REMARKS ON MULTI-COMPARATIVELY QUASI-CONTRACTIONS

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ABSTRACT. In 2025, the author introduced a new class of quasi-contractions, called multi-comparatively quasi-contractions, which includes many known multivalued contractions as special cases. In this work, new sufficient conditions for multi-comparatively quasi-contractions will be established.

1. INTRODUCTION

Let (X, d) be a metric space. For each $a \in X$ and $M \subseteq X$, let

$$d(a, M) = \inf_{b \in M} d(a, b).$$

Denote by $\mathcal{N}(X)$ the class of all nonempty subsets of X, $\mathcal{C}(X)$ the family of all nonempty closed subsets of X and $\mathcal{CB}(X)$ the family of all nonempty closed and bounded subsets of X. A function $\mathcal{H} : \mathcal{CB}(X) \times \mathcal{CB}(X) \to [0, \infty)$ defined by

$$\mathcal{H}(A, B) = \max \left\{ \sup_{x \in B} d(x, A), \sup_{x \in A} d(x, B) \right\}$$

is said to be the Hausdorff metric on $\mathcal{CB}(X)$ induced by the metric d on X. Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$, the sets of positive integers and real numbers, respectively.

Recall that a multivalued mapping $T : X \to \mathcal{N}(X)$ is called

(i) a *Nadler's type contraction*, if there exists a number $k \in [0, 1)$ such that

$$\mathcal{H}(Tx, Ty) \leq kd(x, y) \quad \text{for all } x, y \in X,$$

(ii) a *Kannan's type contraction*, if there exists a number $k \in \left[0, \frac{1}{2}\right)$ such that

$$\mathcal{H}(Tx, Ty) \leq k(d(x, Tx) + d(y, Ty)) \quad \text{for all } x, y \in X,$$

2020 *Mathematics Subject Classification.* 47H09, 47H10, 54H25.

Key words and phrases. w-distance, τ-function, e-distance, e^0-metric, multi-comparatively quasi-contraction.

The author was partially supported by Grant No. NSTC 113-2115-M-017-004 of the National Science and Technology Council of the Republic of China.

(iii) a *Chatterjea's type contraction*, if there exists a number $k \in \left[0, \frac{1}{2}\right)$ such that

$$\mathcal{H}(Tx, Ty) \le k(d(x, Ty) + d(y, Tx)) \quad \text{for all } x,\ y \in X,$$

(iv) a *Mizoguchi-Takahashi's type contraction*, if there exists an $\mathcal{MT}$-function $\alpha : [0, \infty) \to [0, 1)$ such that

$$\mathcal{H}(Tx, Ty) \le \alpha(d(x, y))d(x, y) \quad \text{for all } x, y \in X,$$

(v) a *multivalued* (θ, L)*-almost contraction* [1, 9, 10], if there exist two constants $\theta \in (0, 1)$ and $L \ge 0$ such that

$$\mathcal{H}(Tx, Ty) \le \theta d(x, y) + Ld(y, Tx) \quad \text{for all } x,\ y \in X,$$

(vi) a *Berinde-Berinde's type contraction* (or a *generalized multivalued almost contraction* [1, 9, 10], if there exists an $\mathcal{MT}$-function α : $[0, \infty) \to [0, 1)$ and $L \ge 0$ such that

$$\mathcal{H}(Tx, Ty) \le \alpha(d(x, y))d(x, y) + Ld(y, Tx) \quad \text{for all } x,\ y \in X.$$

It is worth noting that a Berinde-Berinde's type contraction is a real generalization of Mizoguchi-Takahashi's type contraction (see, e.g., [4, Example 2.1]).

A function $p : X \times X \to [0, +\infty)$ is called a *w-distance* [12, 14, 16], if the following are satisfied:

($w1$) $p(a, c) \le p(a, b) + p(b, c)$ for any $a, b, c \in X$;
($w2$) for any $a \in X$, $p(a, \cdot) : X \to [0, +\infty)$ is l.s.c.;
($w3$) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(c, a) \le \delta$ and $p(c, b) \le \delta$ imply $d(a, b) \le \varepsilon$.

