Existence and approximation of a common fixed point on Banach spheres
バナッハ球面における共通不動点の存在と近似

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Abstract

In this paper, we prove a common fixed point theorem and an approximation theorem with the shrinking projection method on Banach spheres. In this work, we became able to generate an iterative sequence even if considered mappings do not have fixed points. Further, if they have a common fixed point, the sequence converges to such a point.

1 Introduction

In 2008, the shrinking projection method was proposed by Takahashi, Takeuchi and Kubota [11]. This iterative scheme has been investigated by many researchers in Hilbert, Banach and geodesic spaces; see [2, 6, 7, 9].

Recently, Kimura showed the following theorem:

Theorem 1.1 (Kimura [4])**.** *Let X be a Hadamard space and suppose that a subset $\{z \in X \mid d(u,z) \leq d(v,z)\}$ of X is convex for any $u, v \in X$. Let $\{T_i : X \to X \mid i = 1, 2, \ldots, m\}$ be a family of nonexpansive mappings. Generate a sequence $\{x_n\}$ in X with a sequence $\{C_n\}$ of subsets X by the following steps:*

Step 0. $x_1 \in X,\ C_1 = X$, *and* $n = 1$*;*
Step 1. $C_{n+1} = \{z \in X \mid d(T_i x_n, z) \leq \rho(x_n, z)\} \cap C_n$*;*
Step 2. (1) *if* $C_{n+1} \neq \emptyset$*, then let* $x_{n+1} = P_{C_{n+1}} x_n$*, increment* n *to* 1*, and go to Step 1;*
(2) *if* $C_{n+1} = \emptyset$*, then* $C_k = \emptyset$ *and leave* x_k *to be undefined for all* $k \geq n+1$*, and terminate the generating process.*

Then, the following conditions are equivalent:

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(a) $\bigcap_{i=1}^{m} \operatorname{Fix} T_i \neq \emptyset$;
(b) $\bigcap_{k=1}^{\infty} C_k \neq \emptyset$.

Further, in this case, $\{x_n\}$ is well defined and Δ-convergent to some $x_0 \in \bigcap_{i=1}^{m} \operatorname{Fix} T_i$.

Usually, we need to assume the existence of common fixed points to generate approximate sequences with the shrinking projection method. In this theorem, we do not suppose it, and in the case where $C_{n+1} = \emptyset$, we terminate its process. Thus, we obtained an equivalent condition for the existence of common fixed points.

On the other hand, in 2024, Kimura and Sudo [8] investigated fixed point theory on the unit sphere of a Banach space, which is called a Banach sphere, and show a fixed point approximation theorem for a spherically nonspreading mapping. In general, such a sphere does not have spherical distance, however, they adapt the following bifunction corresponding to the spherical distance on Hilbert spheres: For a point x in a Banach space and for a bounded linear functional f such that $\|x\| = \|f\| = 1$,

$$\rho(x, f) = \arccos\langle x, f\rangle.$$

In this paper, we show an existence and approximation of a common fixed point of a family of spherically nonspreading mappings on a Banach sphere.

2 Preliminaries

Let X be a nonempty set, C a nonempty subset of X and T a mapping from X into C. We denote the set of all fixed points of T by $\operatorname{Fix} T$, namely,

$$\operatorname{Fix} T = \{x \in X \mid Tx = x\}.$$

In this paper, we always consider real linear spaces. Let E be a Banach space and E^* its dual space. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^*\rangle$. Let

$$S_E = \big\{x \in E \mid \|x\| = 1\big\}$$

be the unit sphere of E. The duality mapping J on E is defined by

$$Jx = \big\{x^* \in E^* \mid \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\big\}$$

for each $x \in E$. We know that Jx is a nonempty bounded closed convex subset of E^* for any $x \in E$. Let E be a Banach space. E is said to be strictly convex if $x = y$ whenever $\|x + y\| = 2$ for $x, y \in S_E$. Further, we say that E is uniformly convex if

$$\lim_{n\to\infty} \|x_n - y_n\| = 0$$

whenever $\lim_{n\to\infty} \|x_n + y_n\| = 2$ for two sequences $\{x_n\}$ and $\{y_n\}$ of S_E. E is said to be smooth if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S_E$. A norm of E is said to be Fréchet differentiable if the limit is attained uniformly for $y \in S_E$ for fixed $x \in S_E$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in S_E$.

