Gap Function Approach to Duality
— discount model vs control model —

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Abstract

This paper discusses two pairs of quadratic optimization problem (primal) and its dual. In particular, we deal with the pair of problems which are called *discount model* and *control model*. For each model, duality is shown through the *gap function*. The method is based upon a complementary identity. Moreover, complete solutions are given through characteristic equations.

1 Introduction

Recently, in [6–14], S.Iwamoto, Y.Kimura, T.Fujita and A.Kira show that a duality for paired optimization problems through several methods. In particular, in [13], we have a method through gap function to show a duality between a primal problem and its dual. As a historical background, see Bellman and others [1–5], [15] for dynamic optimization.

In this paper, we discuss a method through *gap function* to show a duality for pairs of quadratic optimization problems, which are called *discount model* and *control model*. Section 2 considers a pair of n-variable minimization (primal) problem and maximization (dual) problem, which is called discount model. Then we define a gap function and discuss duality and optimal solution (point and value). In section 3, we consider a pair of quadratic optimization problems, which is called control model. As in section 2, we discuss a duality through a gap function and optimal solution for the pair.

Throughout the paper let n be a natural number and $c\,(\in R^1)$ be a constant. c denotes an *initial state* at time 0 of a dynamic system.

2 Discount model

In this section let ρ be a *positive* constant. We consider a pair of n-variable optimization problems :

$$\mathrm{P}_n \qquad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{k=1}^{n} \rho^{k-1}\left[(x_{k-1}-x_k)^2 + x_k^2\right] \\ \text{subject to} & \text{(i)}\ \ x \in R^n, \quad \text{(ii)}\ \ x_0 = c \end{array}$$

$$\mathrm{D}_n \quad \begin{array}{l} \text{Maximize } \ 2x_0\mu_1 - \sum_{k=1}^{n-1} \rho^{k-1}\left[\mu_k^2 + (\mu_k - \rho\mu_{k+1})^2\right] - 2\rho^{n-1}\mu_n^2 \\ \text{subject to } \ \text{(i)} \ \ \mu \in R^n, \quad \text{(ii)} \ \ x_0 = c. \end{array}$$

Let f, $g : R^n \to R^1$ be the respective objective functions of P_n, D_n:

$$\begin{aligned} f(x) &= \sum_{k=1}^{n} \rho^{k-1}\left[(x_{k-1} - x_k)^2 + x_k^2\right] \\ g(\mu) &= 2c\mu_1 - \sum_{k=1}^{n-1}\left[\mu_k^2 + (\mu_k - \rho\mu_{k+1})^2\right] - 2\rho^{n-1}\mu_n^2. \end{aligned}$$

Note that $f(x)$ is convex and $g(\mu)$ is concave. Then it hods that

$$f(x) \ \geq \ g(\mu) \qquad (x, \mu) \in R^n \times R^n. \tag{1}$$

The sign of equality holds iff a linear system of $2n$-equation on $2n$-variable

$$\mathrm{EC}(\rho) \quad \begin{array}{rclrcll} c - x_1 &=& \mu_1 & x_1 &=& \mu_1 - \rho\mu_2 & \\ x_{k-1} - x_k &=& \mu_k & x_k &=& \mu_k - \rho\mu_{k+1} & 2 \leq k \leq n-1 \\ x_{n-1} - x_n &=& \mu_n & x_n &=& \mu_n & \end{array}$$

holds. $\mathrm{EC}(\rho)$ is called an *equality condition* between P_n and D_n. Thus both problems are called *dual* of each other.

The equality condition $\mathrm{EC}(\rho)$ yields a pair of linear systems of n-equation on n-variable:

$$(\mathrm{EQ}_x) \quad \begin{array}{rcll} \gamma x_1 - \rho x_2 &=& c & \\ -x_{k-1} + \gamma x_k - \rho x_{k+1} &=& 0 & 2 \leq k \leq n-1 \\ -x_{n-1} + 2x_n &=& 0, & \end{array}$$

$$(\mathrm{EQ}_\mu) \quad \begin{array}{rcll} 2\mu_1 - \rho\mu_2 &=& c & \\ -\mu_{k-1} + \gamma\mu_k - \rho\mu_{k+1} &=& 0 & 2 \leq k \leq n-1 \\ -\mu_{n-1} + \gamma\mu_n &=& 0 & \end{array}$$

where $\gamma = 2 + \rho$.

2.1 Gap function for discount model

First we present an identity, which takes a fundamental role in analyzing respective pairs of primal and dual. Let $x = \{x_k\}_0^n$, $\mu = \{\mu_k\}_1^n$ be any two sequences of real number with $x_0 = c$. Then an identity

$$(\mathrm{C}_2) \quad \sum_{k=1}^{n-1} \rho^{k-1}\big[\,(x_{k-1} - x_k)\mu_k + x_k(\mu_k - \rho\mu_{k+1})\,\big] + \rho^{n-1}\big[\,(x_{n-1} - x_n)\mu_n + x_n\mu_n\,\big] \ = \ c\mu_1$$

holds true. This identity is called *complementary*.