A function $p : X \times X \to [0, +\infty)$ is said to be a *τ-function* [2, 3, 13, 15], first introduced and studied by Lin and Du, if the following conditions hold:

($\tau1$) $p(a, c) \le p(a, b) + p(b, c)$ for any $a, b, c \in X$;
($\tau2$) if $a \in X$ and $\{b_n\}$ in X with $\lim_{n\to\infty} b_n = b$ such that $p(a, b_n) \le \beta$ for some $\beta = \beta(a) > 0$, then $p(a, b) \le \beta$;
($\tau3$) for any sequence $\{a_n\}$ in X with $\limsup_{n\to\infty}\{p(a_n, a_m) : m > n\} = 0$, if there exists a sequence $\{b_n\}$ in X such that $\lim_{n\to\infty} p(a_n, b_n) = 0$, then $\lim_{n\to\infty} d(a_n, b_n) = 0$;
($\tau4$) for $a, b, c \in X$, $p(a, b) = 0$ and $p(a, c) = 0$ imply $b = c$.

It is well known that the metric d is a w-distance and any w-distance is a τ-function, but the converse is not true (see [2, 13]). Note that not either of the implications $p(x, y) = 0 \iff x = y$ necessarily holds and p is nonsymmetric in general.

In 2016, Du [6] introduced and studied the concept of essential distance as follows:

Definition 1.1 (see [6, Definition 1.2])**.** Let (X, d) be a metric space. A function $p : X \times X \to [0, +\infty)$ is called an *essential distance* (abbreviated as "*e-distance*") if conditions ($\tau1$), ($\tau2$) and ($\tau3$) hold.

It is very obvious that any τ-function is an e-distance.

In 2019, Du introduced and studied the concept of e^0-distance [7] (see also [11]) as follows:

Definition 1.2 (see [7, Definition 1.3])**.** Let (X, d) be a metric space. A function $p : X \times X \to [0, +\infty)$ is called an e^0-*distance* if it is an e-distance on X with $p(x, x) = 0$ for all $x \in X$.

Example 1.3. Let $X = \mathbb{R}$ with the metric $d(a, b) = |a - b|$. Then (X, d) is a metric space. Define the function $p : X \times X \to [0, +\infty)$ by

$$p(x, y) = \max\{5(y - x), 3(x - y)\}.$$

Therefore p is not a metric due to its asymmetry. It is easy to see that p is an e^0-distance on X.

The following definition of e^0-metric was studied by Du in [7] which generalizes the concept of Hausdorff metric.

Definition 1.4 (see [7, Definition 1.4])**.** Let (X, d) be a metric space and p be an e^0-distance. For any E, $F \in \mathcal{CB}(X)$, define a function $\mathcal{D}_p : \mathcal{CB}(X) \times \mathcal{CB}(X) \to [0, +\infty)$ by

$$\mathcal{D}_p(E, F) = \max\{\xi_p(E, F), \xi_p(F, E)\},$$

where $\xi_p(E, F) = \sup_{x \in E} p(x, F)$, then $\mathcal{D}_p$ is said to be the e^0-*metric* on $\mathcal{CB}(X)$ induced by p.

Theorem 1.5 (see [7, Theorem 2.4] or [11, Theorem 6])**.** *Let (X, d) be a metric space and $\mathcal{D}_p$ be an e^0-metric defined as in Definition* **??** *on $\mathcal{CB}(X)$ induced by an e^0-distance p. Then for $E, F, G \in \mathcal{CB}(X)$, the following hold:*

(i) $\xi_p(E, F) = 0 \iff E \subseteq F$;

(ii) $\xi_p(E, F) \leq \xi_p(E, G) + \xi_p(G, F)$;

(iii) *Every e^0-metric $\mathcal{D}_p$ is a metric on $\mathcal{CB}(X)$.*

In 2025, Du introduced a new class of quasi-contractions, called multi-comparatively quasi-contractions (see Definition 2.3 below), which includes Nadler's type contractions, Kannan's type contractions, Chatterjea's type contractions, Mizoguchi-Takahashi's type contractions, multivalued (θ, L)-almost contractions and Berinde-Berinde's type contractions as special cases. In this work, some new sufficient conditions for multi-comparatively quasi-contractions will be established.