Let E be a Banach space with its dual E^* and J the duality mapping on E. Then, we know the following properties of E on J:

- If E is uniformly convex, then it is reflexive and strictly convex;
- if E is uniformly smooth, then it has a Fréchet differentiable norm;
- if E has a Fréchet differentiable norm, then it is smooth;
- E is smooth if and only if J is single-valued, and then

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}=\langle x, Jy\rangle$$

  for each $x, y \in E$;
- if E is smooth, then E is strictly convex if and only if J is injective;
- if E is smooth, then E is reflexive if and only if J is surjective;
- if E has a Fréchet differentiable norm, then J is norm-to-norm continuous;
- if E is reflexive, then E is strictly convex if and only if E^* is smooth;
- if E is reflexive, then E is smooth if and only if E^* is strictly convex;
- E is uniformly convex if and only if E^* is uniformly smooth;
- E is uniformly smooth if and only if E^* is uniformly convex.

Let E be a Banach space with its dual E^*. Let S_E and S_E^* be their unit spheres. Then, we know that

$$|\langle x, y^*\rangle| \le \|x\|\|y^*\| = 1$$

for each $(x, y^*) \in S_E \times S_E^*$. We define a function ρ from $S_E \times S_E^*$ to $[0, \pi]$ by

$$\rho(x, y^*) = \arccos\langle x, y^*\rangle.$$

for each $(x, y^*) \in S_E \times S_E^*$. In general, ρ is not symmetry and it does not satisfy the triangle inequality.

We define a notion of convex combination on Banach spheres. Let E be a Banach space. For $x, y \in S_E$ with $x \neq -y$ and $t \in [0, 1]$, set

$$tx \oplus (1-t)y = \frac{tx + (1-t)y}{\|tx + (1-t)y\|} \in S_E.$$

Let E be a smooth Banach space and J the duality mapping on E. Let X be a nonempty subset of S_E. We say that X is admissible if $\langle x, Jy\rangle > 0$ for all $x, y \in X$. Notice that X is admissible if and only if $\rho(x, y) < \pi/2$ for all $x, y \in X$.

We introduce the spherical projection. We first consider spherical convexity of a subset. Let E be a Banach space and C a subset of S_E. We say that C is spherically convex if

$$tx \oplus (1-t)y \in C$$

for all $x, y \in C$ with $x \neq -y$ and $t \in [0, 1]$. Let E be a smooth and uniformly convex Banach space. Let C be a nonempty, closed and spherically convex subset of S_E. Let

$$\operatorname{Dom} \mathit{\Pi}_C = \left\{ x \in S_E \,\middle|\, \inf_{y\in C}\rho(y, Jx) < \frac{\pi}{2}. \right\}$$

Then, for $x \in \operatorname{Dom} \mathit{\Pi}_C$, there is a unique point $u_x \in C$ such that

$$\rho(u_x, Jx) = \inf_{y\in C}\rho(y, Jx).$$

We call such a mapping $\Pi_C : x \mapsto u_x$ a spherical projection onto C. Note that $\operatorname{Fix} \Pi_C = C$.

Let E be a smooth Banach space and X a nonempty admissible subset of S_E. We call a mapping T from X into itself a spherically nonspreading mapping if

$$\cos \rho(Tx, JTy) + \cos \rho(Ty, JTx) \geq \cos \rho(Tx, Jy) + \cos \rho(Ty, Jx).$$

for all $x, y \in X$, where J is the duality mapping on E. The nonspreadingness of a mapping was first introduced by Kohsaka and Takahashi on smooth Banach spaces; refer to [10]. After that, on geodesic spaces, the spherical nonspreadingness of a mapping has been investigated; see [1, 3, 5].

3 Main Theorem

We show our main result, which generates an approximate sequence of a common fixed point of a family of spherically nonspreading mappings. We also obtain an equivalent condition for the existence of a common fixed point.

