The Solodov–Svaiter type proximal point algorithm for a convex function on a Hadamard space

Shuta Sudo
Department of Information Science, Graduate School of Science
Toho University

Abstract

In this paper, we prove a minimiser approximation theorem with the projection type proximal point algorithm. To generate an iterative sequence, we use a notion of tangent spaces and their metric.

1 Introduction

In this work, we consider a minimisation problem as follows: For a given function f on a nonempty set S, find a point $x \in S$ such that $f(x) = \inf_{y \in S} f(y)$. Particularly, we deal with a convex function on a metric space having some convex structure.

Recently, the fixed point theory on geodesic spaces has rapidly developed. We have many fixed point theorems for nonlinear mappings, and many fixed point approximation methods with iterations. For instance, see [5, 6, 9]. In this work, we consider a geodesic space having nonpositive bounded curvature, which is called a CAT(0) space. A nonempty convex subset of a Hilbert space is an example of CAT(0) spaces. In particular, a complete CAT(0) space is called a Hadamard space. This space has many useful properties that Hilbert spaces have. On the other hand, for a given convex function on a Hadamard space, we can define a resolvent operator, and hence we can apply fixed point approximation techniques to find a minimiser of a given function.

In this paper, we handle the following type approximation method:

Theorem 1.1 (Solodov–Svaiter [11]). *Let H be a Hilbert space and A a maximal monotone operator on H which has a zero point. Let $\{r_n\}$ be a positive real sequence such that $\inf_{k \in \mathbb{N}} r_k > 0$. Let P_K be the metric projection onto a nonempty closed convex subset K of H. For a given point $u = x_1 \in H$, generate a sequence $\{x_n\}$ of H as follows:*

$$\begin{aligned} y_n &= (I + r_n A)^{-1} x_n; \\ H_n &= \{v \in H \mid \langle v - y_n, x_n - y_n \rangle \leq 0\}; \\ W_n &= \{w \in H \mid \langle w - x_n, u - x_n \rangle \leq 0\}; \\ x_{n+1} &= P_{H_n \cap W_n} u \end{aligned}$$

for $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to the closest zero point of A to u.

2020 *Mathematics subject classification.* 52A41, 90C25
Key words and phrases. Hadamard space, convex minimisation, proximal point algorithm, projection method.

Nevertheless, in general, Hadamard spaces cannot equipped inner products. In the Solodov–Svaiter type iteration, we require an inner product to generate two kinds of closed convex subsets. To make that possible, we introduce a notion of tangent spaces and their metic on a Hadamard space.

2 Preliminaries

Let (H, d) be a metric space. For $x, y \in H$ and $l = d(x, y)$, we call an isometric mapping γ_{xy} from $[0, l]$ into H a geodesic from x to y if $\gamma_{xy}(0) = x$ and $\gamma_{xy}(l) = y$. Additionally, H is said to be uniquely geodesic if for all $x, y \in H$, there exists a unique geodesic. In a uniquely geodesic space H, for $x, y \in H$ and $t \in [0, 1]$, we define convex combination of x and y with a ratio t by

$$tx \oplus (1-t)y = \gamma_{xy}((1-t)d(x, y)).$$

We define a CAT(0) space. The classical definition of CAT(0) spaces uses the notion of comparison triangles on the Euclidean space. However, we know an equivalent condition to the definition as follows: A uniquely geodesic space H is a CAT(0) space if and only if

$$d(tx \oplus (1-t)y, z)^2 \leq td(x, z)^2 + (1-t)d(y, z)^2 - t(1-t)d(x, y)^2$$

for $x, y, z \in H$ and $t \in [0, 1]$; see [1, Theorem 1.3.3]. In particular, we call H a Hadamard space if it is a complete CAT(0) space.

Let H be a CAT(0) space. We say that a subset C of H is convex if

$$tx \oplus (1-t)y \in C$$

for $x, y \in C$ and $t \in [0, 1]$. Let C be a nonempty closed convex subset of a Hadamard space H. For $x \in H$, we can find a unique point $P_C x \in C$ such that

$$d(x, P_C x) = \inf_{y \in C} d(x, y).$$

We call a mapping P_C the metric projection onto C.