2. Preliminaries

In 2005, Uderzo [17] introduced directional multivalued $k(\cdot)$-contractions and established many new fixed point theorems for directional multivalued $k(\cdot)$-contractions (see [17] for more details).

Definition 2.1 (**Uderzo** [17])**.** Let U be a nonempty subset of a metric space (X, d). A multivalued map $T : U \to \mathcal{CB}(X)$ is called a *directional multivalued $k(\cdot)$-contraction* if there exists $\lambda \in (0, 1]$, $a : (0, +\infty) \to [\lambda, 1]$

and $k:(0,+\infty)\to[0,1)$ such that for every $x\in U$ with $x\notin Tx$, there is $y\in U\setminus\{x\}$ satisfying the inequalities

$$a(d(x,y))d(x,y)+d(y,Tx)\le d(x,Tx)$$

and

$$\sup_{z\in Tx} d(z,Ty)\le k(d(x,y))d(x,y).$$

Definition 2.2 (see [8, Definition 3.1])**.** Let (X,d) be a metric space and p be an e-distance.

(i) For any $a_1,a_2,a_3,a_4,a_5\in[0,1]$, we define a real-valued mapping $f_{[a_1,a_2,a_3,a_4,a_5]}:X\times X\to[0,+\infty)$ by

$$f_{[a_1,a_2,a_3,a_4,a_5]}(x,y):=a_1p(x,y)+a_2p(x,Tx)+a_3p(y,Ty)+a_4p(x,Ty)+a_5p(y,Tx).$$

(ii) We denote by $\mathcal{O}$ the collection of all mappings $f_{[a_1,a_2,a_3,a_4,a_5]}$ on $X\times X$ satisfying $0<\sum_{i=1}^{5}a_i\le 1$.

(iii) For any $(x,y)\in X\times X$, we define

$$\mathcal{O}_{[x,y]}:=\left\{f_{[a_1,a_2,a_3,a_4,a_5]}(x,y)\in[0,+\infty): f_{[a_1,a_2,a_3,a_4,a_5]}\in\mathcal{O}\right\}.$$

Here, we shall give some examples of such mappings in $\mathcal{O}$:

$$f_{[1,0,0,0,0]}(x,y)=p(x,y),\quad f_{[0,1,0,0,0]}(x,y)=p(x,Tx),\quad f_{[0,0,1,0,0]}(x,y)=p(y,Ty),$$

$$f_{[0,0,0,1,0]}(x,y)=p(x,Ty),\quad f_{[0,0,0,0,1]}(x,y)=p(y,Tx),$$

$$f_{[0,\frac{1}{2},\frac{1}{2},0,0]}(x,y)=\frac{p(x,Tx)+p(y,Ty)}{2},$$

$$f_{[0,0,0,\frac{1}{2},\frac{1}{2}]}(x,y)=\frac{p(x,Ty)+p(y,Tx)}{2},$$

$$f_{[0,\frac{1}{3},0,\frac{2}{3},0]}(x,y)=\frac{p(x,Tx)+2p(x,Ty)}{3},$$

$$f_{[\frac{1}{4},\frac{1}{4},0,0,\frac{1}{4}]}(x,y)=\frac{p(x,y)+p(x,Tx)+p(y,Tx)}{4},$$

$$f_{[\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7}]}(x,y)=\frac{p(x,y)+p(x,Tx)+p(y,Ty)+p(x,Ty)+p(y,Tx)}{7},$$

$$f_{[\frac{1}{4},\frac{1}{8},0,\frac{1}{8},\frac{1}{2}]}(x,y)=\frac{2p(x,y)+p(x,Tx)+p(x,Ty)+4p(y,Tx)}{8},$$

and so on.

It is easy to see that for any $(x,y) \in X \times X$, we have

$$\sup \mathcal{O}_{[x,y]} = \max \{p(x,y), p(x,Tx), p(y,Ty), p(x,Ty), p(y,Tx)\}.$$

Very recently, Du [8] introduced the concept of multi-comparatively quasi-contractions to generalize and improve the concept of directional multivalued $k(\cdot)$-contractions.