Theorem 3.1. *Let E be a uniformly smooth and uniformly convex Banach space. Let X be a nonempty, closed, and spherically convex subset of S_E. Let $\{T_1, T_2, \ldots, T_l\}$ be a family of spherically nonspreading mappings from X into itself. Generate a sequence $\{x_n\}$ in X with a sequence $\{C_n\}$ of subsets of X by the following steps:*

Step 0. $u, x_1 \in X,\ C_1 = X,$ *and* $n = 1$;
Step 1. $C_{n+1} = \bigcap_{i=1}^{l} \{z \in X \mid \rho(z, JT_i x_n) \leq \rho(z, Jx_n)\} \cap C_n$;
Step 2. (1) *if* $C_{n+1} \neq \emptyset$, *then let* $x_{n+1} = \Pi_{C_{n+1}} u$, *increment* n *to* 1, *and go to Step 1;*
(2) *if* $C_{n+1} = \emptyset$, *then* $C_k = \emptyset$ *and leave* x_k *to be undefined for all* $k \geq n+1$, *and terminate the generating process.*

Then, the following conditions are equivalent:

(a) $\bigcap_{i=1}^{l} \operatorname{Fix} T_i \neq \emptyset$;
(b) $\bigcap_{k=1}^{\infty} C_k \neq \emptyset$.

Further, in this case, $\{x_n\}$ is well defined and convergent to $\Pi_{\bigcap_{i=1}^{l} \operatorname{Fix} T_i} u$.

Proof. First we show that

$$\bigcap_{i=1}^{l} \{z \in X \mid \rho(z, JT_i x_n) \leq \rho(z, Jx_n)\} = \bigcap_{i=1}^{l} \{z \in X \mid \cos \rho(z, JT_i x_n) \geq \cos \rho(z, Jx_n)\}$$

is closed and spherically convex. It is obviously closed. Take p, q in the set above and

$t \in [0,1]$. Then,

$$\begin{aligned}\cos\rho(tp \oplus (1-t)q, JT_i x_n) &= \frac{t\cos\rho(p, JT_i x_n) + (1-t)\cos\rho(q, JT_i x_n)}{\|tp + (1-t)q\|} \\ &\geq \frac{t\cos\rho(p, Jx_n) + (1-t)\cos\rho(q, Jx_n)}{\|tp + (1-t)q\|} \\ &= \cos\rho(tp \oplus (1-t)q, Jx_n).\end{aligned}$$

This implies that $\bigcap_{i=1}^{l}\{z \in X \mid \cos\rho(z, JT_i x_n) \geq \cos\rho(z, Jx_n)\}$ is spherically convex. We suppose $\bigcap_{i=1}^{l} \operatorname{Fix} T_i \neq \emptyset$ and show $\bigcap_{k=1}^{\infty} C_k \neq \emptyset$. It is sufficient to show that $\bigcap_{i=1}^{l} \operatorname{Fix} T_i \subset C_k$ for every $k \in \mathbb{N}$. We prove this inclusion by induction. It is obvious for the case $k = 1$. Suppose $\bigcap_{i=1}^{l} \operatorname{Fix} T_i \subset C_k$ and we consider the case $k+1$. Notice that, in this case, x_k is defined. Let $z \in \bigcap_{i=1}^{l} \operatorname{Fix} T_i$. Then, since each T_i is spherically nonspreading, we have

$$\cos\rho(T_i x_k, JT_i z) + \cos\rho(T_i z, JT_i x_k) \geq \cos\rho(T_i x_k, Jz) + \cos\rho(T_i z, Jx_k)$$

for each $i = 1, 2, \dots, l$. Since $z \in \bigcap_{i=1}^{l} \operatorname{Fix} T_i$, we get

$$\cos\rho(T_i x_k, Jz) + \cos\rho(z, JT_i x_k) \geq \cos\rho(T_i x_k, Jz) + \cos\rho(z, Jx_k).$$

Hence,

$$\cos\rho(z, JT_i x_k) \geq \cos\rho(z, Jx_k).$$

The definition of C_{k+1} and the assumption of induction imply $z \in C_{k+1}$. Consequently, we obtain

$$C_k \supset \bigcap_{i=1}^{l} \operatorname{Fix} T_i \neq \emptyset,$$

and this is the desired result. Therefore, we know that C_n is closed and spherically convex, and $\bigcap_{i=1}^{l} \operatorname{Fix} T_i \subset C_{n+1} \subset C_n$ for any $n \in \mathbb{N}$. Hence, since Π_{C_n} is well defined, $\{x_n\}$ is well defined.