Let $\{x_n\}$ be a bounded sequence of a metric space H. We call $w \in H$ an asymptotic centre of $\{x_n\}$ if

$$\limsup_{n\to\infty} d(x_n, w) = \inf_{y \in H} \limsup_{n\to\infty} d(x_n, y).$$

We say that $\{x_n\}$ Δ-converges to a Δ-limit $x \in H$ if x is a unique asymptotic centre for any subsequence of $\{x_n\}$. It is well known that a bounded sequence of a Hadamard space has a unique asymptotic centre, and that such a sequence has a Δ-convergent subsequence. Suppose that H is a Hadamard space. If a bounded sequence $\{x_n\}$ of H Δ-converges to $x \in H$, then

$$d(x, y) \leq \liminf_{n\to\infty} d(x_n, y)$$

for $y \in H$. If a sequence $\{x_n\}$ of H Δ-converges to $x \in H$ and $\{d(x_n, y)\}$ converges to $d(x, y)$ for some $y \in H$, then $\{x_n\}$ converges to x. For more details about Δ-convergence, refer to [1, 4] for instance.

In what follows, we define tangent spaces on a Hadamard space. Let H be a Hadamard space. For $p, x, y \in H$, we define the Alexandrov angle A_p at p as follows:

$$A_p(x, y) = \lim_{t \to 0+} \arccos \left(1 - \frac{d(\gamma_{px}(t), \gamma_{py}(t))^2}{2t^2} \right) \in [0, \pi]$$

if $p \neq x$ and $p \neq y$; $A_p(x, p) = A_p(p, x) = \pi/2$ if $p \neq x$; $A_p(p, p) = 0$. For more details, refer to [2, Proposition 1.14 in Chapter I.1 and Proposition 3.1 in Chapter II.3] for instance. We define an equivalence relation $\sim_p$ on H by $x \sim_p y$ if

$$A_p(x, y) = 0.$$

For the simplicity, we denote an equivalence class $[x]_{\sim_p}$ of $x \in H$ by $[x]_p$. Let

$$D_p H = H/{\sim_p} = \{[x]_p \mid x \in H\}.$$

Then, $(D_p H, A_p)$ is a metric space, where the distance A_p is defined by

$$A_p([x]_p, [y]_p) = A_p(x, y)$$

for $[x]_p, [y]_p \in D_p H$. We next define a function ζ on $D_p H$ by

$$\zeta([x]_p) = \begin{cases} 0 & ([x]_p = [p]_p); \\ 1 & ([x]_p \neq [p]_p) \end{cases}$$

for $[x]_p \in D_p H$. We define an equivalence relation $\simeq_p$ on a Cartesian product

$$[0, \infty[\times D_p H$$

by $(r_1, [x]_p) \simeq_p (r_2, [y]_p)$ if one of the following conditions is satisfied:

- $r_1 \zeta([x]_p) = r_2 \zeta([y]_p) = 0$;
- $r_1 \zeta([x]_p) = r_2 \zeta([y]_p) > 0$ and $[x]_p = [y]_p$.

Let

$$T_p H = ([0, \infty[\times D_p H)/{\simeq_p}.$$

For the simplicity, we denote an element $[(r, [x]_p)]_{\simeq_p}$ of $T_p H$ by $r[x]_p$. In particular, we denote $0[p]_p$ by 0_p. We define a distance function d_p on $T_p H$ by

$$d_p(r[x]_p, s[y]_p) = \sqrt{r^2 \zeta([x]_p) + s^2 \zeta([y]_p) - 2rs\zeta([x]_p)\zeta([y]_p) \cos A_p(x, y)}$$

for $r[x]_p, s[y]_p \in T_p H$. We call this metric space $(T_p H, d_p)$ the tangent space of H at p.

