Set-Valued Fan-Takahashi Inequalities Via Scalarization

(スカラー化関数による集合値関数のミニマックス定理の一般化とその応用)

Ryota Iwamoto[a] and Tamaki Tanaka[b]

[a]Graduate school of Science and Technology, Niigata University,
[b]Faculty of Science, Niigata University

1 Introduction

Real-valued minimax inequality was found by W. Takahashi and Ky. Fan;

Theorem 1.1. ([4]) Let X be a nonempty compact convex subset of a Hausdorff topological vector space and $f\colon X \times X \to \mathbb{R}$. If f satisfies the following conditions:

1. for each fixed $y \in X$, $f(\cdot, y)$ is lower semicontinuous,
2. for each fixed $x \in X$, $f(x, \cdot)$ is quasi concave,
3. $f(x, x) \leq 0$ for all $x \in X$,

then there exists $\bar{x} \in X$ such that $f(\bar{x}, y) \leq 0$ for all $y \in X$.

In a history of set-valued minimax inequalities, Georgiev and Tanaka [2] extended the minimax inequality to set-valued maps. Kuwano, Tanaka, and Yamada [5] constructed the result of four types of set-valued minimax inequalities with set-relations. Our goal is to generalize the result of four types of set-valued minimax inequalities which is not related to the specific set-relations and scalarization functions.

2 Preliminaries

Let X be a topological space, Y a real topological vector space, and θ_Y be a zero vector in Y. Define that $\mathcal{P}_0(Y)$ is the set of all nonempty subsets of Y. The sets of neighborhoods of $x \in X$ and $y \in Y$ is denoted by $\mathcal{N}_X(x)$ and $\mathcal{N}_Y(y)$, respectively.

For $A \in \mathcal{P}_0(Y)$, the interior, the closure, the boundary, and the complement of A are denoted by $\operatorname{int} A$, $\operatorname{cl} A$, $\operatorname{bd} A$, and A^c, respectively. For given $A, B \in \mathcal{P}_0(Y)$ and $t \in \mathbb{R}$, the algebraic sum $A+B$ and the scalar multiplication tA are denoted as follows: $A+B := \{a+b \mid a \in A, b \in$

$B\}$, $tA := \{ta \mid a \in A\}$. In particular, we denote $A + \{y\}$ by $A + y$ and $(-1)A$ by $-A$ for $A \in \mathcal{P}_0(Y)$ and $y \in Y$. Let us recall that A is said to be C-bounded if for each neighborhood U of θ_Y there exists $t > 0$ such that $A \subset tU + C$.

We define binary relations on $\mathcal{P}_0(Y)$ as follows:

Definition 2.1. For $A, B \in \mathcal{P}_0(Y)$, we define two binary relations on $\mathcal{P}_0(Y)$:

$$A \preccurlyeq_1 B \overset{\text{def}}{\Longleftrightarrow} A \cap B \neq \emptyset \quad \text{and} \quad A \preccurlyeq_2 B \overset{\text{def}}{\Longleftrightarrow} B \subset A.$$

Definition 2.2. For $A, B \in \mathcal{P}_0(Y)$ and a convex cone C, we define;

$$A \preccurlyeq_C^{(3L)} B \overset{\text{def}}{\Longleftrightarrow} B \subset A + C \quad \text{and} \quad A \preccurlyeq_C^{(3U)} B \overset{\text{def}}{\Longleftrightarrow} A \subset B - C.$$

We note that recent studies contain six or eight types of binary relations with a convex cone, that is, set-relations. In this paper, we focus on the above two types of set-relations:

For set-valued maps and scalarization functions, we define the following concepts of continuity and semicontinuity, which are works of P. Dechboon and T. Tanaka [1].

Definition 2.3. ([1]) Let $F\colon X \to \mathcal{P}_0(Y)$, $x_0 \in X$, $\preccurlyeq$ a binary relation on $\mathcal{P}_0(Y)$ and $C \subset Y$ a convex cone. We say that F is $(\preccurlyeq, C)$-continuous at x_0 if

$$\forall W \subset Y, W \text{ open}, W \preccurlyeq F(x_0), \exists V \in \mathcal{N}_X(x_0) \quad \text{s.t.} \quad W + C \preccurlyeq F(x), \forall x \in V.$$

Especially, $(\preccurlyeq_1, C)$-continuity and $(\preccurlyeq_2, C)$-continuity coincide with classical "C-lower continuity" and "C-upper continuity" for set-valued maps, respectively.

Definition 2.4. ([1]) Let $\varphi\colon \mathcal{P}_0(Y) \to \mathbb{R} \cup \{\pm\infty\}$, $A_0 \in \mathcal{P}_0(Y)$, $\preccurlyeq$ a binary relation on $\mathcal{P}_0(Y)$, and C a convex cone in Y with $C \neq Y$. Then, we say that φ is $(\preccurlyeq, C)$-lower semicontinuous at A_0 if

$$\forall r < \varphi(A_0), \exists W \in \mathcal{P}_0(Y), W \text{ open}, \quad \text{s.t.} \quad W \preccurlyeq A_0 \text{ and } r > \varphi(A), \forall A \in U(W + C, \preccurlyeq);$$

where $U(V, \preccurlyeq) := \{A \in \mathcal{P}_0(Y) \mid V \preccurlyeq A\}$.