We next suppose that $\bigcap_{k=1}^{\infty} C_k \neq \emptyset$ and prove $\bigcap_{i=1}^{l} \operatorname{Fix} T_i \neq \emptyset$. Let $C_0 = \bigcap_{k=1}^{\infty} C_k$. Since

$$\rho(\Pi_{C_n} u, Ju) \leq \rho(\Pi_{C_{n+1}} u, Ju) \leq \rho(\Pi_{C_0} u, Ju) < \frac{\pi}{2}$$

for any $n \in \mathbb{N}$, there exists a limit of $\{\rho(\Pi_{C_n}, Ju)\}$. Notice that its limit is less than $\pi/2$. Assume that $\{\Pi_{C_n} u\}$ is not a Cauchy sequence of E. Then, there exist $\epsilon > 0, \{m_j\}, \{n_j\} \subset \mathbb{N}$ such that $m_j > n_j > j$ and $\|\Pi_{C_{m_j}} u - \Pi_{C_{n_j}} u\| \geq \epsilon$ for all $j \in \mathbb{N}$. In this way, we can take two subsequences $\{\Pi_{C_{m_j}} u\}$ and $\{\Pi_{C_{n_j}} u\}$ of $\{\Pi_{C_n} u\}$. Fix $j \in \mathbb{N}$. Since $\Pi_{C_{m_j}} u, \Pi_{C_{n_j}} u \in C_{n_j}$, we have

$$\begin{aligned}\cos\rho(\Pi C_n u, Ju) &\geq \cos\rho\left(\frac{1}{2}\Pi_{C_{m_j}} u \oplus \frac{1}{2}\Pi_{C_{n_j}} u, Ju\right) \\ &= \frac{\cos\rho(\Pi_{C_{m_j}} u, Ju) + \cos\rho(\Pi_{C_{n_j}} u, Ju)}{\|\Pi_{C_{m_j}} u + \Pi_{C_{n_j}} u\|}.\end{aligned}$$

We get

$$2 \geq \|\Pi_{C_{m_j}} u + \Pi_{C_{n_j}} u\| \geq \frac{\cos \rho(\Pi_{C_{m_j}} u, Ju)}{\cos \rho(\Pi_{C_{n_j}} u, Ju)} + 1 \to 2.$$

as $j \to \infty$. Since E is uniformly convex, we have $\|\Pi_{C_{m_j}} u - \Pi_{C_{n_j}} u\| \to 0$ as $j \to \infty$. This is a contradiction. thus, $\{\Pi_{C_n} u\}$ is a Cauchy sequence of E. Let $x_0 \in X$ be its strong limit. Fix $k \in \mathbb{N}$. for $n \in \mathbb{N}$ with $n \geq k$, since $C_n \subset C_k$, we have $\Pi_{C_n} u \in C_k$. Thus, since C_k is closed, we get $x_0 \in C_k$. It implies that $x_0 \in C_0 = \bigcap_{k=1}^{\infty} C_k$. Since

$$\rho(\Pi_{C_n} u, Ju) \leq \rho(\Pi_{C_0} u, Ju) \leq \rho(x_0, Ju)$$

for all $n \in \mathbb{N}$ and $\{\Pi_{C_n} u\}$ converges strongly to x_0, we have $\rho(\Pi_{C_0} u, Ju) = \rho(x_0, Ju)$ and hence $x_0 = \Pi_{C_0} u$. Moreover, since $x_n = \Pi_{C_n} u$ and $\Pi_{C_n} u \to x_0$, we get $\{x_n\}$ converges strongly to $\Pi_{C_0} u$. Since $\Pi_{C_0} u \in \bigcap_{k=1}^{\infty} C_k$, we obtain $\rho(\Pi_{C_0} u, JT_i x_n) \leq \rho(\Pi_{C_0} u, Jx_n)$. Then,

$$0 \leq \lim_{n \to \infty} \rho(\Pi_{C_0} u, JT_i x_n) \leq \lim_{n \to \infty} \rho(\Pi_{C_0} u, Jx_n) = \rho(\Pi_{C_0} u, J\Pi_{C_0}).$$