Let H be a Hadamard space and $p \in H$. We define a logarithmic mapping $\log_p$ from H to $T_p H$ by

$$\log_p x = d(p, x)[x]_p \in T_p H$$

for $x \in H$. We notice that $\log_p p = 0_p$. We define a function g_p by

$$g_p(u_p, v_p) = \frac{d_p(u_p, 0_p)^2 + d_p(v_p, 0_p)^2 - d_p(u_p, v_p)^2}{2}$$

for $u_p, v_p \in T_p H$. We have known the following propositions:

Theorem 2.1 (Chaipunya–Kohsaka–Kumam [3], Kimura–Sudo [7]). *For a Hadamard space H and for $p, x, y \in H$,*

$$2g_p(\log_p x, \log_p y) \geq d(p,x)^2 + d(p,y)^2 - d(x,y)^2.$$

Theorem 2.2 (Kimura–Sudo [7]). *For a Hadamard space H and for $p, x, y \in H$,*

$$\lim_{t \to 0+} \frac{d(p,y)^2 - d(tx \oplus (1-t)p, y)^2}{t} = 2g_p(\log_p x, \log_p y).$$

Theorem 2.3 (Sasaki–Sudo [10]). *For a Hadamard space H and $x, y \in H$, a subset*

$$\{z \in H \mid g_x(\log_x z, \log_x y) \leq 0\}$$

is closed.

The identity of the second theorem is called the first variation formula. For more details about tangent spaces on a geodesic space, refer to [2, 3, 7].

3 Solodov–Svaiter type proximal point algorithm

In this section, we show a minimiser approximation theorem with the Solodov–Svaiter type proximal point algorithm.

For a function f from a Hadamard space H to $]-\infty, \infty]$, we suppose the following conditions in this paper: We say that f is lower semicontinuous if

$$f(x) \leq \liminf_{n \to \infty} f(x_n)$$

for a sequence $\{x_n\}$ of H converging to $x \in H$. We say that f is proper if there exists $x \in H$ such that $f(x) \in \mathbb{R}$. We further say that f is convex if

$$f(tx \oplus (1-t)y) \leq tf(x) + (1-t)f(y)$$

for $x, y \in H$ and $t \in]0,1[$. We call $x \in H$ a minimiser of f if

$$f(x) = \inf_{y \in H} f(y).$$

We denote the set of all minimisers of f by $\operatorname{Min} f$. If f is lower semicontinuous, proper and convex, then $\operatorname{Min} f$ is closed and convex even if it is empty, and then

$$f(x) \leq \liminf_{n \to \infty} f(x_n)$$

for a bounded sequence $\{x_n\}$ of H Δ-converging to $x \in H$. For more details, see [1].

In general, we do not know if a lower semicontinuous proper convex function f has a minimiser, and then if it is unique. Nevertheless, for fixed $x \in H$ and $r > 0$, a function $f_{r;x}$ on H defined by

$$f_{r;x}(y) = f(y) + \frac{1}{2r} d(x,y)^2$$

for $y \in H$ definitely has a unique minimiser. We denote such a point by $R_{rf}x$, and we call a mapping R_{rf} on H the resolvent operator of f with r. For more details, see [1, 8].

To prove the desired convergence theorem, we first show the following proposition:

Theorem 3.1. *Let f be a lower semicontinuous proper convex function from a Hadamard space H to $]-\infty, \infty]$. Then, for $x \in H$ and $r > 0$,*

$$rf(R_{rf}x) \leq \inf_{w \in H} \Big(rf(w) - g_{R_{rf}x}(\log_{R_{rf}x} w, \log_{R_{rf}x} x) \Big).$$

Proof. Fix $w, x \in H$ and $t \in]0,1[$. Note that $f(R_{rf}x) \in \mathbb{R}$. Then,

$$\begin{aligned} f(R_{rf}x) + \frac{1}{2r} d(x, R_{rf}x)^2 &\leq f(tw \oplus (1-t)R_{rf}x) + \frac{1}{2r} d(x, tw \oplus (1-t)R_{rf}x)^2 \\ &\leq tf(w) + (1-t)f(R_{rf}x) + \frac{1}{2r} d(x, tw \oplus (1-t)R_{rf}x)^2, \end{aligned}$$

and hence

$$rtf(R_{rf}x) \leq rtf(w) - \frac{d(x, R_{rf}x)^2 - d(x, tw \oplus (1-t)R_{rf}x)^2}{2}.$$