Theorem 2.5. ([1]) Let $F\colon X \to \mathcal{P}_0(Y)$, $\varphi\colon \mathcal{P}_0(Y) \to \mathbb{R} \cup \{\pm\infty\}$, $x_0 \in X$, $\preccurlyeq$ a binary relation on $\mathcal{P}_0(Y)$, and C a convex cone. If F is $(\preccurlyeq, C)$-continuous at x_0 and φ is $(\preccurlyeq, C)$-lower semicontinuous at $F(x_0)$, then $(\varphi \circ F)$ is lower semicontinuous at x_0.

We define concepts of convexity and concavity for set-valued maps. These notions are utilized to extend minimax inequality for real-valued maps to that for set-valued maps.

Definition 2.6. ([3]) Let X be a nonempty set, Y a real topological vector space, C a convex cone in Y, and $F\colon X \to \mathcal{P}_0(Y)$ a set-valued map where $j = 3U, 3L$.

1. F is called type (j) properly quasi C-concave if for each $x, y \in X$ and $\lambda \in (0,1)$,

$$F(x) \preccurlyeq_C^{(j)} F(\lambda x + (1-\lambda)y) \quad \text{or} \quad F(y) \preccurlyeq_C^{(j)} F(\lambda x + (1-\lambda)y)$$

2. F is called type (j) naturally quasi C-concave if for each $x, y \in X$ and $\lambda \in (0,1)$, there exists $\mu \in [0,1]$ such that

$$\mu F(x) + (1-\mu)F(y) \preccurlyeq_C^{(j)} F(\lambda x + (1-\lambda)y).$$

If F is type (j) properly quasi C-concave, clearly F is type (j) naturally quasi C-concave.

Definition 2.7. ([2]) Let $\mathcal{A} \subset \mathcal{P}_0(Y)$. $\mathcal{A}$ is said to be convex if for each $A_1, A_2 \in \mathcal{A}$ and $\lambda \in (0,1)$,

$$\lambda A_1 + (1-\lambda)A_2 \in \mathcal{A}.$$

Definition 2.8. ([2]) Let $\varphi\colon \mathcal{P}_0(Y) \to \mathbb{R} \cup \{\pm\infty\}$. Then,

1. φ is quasi convex if for any $\alpha \in \mathbb{R}$, $\operatorname{lev}(\varphi, \leq, \alpha) := \{A \in \mathcal{P}_0(Y) \mid \varphi(A) \leq \alpha\}$ is convex.
2. φ is quasi concave if for any $\alpha \in \mathbb{R}$, $\operatorname{lev}(\varphi, \geq, \alpha) := \{A \in \mathcal{P}_0(Y) \mid \varphi(A) \geq \alpha\}$ is convex.

Definition 2.9. ([2]) Let C be a convex cone in Y and $j = 3U, 3L.$. For a given binary relation $\preccurlyeq$, a scalarization function φ is $(\preccurlyeq_C^{(j)})$-monotone if for any $A, B \in \mathcal{P}_0(Y)$ with $A \preccurlyeq_C^{(j)} B$, $\varphi(A) \leq \varphi(B)$.

Proposition 2.10. ([2]) Let φ be $(\preccurlyeq_C^{(j)})$-monotone and quasi convex where $j = 3U, 3L$. If F is type (j) naturally quasi C-convex, then $(\varphi \circ F)$ is quasi convex.

Proposition 2.11. ([2]) Let φ be $(\preccurlyeq_C^{(j)})$-monotone and quasi concave where $j = 3U, 3L$. If F is type (j) naturally quasi C-concave, then $(\varphi \circ F)$ is quasi concave.

3 Main Results

Let $\varphi : \mathcal{P}_0(Y) \to \mathbb{R} \cup \{\pm\infty\}$, $\preccurlyeq$ a binary relation on $\mathcal{P}_0(Y)$, and $C' \subset Y$ a convex cone. To generalize four types of set-valued minimax inequalities [3], we provide a new class of scalarization functions that satisfy;

1. φ is $(\preccurlyeq, C')$-lower semicontinuous,
2. φ is quasi concave,
3. $\varphi(\{\theta_Y\}) = 0$.

In addition, we give necessary conditions between inequalities and set-relations as follows;

(B1) φ is $(\preccurlyeq_{\operatorname{int} C}^{(j)})$-monotone, (B2) $\varphi(A) > 0 \Rightarrow \{\theta_Y\} \preccurlyeq^{(j)}_{\operatorname{int} C} A$ for any $A \in \mathcal{P}_0(Y)$,

where $j = 3U, 3L$. If φ satisfies conditions (i)–(iii), (B1), and (B2), we write the notation as $\varphi \in \Phi(\preccurlyeq^{(j)}_{\operatorname{int} C}, \preccurlyeq, C')$.

