Refinements of the triangle inequality in geodesic spaces

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1. Introduction

In this report, we investigate how refinements of the triangle inequality, which are known to hold in normed spaces, can be extended to the setting of geodesic spaces, with a particular focus on complete Busemann spaces. This study is based on original research and is intended for future submission as an academic paper.

2. Preliminaries

Let (X, d) be a metric space. A *path* in X is a continuous map $\gamma : [\alpha, \beta] \subset \mathbb{R} \to X$. Given a pair of points $x, y \in X$, we say that a path $\gamma : [\alpha, \beta] \to X$ *joins* x *and* y if $\gamma(\alpha) = x$ and $\gamma(\beta) = y$. A *geodesic path* in X is an isometry $\gamma : [\alpha, \beta] \to X$ such that $d(\gamma(s), \gamma(t)) = |s - t|$ for every $s, t \in [\alpha, \beta]$. A *geodesic segment* $\gamma([\alpha, \beta]) \subset X$ *from* x *to* y is the image of a geodesic path $\gamma : [\alpha, \beta] \to X$ joining x and y. Note that a geodesic segment from x to y is not necessarily unique in general. If no confusion arises, then $[x, y]$ denotes a unique geodesic segment from x to y. A *uniquely geodesic space* is a metric space X if every two points in X can be joined by a unique geodesic path.

In a uniquely geodesic space (X, d), every point on a geodesic segment is naturally parametrized by $[0, 1] \subset \mathbb{R}$. For two distinct points $x, y \in X$, a point $z \in X$ belongs to $[x, y]$ if and only if there exists $t \in [0, 1]$ such that $d(x, z) = td(x, y)$ and $d(z, y) = (1 - t)d(x, y)$. For such a point, we use the notation $z = (1 - t)x \oplus ty$, and say that z is a *convex combination* of x and y.

Let (X, d) be a metric space and $x, y \in X$ two distinct points. A *metric midpoint* of x and y is a point $m \in X$ if $d(x, y) = 2d(x, m) = 2d(m, y)$. A complete metric space X is geodesic space if and only if every pair of points in X has a metric midpoint [1, pp. 2–3, Prop. 1.1.3]. From this, it follows that in a complete uniquely geodesic space X, a convex combination $(1 - t)x \oplus ty \in X$ exists for every two distinct points $x, y \in X$ and $t \in [0, 1]$.

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A *Busemann space* is a geodesic space (X, d) such that for every two geodesic paths $\gamma_1 : [\alpha_1, \beta_1] \subset \mathbb{R} \to X$ and $\gamma_2 : [\alpha_2, \beta_2] \subset \mathbb{R} \to X$, the map $D_{\gamma_1,\gamma_2} : [\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \to \mathbb{R}$ defined by

$$D_{\gamma_1,\gamma_2}(t_1, t_2) = d\big(\gamma_1(t_1), \gamma_2(t_2)\big)$$

is convex [3, pp. 576–577][8, pp. 203–204]. Every Busemann space is a uniquely geodesic space [8, p. 210, Prop. 8.1.4]. Basic examples of Busemann spaces are the Euclidean space $\mathbb{E}^n$ [8, p. 211, Ex. 8.1.7], normed strictly convex vector spaces [8, p. 210, Prop. 8.1.6], hyperbolic spaces [1, p. 10, Ex. 1.2.11], $\mathbb{R}$-trees [9], and Riemannian manifolds of global nonpositive sectional curvature [4]. A large subclass of Busemann spaces consists of non-positively curved spaces in the sense of Alexandrov, also known as CAT(0) spaces [6].

Let (X, d) be a metric space. A *geodesic line* in X is a distance-preserving map $\gamma : \mathbb{R} \to X$. A *local geodesic* is a map $\gamma : [\alpha, \beta] \subset \mathbb{R} \to X$ with the property that for every $t \in [\alpha, \beta]$ there exists $\epsilon > 0$ such that $d\big(\gamma(s_1), \gamma(s_2)\big) = |s_1 - s_2|$ for all $s_1, s_2 \in [\alpha, \beta]$ with $|t - s_1| + |t - s_2| \leq \epsilon$. A geodesic space X is said to have *the geodesic extension property* if for every local geodesic $\gamma : [\alpha, \beta] \to X$ with $\alpha \neq \beta$, there exist $\epsilon > 0$ and a local geodesic $\gamma' : [\alpha, \beta + \epsilon] \to X$ such that $\gamma'|_{[\alpha,\beta]} = \gamma$ [4, p. 208, Def. 5.7]. In a Busemann space, every local geodesic is a geodesic path [8, p. 212, Cor. 8.2.3]. From this, it follows that if X is a Busemann space, then X has the geodesic extension property if and only if every non-constant geodesic path can be extended to a geodesic line.