Now we derive both P_n and D_n through gap function. Let us define a *gap function* $h = h(x, \mu)$ between $x \in R^n$ and $\mu \in R^n$ by

$$h(x,\mu) \;=\; \sum_{k=1}^{n-1} \rho^{k-1}\big[\,(x_{k-1} - x_k - \mu_k)^2 + \{x_k - (\mu_k - \rho\mu_{k+1})\}^2\,\big] \\ + \rho^{n-1}\big[\,(x_{n-1} - x_n - \mu_n)^2 + (x_n - \mu_n)^2\,\big]. \tag{2}$$

Thus $h(x, \mu)$ denotes a *total difference* between x and μ. It turns out that the quadratic function $h = h(x, \mu)$ is convex in (x, μ).

Lemma 1

$$\text{(i)} \qquad f(x) - g(\mu) \;=\; h(x,\mu) \;\geq\; 0 \quad \forall (x,\mu) \in R^n \times R^n$$

$$\text{(ii)} \qquad h(x,\mu) \;=\; 0 \quad \Longrightarrow \quad (x,\mu) \textit{ satisfies } \mathrm{EC}(\rho).$$

Theorem 1 (i) *It holds that*

$$f(x) \;\geq\; g(\mu) \quad \textit{on} \quad R^n \times R^n.$$

(ii) *It holds that*

$$f(x) \;=\; g(\mu) \;\Longleftrightarrow\; (x,\mu) \textit{ satisfies } \mathrm{EC}(\rho).$$

Then P_n *attains a minimum* $f(x)$*, while* D_n *attains a maximum* $g(\mu)$*.*

Hence a solution (x, μ) to $\mathrm{EC}(\rho)$ yields a minimum point x for P_n and a maximum point μ for D_n.

Theorem 2 *Let* (x, μ) *satisfy* $\mathrm{EC}(\rho)$*. Then both sides become a common value with five expressions:*

$$(5\mathrm{V}_2) \qquad \begin{aligned} f(x) \;&=\; c(c - x_1) \\ =\; g(\mu) \;&=\; \sum_{k=1}^{n-1} \rho^{k-1}\left[\mu_k^2 + (\mu_k - \rho\mu_{k+1})^2\right] + 2\rho^{n-1}\mu_n^2 \;=\; c\mu_1. \end{aligned}$$

The primal P_n *has a minimum value*

$$m \;=\; f(x) \;=\; c(c - x_1)$$

at x*, while the dual* D_n *has a maximum value*

$$M \;=\; g(\mu) \;=\; \sum_{k=1}^{n-1} \rho^{k-1}\left[\mu_k^2 + (\mu_k - \rho\mu_{k+1})^2\right] + 2\rho^{n-1}\mu_n^2 \;=\; c\mu_1$$

at μ.

2.2 Characteristic equation for discount model

Now let us solve the pair of linear systems (EQ_x) and (EQ_μ). We introduce a second-order linear difference equation

$$\rho x_{n+2} - \gamma x_{n+1} + x_n \;=\; 0, \quad x_1 = 1,\; x_0 = 0. \tag{3}$$

Lemma 2 *Eq (3) has a unique solution*

$$x_n \;=\; \frac{\beta^n - \alpha^n}{\beta - \alpha} \tag{4}$$

where $\alpha\,(<)\,\beta$ *are two positive solutions*

$$\alpha \;=\; \frac{\gamma - \sqrt{D}}{2\rho}, \quad \beta \;=\; \frac{\gamma + \sqrt{D}}{2\rho}\;;\; D \;=\; \rho^2 + 4\ (> 4) \tag{5}$$

to the associated characteristic equation

$$(\mathrm{CE}) \qquad \rho t^2 - \gamma t + 1 \;=\; 0. \tag{6}$$

Definition 1 *Let us define the sequence* $\{K_n\}$ *by*

$$K_n \;=\; \frac{\beta^n - \alpha^n}{\beta - \alpha}. \tag{7}$$

We call $\{K_n\}$ a *Kibonacci* sequence[1][13]. Thus $\{K_n\}$ satisfies a second-order linear difference equation

$$\rho K_{n+1} \;=\; \gamma K_n - K_{n-1}, \quad K_1 = 1,\; K_0 = 0. \tag{8}$$

This has a unique solution (7).