Definition 2.3 (see [8, Definition 3.3])**.** Let U be a nonempty subset of a metric space (X,d), $p : X \times X \to [0,+\infty)$ and $\phi : [0,+\infty) \to [0,1)$ be functions. Assume that $c \in [0,1)$ and let $\eta : [0,+\infty) \to (c,1]$ be a function. A multivalued mapping $T : U \to \mathcal{N}(X)$ is called a *multi-comparatively quasi-contraction* with respect to p, ϕ, c and η (abbreviated as (p,ϕ,c,η)-**MCQC**) if for any $x \in U$ with $x \notin Tx$, there exist $y \in U \setminus \{x\}$ and $z \in Tx$ such that

$$\eta(p(x,y))p(x,y) + p(y,z) \le p(x,Tx)$$

and

$$p(z,Ty) \le \phi(p(x,y))G(x,y), \tag{2.1}$$

where Ω is a nonempty subset of $\mathcal{O}$,

$$\Omega_{[x,y]} := \left\{ f_{[a_1,a_2,a_3,a_4,a_5]}(x,y) \in [0,+\infty) : f_{[a_1,a_2,a_3,a_4,a_5]} \in \Omega \right\}$$

and $G(x,y) = \sup \Omega_{[x,y]}$ for $(x,y) \in X \times X$. In particular, if $p \equiv d$, then we use the notation (ϕ,c,η)-**MCQC** instead of (d,ϕ,c,η)-**MCQC**.

Remark 2.4. In Definition 2.3, if condition (2.1) is replaced by the following condition

$$p(z,Ty) \le \phi(p(x,y))p(x,y),$$

Then T is called a directional hidden contraction with respect to p, ϕ, c and η, which was studied by Du in [4, Definition 2.1]. It is obvious that a directional hidden contraction is (p,ϕ,c,η)-**MCQC** with $\Omega_{[x,y]} := \{d(x,y)\}$.

Theorem 2.5 (**Du** [8, Theorem 3.5])**.** *Any directional multivalued $k(\cdot)$-contraction is a multi-comparatively quasi-contraction.*

3. Sufficient conditions for multi-comparatively quasi-contractions

Theorem 3.1 (see [8, Theorem 3.6])**.** *Let (X,d) be a metric space, p be a e^0-distance, $T : X \to \mathcal{C}(X)$ be a multivalued mapping and $\varphi : [0,+\infty) \to [0,1)$ be a function. Suppose that*

$(H1)$ there exists $\beta \in [0,+\infty)$ such that

$$\limsup_{t \to \beta^+} \varphi(t) < 1;$$

$(H2)$ for each $x \in X$ with $x \notin Tx$, it holds

$$p(y,Ty) \le \varphi(p(x,y)) \max \{p(x,y), p(x,Tx), p(y,Ty), p(x,Ty)\} \quad \text{for all } y \in Tx.$$

Then there exist $c \in [0,1)$ and functions $\eta : [0,+\infty) \to (c,1]$ and $\phi : [0,+\infty) \to [0,1)$ such that

(a) $\limsup_{t\to\beta^+} \phi(t) \le c < \liminf_{t\to\beta^+} \eta(t)$;

(b) T *is a* (p, ϕ, c, η)*-**MCQC**.*

Theorem 3.2 (see [8, Theorem 3.7])**.** *Let (X, d) be a metric space, p be a e^0-distance, $\mathcal{D}_p$ be a e^0-metric on $\mathcal{CB}(X)$ induced by p, $T : X \to \mathcal{CB}(X)$ be a multivalued mapping and $\varphi : [0, +\infty) \to [0, 1)$ and $h : X \times X \to [0, +\infty)$ be functions. Suppose that*

$(K1)$ *there exists $\beta \in [0, +\infty)$ such that*

$$\limsup_{t\to\beta^+} \varphi(t) < 1;$$

$(K2)$ $\mathcal{D}_p(Tx, Ty) \le \varphi(p(x, y))S(x, y) + h(x, y)p(y, Tx)$ *for all $x, y \in X$ with $x \neq y$, where*

$$S(x, y) = \max\left\{p(x, y), \frac{p(x, Tx) + p(y, Ty)}{2}, \frac{p(x, Ty) + p(y, Tx)}{2}\right\}.$$

Then there exist $c \in [0, 1)$ and functions $\eta : [0, +\infty) \to (c, 1]$ and $\phi : [0, +\infty) \to [0, 1)$ such that

(a) $\limsup_{t\to\beta^+} \phi(t) \le c < \liminf_{t\to\beta^+} \eta(t)$;

(b) T *is a* (p, ϕ, c, η)*-**MCQC**.*

The following result follows directly from Theorem 3.2.

