satisfy the conditions (i), (ii), and (iii) in Theorem 3.1. For (i), it follows from Lemma 3.2 that

$$\operatorname*{argmin}_X f_{n+1} = \operatorname*{argmin}_X (f + i_{C_n}) = \operatorname*{argmin}_X f \cap C_n.$$

Since $\{\operatorname{argmin}_X f \cap C_n\}$ is a decreasing sequence of subsets of X with respect to inclusion, we have

$$\Delta\text{M-}\lim_{n\to\infty} \operatorname*{argmin}_X f_n = \Delta\text{M-}\lim_{n\to\infty} \left(\operatorname*{argmin}_X f \cap C_n \right) = \operatorname*{argmin}_X f \cap C_\infty = \operatorname*{argmin}_X f_\infty,$$

which is the condition (i).

Next, we prove (ii). Fix $b \in X$ arbitrarily. If $f_\infty(b) = \infty$, then the inequality $\limsup_{n\to\infty} f_n(b_n) \le f_\infty(b)$ obviously holds for any choice of $\{b_n\}$ with $b_n \to b$. If $f_\infty(b) = (f + i_{C_\infty})(b) < \infty$, then we have $f_\infty(b) = f(b) < \infty$ and $b \in C_\infty$. It follows that $b \in C_n$ for every $n \in \mathbb{N}$, and thus we have

$$f_{n+1}(b) = f(b) + i_{C_n}(b) = f(b)$$

for any $n \in \mathbb{N}$. Therefore, letting $b_n = b$ for every $n \in \mathbb{N}$, we obtain $b_n \to b$ and $\limsup_{n\to\infty} f_n(b_n) = f(b) = f_\infty(b)$. Hence the condition (ii) holds.

For (iii), let $\{f_{n_i}\}$ be a subsequence of $\{f_n\}$ and $\{c_i\} \subset X$ a Δ-convergent sequence with its Δ-limit $c \in X$. If $\liminf_{i\to\infty} f_{n_i}(c_i) = \infty$, then the inequality $f_\infty(c) \le \liminf_{i\to\infty} f_{n_i}(c_i)$ obviously holds. Therefore, we may assume $\liminf_{i\to\infty} f_{n_i}(c_i) < \infty$. Let $\{i_j\}$ be an increasing sequence of $\mathbb{N}$ such that $f_{n_{i_j}}(c_{i_j}) \in \mathbb{R}$ for all $j \in \mathbb{N}$ and that

$$\lim_{j\to\infty} f_{n_{i_j}}(c_{i_j}) = \liminf_{i\to\infty} f_{n_i}(c_i) < \infty.$$

Since $f_{n_{i_j}}(c_{i_j}) = f(c_{i_j}) + i_{C_{n_{i_j}-1}}(c_{i_j}) < \infty$, we have $f_{n_{i_j}}(c_{i_j}) = f(c_{i_j})$ and $c_{i_j} \in C_{n_{i_j}-1}$ for all $j \in \mathbb{N}$. To show the Δ-limit c of $\{c_i\}$ belongs to $C_\infty = \bigcap_{k=1}^\infty X_k = \bigcap_{k=1}^\infty C_k$, fix $k \in \mathbb{N}$ arbitrarily. Then, there exists $j_0 \in \mathbb{N}$ such that $n_{i_{j_0}} - 1 \ge k$. Since $\{C_n\}$ is decreasing with respect to inclusion, we have

$$c_{i_j} \in C_{n_{i_j}-1} \subset C_{n_{i_{j_0}}-1} \subset C_k$$

for every $j \geq j_0$. Since C_k is a closed convex subset of X, it is Δ-closed. It implies that $c \in C_k$. Since k is arbitrary, we obtain $c \in \bigcap_{k=1}^{\infty} C_k = C_\infty$. It follows that $f_\infty(c) = f(c) + i_{C_\infty}(c) = f(c)$. Since f is lower semicontinuous and convex, it is Δ-lower semicontinuous and it implies that

$$f_\infty(c) = f(c) \leq \liminf_{j\to\infty} f(c_{i_j}) = \liminf_{j\to\infty} f_{n_{i_j}}(c_{i_j}) = \lim_{j\to\infty} f_{n_{i_j}}(c_{i_j}) = \liminf_{i\to\infty} f_{n_i}(c_i).$$

Thus the condition (iii) holds.