Lemma 3 *The system* (EQ_x) *has a unique solution*

$$x_k \;=\; \frac{c}{\rho^k} \cdot \frac{K_{n+1-k} - K_{n-k}}{K_{n+1} - K_n} \qquad 0 \le k \le n$$

, while the system (EQ_μ) *has a unique solution*

$$\mu_k \;=\; \frac{c}{\rho^{k-1}} \cdot \frac{K_{n+1-k}}{2K_n - K_{n-1}} \qquad 1 \le k \le n.$$

Theorem 3 *The equality condition* $\mathrm{EC}(\rho)$ *has a unique solution* $(x,\,\mu)$*;*

$$x_k \;=\; \frac{c}{\rho^k} \cdot \frac{K_{n+1-k} - K_{n-k}}{K_{n+1} - K_n}$$

$$\mu_k \;=\; \frac{c}{\rho^{k-1}} \cdot \frac{K_{n+1-k}}{2K_n - K_{n-1}}.$$

Hence the gap function h *attains the zero minimum at* $(x,\,\mu)$*.*

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[1]Strictly speaking, ρ-*Kibonacci* sequence.

3 Control model

In this section let $b\,(\in R^1)$ be a constant. We consider a pair of n-variable optimization problems :

$$
\mathrm{P}_n \qquad \begin{array}{l} \text{minimize} \quad \displaystyle\sum_{k=1}^{n}\left[(x_{k-1}-bx_k)^2+x_k^2\right] \\ \text{subject to} \quad \text{(i)} \;\; x\in R^n, \quad \text{(ii)} \;\; x_0=c \end{array}
$$

$$
\mathrm{D}_n \qquad \begin{array}{l} \text{Maximize} \;\; 2x_0\mu_1-\displaystyle\sum_{k=1}^{n-1}\left[\mu_k^2+(b\mu_k-\mu_{k+1})^2\right]-(1+b^2)\mu_n^2 \\ \text{subject to} \;\; \text{(i)} \quad \mu\in R^n, \quad \text{(ii)} \;\; x_0=c. \end{array}
$$

Let $f,\, g : R^n \to R^1$ be the respective objective functions of P_n, D_n:

$$
\begin{aligned}
f(x) &= \sum_{k=1}^{n}\left[(x_{k-1}-bx_k)^2+x_k^2\right] \\
g(\mu) &= 2c\mu_1-\sum_{k=1}^{n-1}\left[\mu_k^2+(b\mu_k-\mu_{k+1})^2\right]-(1+b^2)\mu_n^2.
\end{aligned}
$$

Note that $f(x)$ is convex and $g(\mu)$ is concave. Then it hods that

$$
f(x) \;\geq\; g(\mu) \qquad (x,\mu)\in R^n{\times}R^n. \tag{9}
$$

The sign of equality holds iff a linear system of $2n$-equation on $2n$-variable

$$
\mathrm{EC}(b) \qquad \begin{array}{rclrcll} c-bx_1 &=& \mu_1 & x_1 &=& b\mu_1-\mu_2 & \\ x_{k-1}-bx_k &=& \mu_k & x_k &=& b\mu_k-\mu_{k+1} & 2\leq k\leq n-1 \\ x_{n-1}-bx_n &=& \mu_n & x_n &=& b\mu_n & \end{array}
$$

holds. $\mathrm{EC}(b)$ is called an *equality condition* between P_n and D_n. Thus both problems are called *dual* of each other.

The equality condition $\mathrm{EC}(b)$ yields a pair of linear systems of n-equation on n-variable:

$$
(\mathrm{EQ}_x) \qquad \begin{array}{rcll} \gamma x_1-bx_2 &=& bc & \\ -bx_{k-1}+\gamma x_k-bx_{k+1} &=& 0 & 2\leq k\leq n-1 \\ -bx_{n-1}+\xi x_n &=& 0, & \end{array}
$$

$$
(\mathrm{EQ}_\mu) \qquad \begin{array}{rcll} \xi\mu_1-b\mu_2 &=& c & \\ -b\mu_{k-1}+\gamma\mu_k-b\mu_{k+1} &=& 0 & 2\leq k\leq n-1 \\ -b\mu_{n-1}+\gamma\mu_n &=& 0 & \end{array}
$$

where $\gamma \,=\, 2+b^2,\ \xi \,=\, 1+b^2$.