Dividing both sides by t, and letting $t \to 0+$, from the first variation formula, we have

$$rf(R_{rf}x) \leq rf(w) - g_{R_{rf}x}(\log_{R_{rf}x} w, \log_{R_{rf}x} x).$$

Since $w \in H$ is arbitrary, we obtain the desired result. □

We finally prove the following approximation theorem:

Theorem 3.2. *Let f be a lower semicontinuous proper convex function from a Hadamard space H to $]-\infty, \infty]$ which has a minimiser. Suppose that for $x, y \in H$, a subset*

$$\{z \in H \mid g_x(\log_x z, \log_x y) \leq 0\}$$

is convex. Let $\{r_n\}$ be a positive real sequence such that $\inf_{k \in \mathbb{N}} r_k > 0$. Let P_K be the metric projection onto a nonempty closed convex subset K of H. For a given anchor point $u = x_1 \in H$, generate a sequence $\{x_n\}$ of H as follows:

$$\begin{aligned} y_n &= R_{r_n f} x_n; \\ H_n &= \{v \in H \mid g_{y_n}(\log_{y_n} v, \log_{y_n} x_n) \leq 0\}; \\ W_n &= \{w \in H \mid g_{x_n}(\log_{x_n} w, \log_{x_n} u) \leq 0\}; \\ x_{n+1} &= P_{H_n \cap W_n} u \end{aligned}$$

for $n \in \mathbb{N}$. Then, the generated sequence $\{x_n\}$ converges to $P_{\operatorname{Min} f} u$.

Proof. To show that $\{x_n\}$ is well defined, we confirm $H_n \cap W_n$ is closed and convex, and

$$\operatorname{Min} f \subset H_n \cap W_n$$

for $n \in \mathbb{N}$. Henceforth, for the simplicity, let $E_n = H_n \cap W_n$ for $n \in \mathbb{N}$. If x_n is defined, then from Theorem 3.1, for $p \in \operatorname{Min} f$, we have

$$r_n f(y_n) \leq \inf_{w \in H} \big(r_n f(w) - g_{y_n}(\log_{y_n} w, \log_{y_n} x_n) \big) \leq r_n f(p) - g_{y_n}(\log_{y_n} p, \log_{y_n} x_n).$$

It implies that

$$g_{y_n}(\log_{y_n} p, \log_{y_n} x_n) \leq r_n(f(p) - f(y_n)) \leq 0,$$

and hence $p \in H_n$. Thus, $\operatorname{Min} f \subset H_n$ if x_n is defined. Now, we prove that

$$\operatorname{Min} f \subset E_n$$

and E_n is closed and convex for $n \in \mathbb{N}$ by induction. Since $u = x_1$, we have

$$W_1 = \{w \in H \mid g_{x_1}(\log_{x_1} w, 0_{x_1}) \leq 0\} = H,$$

and thus $\operatorname{Min} f \subset E_1 = H_1$. Further, from the assumption, H_1 is closed and convex, and therefore so is E_1. For fixed $k \in \mathbb{N}$, we assume that $\operatorname{Min} f \subset E_k$ and that E_k is a closed convex set. Recall that x_{k+1} is defined in this case, and therefore $\operatorname{Min} f \subset H_{k+1}$. Let $q \in \operatorname{Min} f$ and $t \in]0,1[$. Then, $q \in E_k$ since $\operatorname{Min} f \subset E_k$. From the convexity of E_k, we have

$$d(u, x_{k+1}) = d(u, P_{E_k} u) \leq d(u, tq \oplus (1-t)P_{E_k} u) = d(u, tq \oplus (1-t)x_{k+1}),$$

and hence

$$\frac{d(u, x_{k+1})^2 - d(u, tq \oplus (1-t)x_{k+1})^2}{2t} \leq 0.$$

Letting $t \to 0+$, from the first variation formula, we get

$$0 \geq \lim_{t \to 0+} \frac{d(u, x_{k+1})^2 - d(u, tq \oplus (1-t)x_{k+1})^2}{2t} = g_{x_{k+1}}(\log_{x_{k+1}} q, \log_{x_{k+1}} u).$$

It means that $q \in W_{k+1}$, and thus $\operatorname{Min} f \subset W_{k+1}$. Therefore,

$$\operatorname{Min} f \subset H_{k+1} \cap W_{k+1} = E_{k+1}.$$

On the other hand, from the assumption, since both of H_{k+1} and W_{k+1} are closed and convex, E_{k+1} is also closed and convex. Therefore, the sequence $\{x_n\}$ is well defined since E_n is nonempty, closed and convex for all $n \in \mathbb{N}$. We also obtain $\operatorname{Min} f \subset E_n$ for $n \in \mathbb{N}$.