Now we apply Theorem 3.1 and obtain

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} R_{\lambda_n f_n} u = P_{\operatorname{argmin}_X f_\infty} u.$$

To conclude the proof, we show $P_{\operatorname{argmin}_X f_\infty} u = P_S u$. Let $z_0 = P_{\operatorname{argmin}_X f_\infty} u$. Since z_0 belongs to $\operatorname{argmin}_X f_\infty = \operatorname{argmin}_X f \cap C_\infty$, from the definition of C_∞, we have $z_0 \in C_n$ for every $n \in \mathbb{N}$. It follows that $d(Tx_n, z_0) \leq d(x_n, z_0)$ for all $n \in \mathbb{N}$, and thus

$$0 \leq d(Tz_0, z_0) \leq d(Tz_0, Tx_n) + d(Tx_n, z_0) \leq d(z_0, x_n) + d(Tx_n, z_0) \leq 2d(x_n, z_0).$$

Letting $n \to \infty$, we get $d(Tz_0, z_0) = 0$, or equivalently, $z_0 \in \operatorname{Fix} T$. Therefore, we have $z_0 \in S$. Since $\operatorname{Fix} T \subset \bigcap_{k=1}^{\infty} X_k = C_\infty$, we have $S = \operatorname{argmin}_X f \cap \operatorname{Fix} T \subset \operatorname{argmin}_X f \cap C_\infty$ and

$$d(z_0, u) = d(P_{\operatorname{argmin}_X f_\infty} u, u) = d(P_{\operatorname{argmin}_X f \cap C_\infty} u, u) \leq d(P_S u, u) \leq d(z_0, u).$$

It implies that $d(z_0, u) = d(P_S u, u)$. By the uniqueness of the closest point to a nonempty closed convex subset S of X, we obtain $z_0 = P_S u$, which is the desired result. □

We note that the given point u in this theorem is called an anchor point of the iterative sequence.

From this theorem, we can deduce some known results of approximation methods to solve a convex minimization problem with no constraint and a fixed point problem.

The following result generates an approximation sequence of a minimizer of a given function f. Also, it can be regarded as a discrete version of the asymptotic behavior of a resolvent $R_{\lambda f}$ as $\lambda \to \infty$.

Corollary 3.4 (Kimura–Shindo [6])**.** *Let X be a Hadamard space, $f\colon X \to]-\infty, \infty]$ a proper lower semicontinuous convex function having a minimizer on X. Let $\{\lambda_n\} \subset]0, \infty[$ be an increasing real sequence diverging to ∞. Then $\{R_{\lambda_n f} u\}$ converges to $P_{\operatorname{argmin}_X f} u$ for each $u \in X$.*

Proof. Suppose that a mapping T is the identity mapping on X. Applying Theorem 3.3, we obtain the desired result. □

The following approximation technique is known as the shrinking projection method for a nonexpansive mapping, originally proved by Takahashi, Takeuchi, and Kubota [8] in the setting of a Hilbert space.

Corollary 3.5 (Kimura [4]). *Let X be a Hadamard space such that a subset $\{z \in X \mid d(u,z) \le d(v,z)\}$ is convex for all $u, v \in X$. Let $T\colon X \to X$ be a nonexpansive mapping such that $\operatorname{Fix} T \neq \varnothing$. For $u \in X$, generate an iterative sequence $\{x_n\}$ as follows: $x_1 \in X$, $C_1 = X$, and*

$$\begin{aligned} X_n &= \{z \in X \mid d(Tx_n, z) \le d(x_n, z)\}, \\ C_{n+1} &= C_n \cap X_n, \\ x_{n+1} &= P_{C_{n+1}} u \end{aligned}$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges to $P_{\operatorname{Fix} T} u \in X$.

Proof. Define $f\colon X \to]-\infty, \infty]$ by $f(x) = 0$ for all $x \in X$. Applying Theorem 3.3, we obtain the desired result. □

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