3.1 Gap function for control model

Let $x = \{x_k\}_0^n$, $\mu = \{\mu_k\}_1^n$ be any two sequences of real number with $x_0 = c$. Then a complementary identity

$$(\mathrm{C}_3) \quad \sum_{k=1}^{n-1}[(x_{k-1} - bx_k)\mu_k + x_k(b\mu_k - \mu_{k+1})] + (x_{n-1} - bx_n)\mu_n + x_n \cdot b\mu_n \;=\; c\mu_1$$

holds true. Let us define a *gap function* $h = h(x, \mu)$ by

$$h(x, \mu) \;=\; \sum_{k=1}^{n-1} \left[\, (x_{k-1} - bx_k - \mu_k)^2 + \{x_k - (b\mu_k - \mu_{k+1})\}^2 \,\right] + \left[\, (x_{n-1} - bx_n - \mu_n)^2 + (x_n - b\mu_n)^2 \,\right]. \tag{10}$$

Thus $h(x, \mu)$ denotes a *total difference*. It turns out that the quadratic function $h = h(x, \mu)$ is convex.

Lemma 4

$$\text{(i)} \qquad f(x) - g(\mu) \;=\; h(x, \mu) \;\geq\; 0 \quad \forall (x, \mu) \in R^n \times R^n$$

$$\text{(ii)} \qquad h(x, \mu) \;=\; 0 \quad \Longrightarrow \quad (x, \mu) \ \textit{satisfies} \ \mathrm{EC}(b).$$

Theorem 4 (i) *It holds that*

$$f(x) \;\geq\; g(\mu) \quad \textit{on} \quad R^n \times R^n.$$

(ii) *It holds that*

$$f(x) \;=\; g(\mu) \iff (x, \mu) \ \textit{satisfies} \ \mathrm{EC}(b).$$

Then P_n *attains a minimum* $f(x)$*, while* D_n *attains a maximum* $g(\mu)$*.*

Hence a solution (x, μ) to $\mathrm{EC}(b)$ yields a minimum point x for P_n and a maximum point μ for D_n.

Theorem 5 *Let* (x, μ) *satisfy* $\mathrm{EC}(b)$*. Then both sides become a common value with five expressions:*

$$(5\mathrm{V}_3) \qquad \begin{aligned} f(x) \;&=\; c(c - bx_1) \\ \;=\; g(\mu) \;&=\; \sum_{k=1}^{n-1} \left[\mu_k^2 + (b\mu_k - \mu_{k+1})^2\right] + (1 + b^2)\mu_n^2 \;=\; c\mu_1. \end{aligned}$$

The primal P_n *has a minimum value*

$$m \;=\; f(x) \;=\; c(c - bx_1)$$

at x*, while the dual* D_n *has a maximum value*

$$M \;=\; g(\mu) \;=\; \sum_{k=1}^{n-1} \left[\mu_k^2 + (b\mu_k - \mu_{k+1})^2\right] + (1 + b^2)\mu_n^2 \;=\; c\mu_1$$

at μ.

3.2 Characteristic equation for control model

We introduce a second-order linear difference equation

$$bx_{n+2} - \gamma x_{n+1} + bx_n \;=\; 0, \quad x_1 = 1,\; x_0 = 0. \tag{11}$$

Lemma 5 *Eq (11) has a unique solution*

$$x_n \;=\; \frac{\beta^n - \alpha^n}{\beta - \alpha} \tag{12}$$

where $\alpha\,(<)\,\beta$ *are two positive solutions*

$$\alpha \;=\; \frac{\gamma - \sqrt{D}}{2b}, \quad \beta \;=\; \frac{\gamma + \sqrt{D}}{2b}\;;\; D \;=\; b^4 + 4\;(> 4) \tag{13}$$

to the associated characteristic equation

$$\text{(CE)} \qquad bt^2 - \gamma t + b \;=\; 0. \tag{14}$$

Now let us define Kibonacci sequence[2][13] $\{K_n\}$ by

$$K_n \;=\; \frac{\beta^n - \alpha^n}{\beta - \alpha}.$$

Then the sequence $\{K_n\}$ satisfies a second-order linear difference equation

$$bK_{n+1} \;=\; \gamma K_n - bK_{n-1}, \quad K_1 = 1,\; K_0 = 0. \tag{15}$$

Lemma 6 *The system* (EQ_x) *has a unique solution*

$$x_k \;=\; c\,\frac{bK_{n+1-k} - K_{n-k}}{bK_{n+1} - K_n} \qquad 0 \le k \le n$$

, while the system (EQ_μ) *has a unique solution*

$$\mu_k \;=\; c\,\frac{K_{n+1-k}}{\xi K_n - bK_{n-1}} \qquad 1 \le k \le n.$$

Theorem 6 *The equality condition* $\mathrm{EC}(b)$ *has a unique solution* $(x,\,\mu)$*;*

$$x_k \;=\; c\,\frac{bK_{n+1-k} - K_{n-k}}{bK_{n+1} - K_n}$$

$$\mu_k \;=\; c\,\frac{K_{n+1-k}}{\xi K_n - bK_{n-1}}.$$

Hence the gap function h *attains the zero minimum at* $(x,\,\mu)$*.*

[2]Strictly speaking, b-*Kibonacci* sequence.

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