We next show that $\{x_n\}$ converges to $P_{\operatorname{Min} f} u$. Note that $P_{\operatorname{Min} f} u \in E_n$ for $n \in \mathbb{N}$. Since

$$d(u, P_{E_n} u) \leq d(u, P_{\operatorname{Min} f} u)$$

for $n \in \mathbb{N}$, we have

$$\sup_{n \in \mathbb{N}} d(u, x_n) = \sup_{n \in \mathbb{N}} d(u, x_{n+1}) = \sup_{n \in \mathbb{N}} d(u, P_{E_n} u) \leq d(u, P_{\operatorname{Min} f} u) < \infty,$$

and thus $\{x_n\}$ is bounded. For $n \in \mathbb{N}$, since $x_{n+1} \in W_n$, we obtain

$$\begin{aligned} 0 &\geq 2g_{x_n}(\log_{x_n} x_{n+1}, \log_{x_n} u) \geq d(x_n, x_{n+1})^2 + d(x_n, u)^2 - d(x_{n+1}, u)^2 \\ &\geq d(x_n, u)^2 - d(x_{n+1}, u)^2, \end{aligned}$$

which implies that $d(x_n, u) \leq d(x_{n+1}, u)$, and that

$$d(x_n, x_{n+1})^2 + d(x_n, u)^2 - d(x_{n+1}, u)^2 \leq 0.$$

Therefore, $\{d(x_n, u)\}$ is a convergent real sequence. Since

$$d(x_n, x_{n+1})^2 \leq d(x_{n+1}, u)^2 - d(x_n, u)^2$$

for $n \in \mathbb{N}$, we obtain

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Additionally, since $x_{n+1} \in H_n$ for $n \in \mathbb{N}$, we obtain

$$\begin{aligned} 0 &\geq 2g_{y_n}(\log_{y_n} x_{n+1}, \log_{y_n} x_n) \geq d(y_n, x_{n+1})^2 + d(y_n, x_n)^2 - d(x_{n+1}, x_n)^2 \\ &\geq d(y_n, x_n)^2 - d(x_{n+1}, x_n)^2, \end{aligned}$$

and thus $d(y_n, x_n) \leq d(x_{n+1}, x_n)$. Consequently, we get

$$\lim_{n\to\infty} d(y_n, x_n) = 0.$$

Take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ arbitrarily. Since $\{x_{n_i}\}$ is bounded, we can take its Δ-convergent subsequence $\{x_{n_{i_j}}\}$. Let $w \in H$ be its Δ-limit. For the sake of simplicity, henceforth, let us write $x_{n_{i_j}}$ as x_j and $y_{n_{i_j}}$ as y_j for $j \in \mathbb{N}$. Note that

$$\lim_{j\to\infty} d(y_j, x_j) = 0.$$

We notice that $\{y_j\}$ also Δ-converges to w. Now, we show that w is a minimiser of f. Fix $j \in \mathbb{N}$ arbitrarily. From Theorem 3.1,

$$r_{n_{i_j}} f(y_j) \leq r_{n_{i_j}} f(P_{\mathrm{Min}\, f} u) - g_{y_j}(\log_{y_j} P_{\mathrm{Min}\, f} u, \log_{y_j} x_j).$$

Therefore, since $P_{\mathrm{Min}\, f} u$ is a minimiser of f, we have

$$\begin{aligned} \inf_{k\in\mathbb{N}} r_k \left| f(y_j) - \inf_{y\in H} f(y) \right| &\leq r_{n_{i_j}}(f(y_j) - f(P_{\mathrm{Min}\, f} u)) \leq -g_{y_j}(\log_{y_j} P_{\mathrm{Min}\, f} u, \log_{y_j} x_j) \\ &\leq \frac{d(P_{\mathrm{Min}\, f} u, x_j)^2 - d(y_j, P_{\mathrm{Min}\, f} u)^2 - d(y_j, x_j)^2}{2}. \end{aligned}$$