Therefore, $\lim_{n \to \infty} \rho(\Pi_{C_0} u, JT_i x_n) = 0$. Thus, we have

$$\begin{aligned} \lim_{n \to \infty} \langle \Pi_{C_0} u, J\Pi_{C_0} u + JT_i x_n \rangle &= \lim_{n \to \infty} \langle \Pi_{C_0} u, J\Pi_{C_0} u \rangle + \lim_{n \to \infty} \langle \Pi_{C_0} u, JT_i x_n \rangle \\ &= 1 + \langle \Pi_{C_0} u, JT_i x_n \rangle = 2. \end{aligned}$$

We obtain $\|J\Pi_{C_0} u\| = \|JT_i x_n\| = 1$ and

$$\begin{aligned} 2 &= \lim_{n \to \infty} \langle \Pi_{C_0} u, J\Pi_{C_0} u + JT_i x_n \rangle \\ &\leq \liminf_{n \to \infty} \|\Pi_{C_0} u\| \|\Pi_{C_0} u, J\Pi_{C_0} u + JT_i x_n\| \\ &= \liminf_{n \to \infty} \|J\Pi_{C_0} u + JT_i x_n\| \\ &\leq \limsup_{n \to \infty} \|J\Pi_{C_0} u + JT_i x_n\| \\ &= \limsup_{n \to \infty} (\|J\Pi_{C_0} u\| + \|JT_i x_n\|) = 2. \end{aligned}$$

From uniform smoothness of E, since E^* is uniformly convex, $\{JT_i x_n\}$ converges strongly to $J\Pi_{C_0} u$. Since J^{-1} is norm-to-norm continuous, $\{T_i x_n\}$ converges strongly to $\Pi_{C_0} u$. Since T_i is spherically nonspreading, for any $n \in \mathbb{N}$ and each i, we get

$$\begin{aligned} &\cos \rho(T_i x_n, JT_i \Pi_{C_0} u) + \cos \rho(T_i \Pi_{C_0} u, JT_i x_n) \\ &\qquad \geq \cos \rho(T_i x_n, J\Pi_{C_0} u) + \cos \rho(T_i \Pi_{C_0} u, Jx_n). \end{aligned}$$

Since J is norm-to-norm continuous, $\{Jx_n\}$ converges strongly to $J\Pi_{C_0} u$. Therefore, letting $n \to \infty$, we have

$$\cos \rho(\Pi_{C_0} u, JT_i \Pi_{C_0} u) + \cos \rho(T_i \Pi_{C_0} u, J\Pi_{C_0} u) \geq 1 + \cos \rho(T_i \Pi_{C_0} u, J\Pi_{C_0} u)$$

and thus $\cos \rho(\Pi_{C_0} u, JT_i \Pi_{C_0} u) = 1$. Therefore, $\Pi_{C_0} u = T_i \Pi_{C_0} u$. It implies that $\Pi_{C_0} u \in \operatorname{Fix} T_i$. Thus, if $\bigcap_{k=1}^{\infty} C_k \neq \emptyset$, then $\bigcap_{i=1}^{l} \operatorname{Fix} T_i \neq \emptyset$.

We finally show that $\{x_n\}$ converges strongly to $\Pi_{\bigcap_{i=1}^{l} \operatorname{Fix} T_i} u$. Since $\operatorname{Fix} T_i \subset C_0$, we obtain

$$\rho(\mathit{\Pi}_{C_0}u, Ju) = \rho(\mathit{\Pi}_{\bigcap_{i=1}^{l} \operatorname{Fix} T_i}u, Ju).$$

Therefore $\mathit{\Pi}_{C_0}u = \mathit{\Pi}_{\bigcap_{i=1}^{l} \operatorname{Fix} T_i}u$. We know $x_n \to x_0 = \mathit{\Pi}_{C_0}u$. Consequently, $\{x_n\}$ converges strongly to $\mathit{\Pi}_{\bigcap_{i=1}^{l} \operatorname{Fix} T_i}u$. □

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