Given any two distinct points x and y in a Busemann space having the geodesic extension property, there exists a unique geodesic line whose image contains $[x, y]$. For $r \geq 0$, $(1 + r)x \ominus ry$ denotes a unique point z on this geodesic line satisfying $d(z, x) = rd(x, y)$ and $d(z, y) = (1 + r)d(x, y)$.

3. A study of refinements of the triangle inequality in geodesic spaces

In this section, we consider refinements of the triangle inequality in complete Busemann spaces. In normed spaces, the following strengthened triangle inequality and its reverse hold.

Theorem 3.1. [7, p. 257, Thm. 1] *For nonzero vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in a normed space $(E, \|\cdot\|)$ it is true that*

$$\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\| - \left(2 - \left\|\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} + \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|}\right\|\right) \min(\|\boldsymbol{x}\|, \|\boldsymbol{y}\|) \tag{3.1}$$

and

$$\|\boldsymbol{x} + \boldsymbol{y}\| \geq \|\boldsymbol{x}\| + \|\boldsymbol{y}\| - \left(2 - \left\|\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} + \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|}\right\|\right) \max(\|\boldsymbol{x}\|, \|\boldsymbol{y}\|). \tag{3.2}$$

Hereafter, we may assume without loss of generality that $\|\boldsymbol{x}\| \leq \|\boldsymbol{y}\|$. (3.1) and (3.2) can be rewritten in the following form.

$$\|\boldsymbol{x}+\boldsymbol{y}\| \leq \|\boldsymbol{y}\| - \|\boldsymbol{x}\| + \left\| \|\boldsymbol{x}\| \left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} + \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right) \right\|, \tag{3.3}$$

$$\|\boldsymbol{x}+\boldsymbol{y}\| \geq \|\boldsymbol{x}\| - \|\boldsymbol{y}\| + \left\| \|\boldsymbol{y}\| \left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} + \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right) \right\|. \tag{3.4}$$

We consider how these inequalities are represented in complete Busemann spaces. Let (X,d) be a complete Busemann space and $x,y,z \in X$ three distinct points with $d(z,x) \leq d(z,y)$. Put $r := d(z,x)/d(z,y)$. By the triangle inequality, we have

$$\begin{aligned} d(x,y) &\leq d(x, ry \oplus (1-r)z) + d(ry \oplus (1-r)z, y) \\ &= d(x, ry \oplus (1-r)z) + (1-r)d(z,y) \\ &= d(z,y) - d(z,x) + d(x, ry \oplus (1-r)z). \end{aligned} \tag{3.5}$$

This holds whether the three points x,y,z form a triangle or lie on the same geodesic segment. Moreover, if X is a normed space and we associate z with the initial point of $\boldsymbol{x}$ and the terminal point of $\boldsymbol{y}$, as well as x with the terminal point of $\boldsymbol{x}$ and y with the initial point of $\boldsymbol{y}$, then (3.5) corresponds to (3.3). In fact, $d(x, ry \oplus (1-r)z)$ represents the length of the vector from the terminal point of $(1 - \|\boldsymbol{x}\|/\|\boldsymbol{y}\|)\boldsymbol{y}$ to the terminal point of $\boldsymbol{x}$ in the normed space (see Figure 1, with $z_r := ry \oplus (1-r)z$).

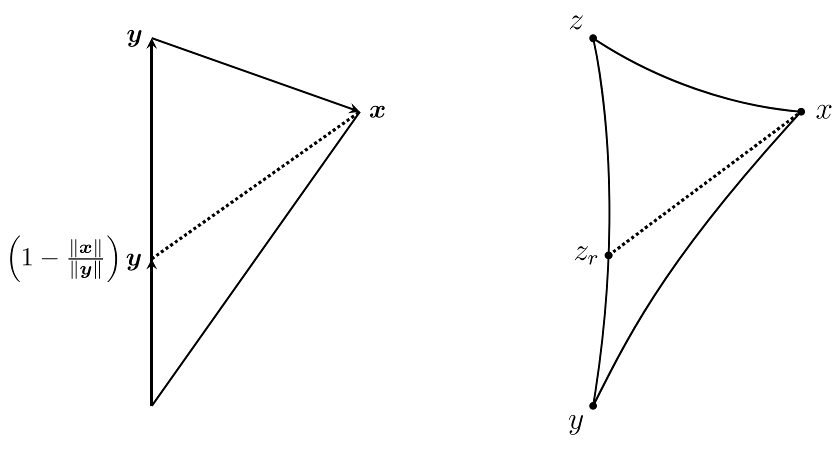

FIGURE 1. A comparison of (3.3) and (3.5)

Returning to the complete Busemann space X, let $s := d(z,x)$, $s' := d(z,y)$, and $\gamma : [0,s] \to X$ be a geodesic path joining z and x. If X has the geodesic extension property, there exists a geodesic path $\gamma' : [0,s'] \to X$

such that $\gamma'|_{[0,s]} = \gamma$. This gives a point $w \in X$ satisfying $\gamma'(s') = w$ and $d(z,w) = d(z,y)$. Moreover, we can write

$$w = \left(1 + \frac{s'-s}{s}\right) x \ominus \frac{s'-s}{s} z = \frac{1}{r} x \ominus \left(\frac{1}{r} - 1\right) z.$$