Since $\inf_{k\in\mathbb{N}} r_k > 0$ and $\{d(y_j, x_j)\}$ converges to 0, we have

$$\lim_{j\to\infty} \left| f(y_j) - \inf_{y\in H} f(y) \right| \leq \lim_{j\to\infty} \frac{d(P_{\mathrm{Min}\, f} u, x_j)^2 - d(y_j, P_{\mathrm{Min}\, f} u)^2 - d(y_j, x_j)^2}{2 \inf_{k\in\mathbb{N}} r_k} = 0,$$

which means that $\{f(y_j)\}$ converges to $\inf_{y\in H} f(y)$. Hence, since $\{y_j\}$ Δ-converges to w, we have

$$f(w) \leq \liminf_{j\to\infty} f(y_j) = \inf_{y\in H} f(y),$$

which implies that w is a minimiser of f. Since

$$d(u, P_{\mathrm{Min}\, f} u) \leq d(u, w) \leq \liminf_{j\to\infty} d(u, x_j) \leq \limsup_{j\to\infty} d(u, x_j) \leq d(u, P_{\mathrm{Min}\, f} u),$$

we obtain $w = P_{\mathrm{Min}\, f} u$ and

$$\lim_{j\to\infty} d(u, x_j) = d(u, P_{\mathrm{Min}\, f} u).$$

Since $\{x_j\}$ is Δ-convergent to $P_{\mathrm{Min}\, f} u$ and $\{d(u, x_j)\}$ converges to $d(u, P_{\mathrm{Min}\, f} u)$, we have $\{x_j\}$ converges to $P_{\mathrm{Min}\, f} u$. Consequently, any subsequence of $\{x_n\}$ has a subsequence converging to $P_{\mathrm{Min}\, f} u$, and therefore $\{x_n\}$ converges to $P_{\mathrm{Min}\, f} u$. $\square$

References

[1] M. Bačák, *Convex analysis and optimization in Hadamard spaces*, De Gruyter, Berlin, 2014.

[2] M. R. Bridson and Haefliger, *Metric spaces of non-positive curvature*, Springer-Verlag, Berlin, 1999.

[3] P. Chaipunya, F. Kohsaka, and P. Kumam, *Monotone vector fields and generation of nonexpansive semigroups in complete* CAT(0) *spaces*, Numer. Funct. Anal. Optim. **42** (2021), 989–1018.

[4] R. Espínola and A. Fernández-León, CAT(k)*-spaces, weak convergence and fixed points*, J. Math. Anal. Appl. **353** (2009), 410–427.

[5] J. S. He, D. H. Fang, G. López, and C. Li, *Mann's algorithm for nonexpansive mappings in* CAT(κ) *spaces*, Nonlinear Anal. **75** (2012), 445–452.

[6] Y. Kimura, *A shrinking projection method for nonexpansive mappings with non-summable errors in a Hadamard space*, Ann. Oper. Res. **243** (2016), 1–6.

[7] Y. Kimura and S. Sudo, *Tangent spaces and a metric on geodesic spaces*, RIMS Kôkyûroku **2240** (2023), 7–19.

[8] U. F. Mayer, *Gradient flows on nonpositively curved metric spaces and harmonic map*, Comm. Anal. Geom. **6** (1998), 199–253.

[9] S. Saejung, *Halpern's iteration in* CAT(0) *spaces*, Fixed Point Theory Appl. (2010), 13pp.

[10] K. Sasaki and S. Sudo, *The Nakajo–Takahashi type projection method for a nonexpansive mapping on a Hadamard space*, submitted.

[11] M. V. Solodov and B. F. Svaiter, *Forcing strong convergence of proximal point iterations in a Hilbert space*, Math. Program. **87** (2000), 189–202.

Shuta Sudo
Department of Information Science
Toho University
Japan
Email address: 7523001s@st.toho-u.ac.jp