Theorem 3.3 (see [8, Theorem 3.8])**.** *Let (X, d) be a metric space and $T : X \to \mathcal{CB}(X)$ be a multivalued mapping. Assume that one of the following conditions holds:*

(1) *T is a Berinde-Berinde's type contraction;*
(2) *T is a multivalued (θ, L)-almost contraction;*
(3) *T is a Mizoguchi-Takahashi's type contraction;*
(4) *T is a Nadler's type contraction;*
(5) *T is a Kannan's type contraction;*
(6) *T is a Chatterjea's type contraction.*

*Then there exist $c \in [0, 1)$ and functions $\eta : [0, +\infty) \to (c, 1]$ and $\phi : [0, +\infty) \to [0, 1)$ such that T is a (ϕ, c, η)-**MCQC**.*

Finally, we establish the following new existence theorem which provides new sufficient conditions for multi-comparatively quasi-contractions.

Theorem 3.4. *Let (X, d) be a metric space, p be a e^0-distance, $\mathcal{D}_p$ be a e^0-metric on $\mathcal{CB}(X)$ induced by p, $T : X \to \mathcal{CB}(X)$ be a multivalued mapping and $\varphi : [0, +\infty) \to [0, 1)$ and $h : X \times X \to [0, +\infty)$ be functions. Suppose that*

$(V1)$ *there exists $\beta \in [0, +\infty)$ such that*

$$\limsup_{t\to\beta^+} \varphi(t) < 1;$$

(V2) $\mathcal{D}_p(Tx,Ty) \le \varphi(p(x,y))S(x,y) + h(x,y)p(y,Tx)$ *for all* $x,y \in X$ *with* $x \neq y$*, where*

$$S(x,y) = \max\Big\{p(x,y), \frac{p(x,Tx)+p(y,Ty)}{2}, \frac{p(x,Ty)+p(y,Tx)}{2}, \frac{p(x,Tx)+p(y,Tx)}{2},$$
$$\frac{p(y,Tx)+p(y,Ty)}{2}, \frac{p(x,Tx)+p(y,Ty)+p(y,Tx)}{3},$$
$$\frac{p(x,Tx)+p(x,Ty)+p(y,Tx)}{3}, \frac{p(y,Ty)+p(x,Ty)+p(y,Tx)}{3},$$
$$\frac{p(x,y)+p(y,Ty)+p(x,Tx)}{3}, \frac{p(x,y)+p(x,Tx)+p(y,Tx)}{3},$$
$$\frac{2p(y,Tx)+p(y,Ty)+p(x,Ty)}{4}, \frac{p(x,y)+p(y,Ty)+p(y,Tx)}{4},$$
$$\frac{p(x,Tx)+p(y,Ty)+p(x,Ty)+p(y,Tx)}{4},$$
$$\frac{p(x,y)+p(y,Ty)+p(x,Ty)}{4}, \frac{p(x,y)+2p(x,Tx)+p(y,Ty)}{5},$$
$$\frac{p(x,y)+p(x,Tx)+p(y,Ty)+p(x,Ty)+p(y,Tx)}{5}\Big\}.$$

Then there exist $c \in [0,1)$ *and functions* $\eta : [0,+\infty) \to (c,1]$ *and* $\phi : [0,+\infty) \to [0,1)$ *such that*

(a) $\limsup_{t\to\beta^+} \phi(t) \le c < \liminf_{t\to\beta^+} \eta(t)$*;*

(b) T *is a* (p,ϕ,c,η)*-**MCQC**.*

Remark 3.5. (a) Theorem 3.4 is a real generalization of Theorem 3.2.

(b) In fact, Theorem 3.4 can lead to some new sufficient conditions for multi-comparatively quasi-contractions.

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