Theorem 3.1. Let X be a nonempty compact convex subset of a Hausdorff topological vector space, Y a real topological vector space, $\preccurlyeq$ a binary relation on $\mathcal{P}_0(Y)$, C a convex cone in Y, C' a convex cone in Y, $\varphi\colon \mathcal{P}_0(Y) \to \mathbb{R} \cup \{\pm\infty\}$, and $F\colon X \times X \to \mathcal{P}_0(Y)$. For the scaralization function $\varphi \in \Phi(\preccurlyeq^{(j)}_{\operatorname{int} C}, \preccurlyeq, C')$ satisfying Assumption (A2) where $j = 3U, 3L$, if F satisfies the following conditions:

1. $(\varphi \circ F)(x, y) \in \mathbb{R}$ for all $x, y \in X$,
2. for each fixed $y \in X$, $F(\cdot, y)$ is $(\preccurlyeq, C')$-continuous,
3. for each fixed $x \in X$, $F(x, \cdot)$ is type (j) naturally quasi C-concave,
4. for all $x \in X$, $\{\theta_Y\} \not\preccurlyeq^{(j)}_{\operatorname{int} C} F(x, x)$,

then there exists $\bar{x} \in X$ such that $\{\theta_Y\} \not\preccurlyeq^{(j)}_{\operatorname{int} C} F(\bar{x}, y)$ for all $y \in X$.

In the first part of this section, we provide conditions under which semicontinuity and convexity can be preserved when considering composite functions of a set-valued map and a scalarization function.

Let $\hat{Y}$ be a real normed vector space equipped with $\|y\|$ the norm of $y \in \hat{Y}$ and $\theta_{\hat{Y}}$ the zero vector of $\hat{Y}$. As a scalarization function, we introduce the Hiriart-Urruty oriented distance function.

Definition 3.2. ([5]) For the set $A \in Y$, let generalized oriented distance functions $\mathcal{D}^{(1)}_A\colon \mathcal{P}_0(\hat{Y}) \to \mathbb{R} \cup \{\pm\infty\}$ and $\mathcal{D}^{(2)}_A\colon \mathcal{P}_0(\hat{Y}) \to \mathbb{R} \cup \{\pm\infty\}$ be defined as

$$\mathcal{D}^{(1)}_A(B) := \sup\{\Delta_A(b) \mid b \in B\}, \text{ for all } B \in \mathcal{P}_0(\hat{Y}),$$
$$\mathcal{D}^{(2)}_A(B) := \inf\{-\Delta_A(b) \mid b \in B\} = -\mathcal{D}^{(1)}_A(B), \text{ for all } B \in \mathcal{P}_0(\hat{Y}).$$

As $\mathcal{D}^{(2)}_A(B)$ satisfies the conditions conditions (i)–(iii), (B1), and (B2), the following result is obtained by Theorem 3.1.

Proposition 3.3. Let X be a nonempty compact convex subset of a topological vector space, $\hat{Y}$ a real normed vector space, C a closed convex cone in $\hat{Y}$ with $\operatorname{int} C \neq \emptyset$, and, $F\colon X \times X \to \mathcal{P}_0(\hat{Y})$. If F satisfies the following conditions:

1. F is C-bounded on $X \times X$,
2. for each fixed $y \in X$, $F(\cdot, y)$ is $(\preccurlyeq_2, C)$-continuous (that is, C-upper continuous),
3. for each fixed $x \in X$, $F(x, \cdot)$ is type (j) naturally quasi C-concave,
4. for all $x \in X$, $F(x, x) \preccurlyeq^{(3L)}_C \{\theta_{\hat{Y}}\}$,

then there exists $\bar{x} \in X$ such that $\{\theta_{\hat{Y}}\} \not\leq^{(3L)}_{\operatorname{int} C} F(\bar{x}, y)$ for all $y \in X$.

We remark that the above concequence was found by using another scalarization function. However, Our main result avoids to depend on the specific scalarization function. If we find a scalarization function that satisfies the conditions (i)–(iii), (B1), and (B2), Theorem 3.1 can be applied to obtain minimax inequalities for set-valued maps.

References

[1] P. Dechboon and T. Tanaka, Inheritance properties on cone continuity for set-valued maps via scalarization, Minimax Theory Appl. **9** (2024), no. 2, 201–224.

[2] S. Kobayashi, Y. Saito and T. Tanaka, Convexity for compositions of set-valued map and monotone scalarizing function, Pac. J. Optim. **12** (2016), no. 1, 43–54.

[3] I. Kuwano, T. Tanaka and S. Yamada, Unified scalarization for sets and set-valued Ky Fan minimax inequality, J. Nonlinear Convex Anal. **11** (2010), no. 3, 513–525.

[4] W. Takahashi, Nonlinear variational inequalities and fixed point theorems, J. Math. Soc. Japan **28** (1976), no. 1, 168–181.

[5] Y. D. Xu and S. J. Li, A new nonlinear scalarization function and applications, Optimization **65** (2016), no. 1, 207–231.