By the triangle inequality, we have

$$\begin{aligned} d(x,y) &\geq d(y,w) + d(w,x) \\ &= d(y,w) - \{d(z,w) - d(z,x)\} \\ &= d(z,x) - d(z,y) + d(y,w). \end{aligned} \tag{3.6}$$

This holds whether the three points x, y, z form a triangle or lie on the same geodesic segment. Moreover, if X is a normed space and we associate z with the initial point of $\boldsymbol{x}$ and the terminal point of $\boldsymbol{y}$, as well as x with the terminal point of $\boldsymbol{x}$ and y with the initial point of $\boldsymbol{y}$, then (3.6) corresponds to (3.4). In fact, $d(y,w)$ represents the length of the vector from the initial point of $\boldsymbol{y}$ to the terminal point of $(1/r)\boldsymbol{x}$ in the normed space (see Figure 2).

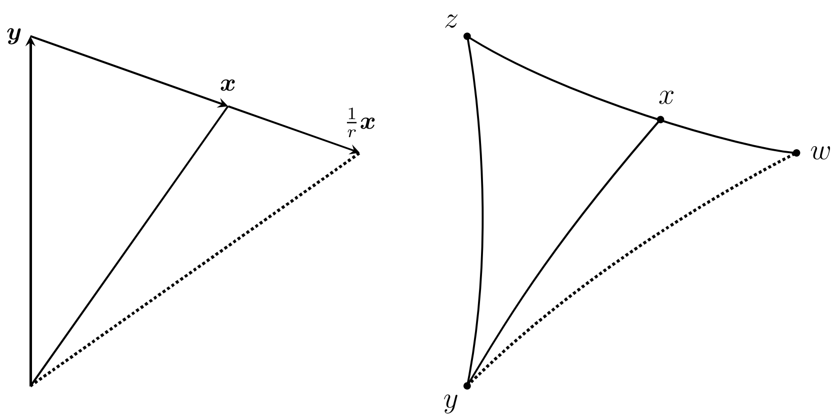

FIGURE 2. A comparison of (3.4) and (3.6)

4. In pursuit of inequalities for lengths of geodesic segments

The above results suggest the existence of further inequalities for lengths of geodesic segments in complete Busemann spaces. To explore this direction further, we review some known inequalities in normed spaces. The following result is a combined inequality of (3.1) and (3.2).

Proposition 4.1. [7, p. 257, Remark 2] *For nonzero vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in a normed space $(E, \|\cdot\|)$, let $\alpha(\boldsymbol{x}, \boldsymbol{y})$ be the angular distance between $\boldsymbol{x}$ and $\boldsymbol{y}$, defined by*

$$\alpha(\boldsymbol{x}, \boldsymbol{y}) := \left\| \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} - \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\|$$

(*see* [5, p. 403]). *Then*

$$(2-\alpha(\boldsymbol{x},\mp\boldsymbol{y}))\min(\|\boldsymbol{x}\|,\|\boldsymbol{y}\|) \le \|\boldsymbol{x}\|+\|\boldsymbol{y}\|-\|\boldsymbol{x}\pm\boldsymbol{y}\| \\ \le (2-\alpha(\boldsymbol{x},\mp\boldsymbol{y}))\max(\|\boldsymbol{x}\|,\|\boldsymbol{y}\|).$$

Moreover, the following inequalities hold in normed spaces.

Theorem 4.2. [7, p. 258, Thm. 2] *For* $p \in [0.\infty)$ *and nonzero vectors* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *in a normed space* $(E, \|\cdot\|)$*, let*

$$\alpha_p(\boldsymbol{x},\boldsymbol{y}) := \left\| \|\boldsymbol{x}\|^{p-1}\boldsymbol{x} - \|\boldsymbol{y}\|^{p-1}\boldsymbol{y} \right\|.$$

(i) If $0 \le p \le 1$*, then*

$$\alpha_p(\boldsymbol{x},\boldsymbol{y}) \le (2-p)\frac{\|\boldsymbol{x}-\boldsymbol{y}\|}{\max(\|\boldsymbol{x}\|,\|\boldsymbol{y}\|)^{1-p}}.$$

(ii) If $p \ge 1$*, then*

$$\alpha_p(\boldsymbol{x},\boldsymbol{y}) \le p\max(\|\boldsymbol{x}\|,\|\boldsymbol{y}\|)^{p-1}\|\boldsymbol{x}-\boldsymbol{y}\|.$$

Whether the above inequalities can be generalized to a complete Busemann space and what forms such generalized inequalities would take remain subjects for future research.

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