Common fixed point theorems using haltable shrinking projection method on an Hadamard space
Hadamard 空間における
停止可能な収縮射影法を用いた共通不動点定理

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Abstract

In this paper, we show common fixed point theorems using haltable shrinking projection method on an Hadamard space. Moreover, we generalized the assumptions of previous results.

1 Introduction

In nonlinear analysis, fixed point theory is one of the most important topics. There are many results about the existence of a fixed point of nonlinear mappings and their approximation techniques.

The shrinking projection method is an approximation scheme for fixed points. It was proposed by Takahashi, Takeuchi and Kubota [7]. In 2023, Kimura [3] showed the following theorem:

Theorem 1.1 (Kimura [3]). *Let X be a Hadamard space and suppose that a subset $\{z \in X \mid d(u,z) \le d(v,z)\}$ is convex for any $u, v \in X$. Let $T\colon X \to X$ be a nonexpansive mapping with $\mathcal{F}(T) \neq \varnothing$. Generate a sequence $\{x_n\} \subset X$ as follows: $x_1 \in X$ is given, $C_1 = X$, and*

$$\begin{aligned} C_{n+1} &= \{z \in X \mid d(Tx_n, z) \le d(x_n, z)\} \cap C_n, \\ x_{n+1} &= P_{C_n} x_n \end{aligned}$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ is Δ-convergent to $x_0 \in \mathcal{F}(T)$.

In 2024, Kimura proposed a haltable shrinking projection method.

Theorem 1.2 (Kimura [4]). *Let X be a Hadamard space and suppose that*

$$\{z \in X \mid d(u,z) \le d(v,z)\}$$

of X is convex for any $u, v \in X$. Let $\{T_i \colon X \to X \mid i = 1, 2, \dots, m\}$ be a family of nonexpansive mappings. Generate a sequence $\{x_n\}$ in X with a sequence $\{C_n\}$ of subsets of X by the following steps:

Step 0. $x_1 \in X$, $C_1 = X$, *and* $n = 1$;
Step 1. $C_{n+1} = \bigcap_{i=1}^{m} \{z \in X \mid d(T_i x_n, z) \le d(x_n, z)\} \cap C_n$;
Step 2. (1) *if* $C_{n+1} \neq \varnothing$, *then let* $x_{n+1} = P_{C_{n+1}} x_n$, *increment n to 1, and go to Step 1;*
(2) *if* $C_{n+1} = \varnothing$, *then* $C_k = \varnothing$ *and leave x_k to be undefined for all* $k \ge n+1$, *and terminate the generating process.*

Then, the following conditions are equivalent:

(a) $\bigcap_{i=1}^{m} \mathcal{F}(T_i) \neq \varnothing$;
(b) $\bigcap_{k=1}^{\infty} C_k \neq \varnothing$.

Further, in this case, $\{x_n\}$ is well defined and Δ-convergent to $x_0 \in \bigcap_{i=1}^{m} \mathcal{F}(T_i)$.

Under the assumption that the space is bounded, we can judge the nonexistence of common fixed points in finite time.

In this paper, we show common fixed point theorems using haltable shrinking projection method on an Hadamard space. Unlike Theorem 1.1 and 1.2, we show that the sequence strongly converge. Moreover, we generalized the assumptions of previous results.

2 Preliminaries

Let X be a metric space with metric d. We call $x \in X$ a fixed point of a mapping T on X if $Tx = x$. We denote the set of all fixed points of T by $\mathcal{F}(T)$.

For $x, y \in X$, a mapping $c \colon [0, l] \to X$ is called a geodesic joining x and y if c satisfies $c(0) = x$, $c(l) = y$ and $d(c(s), c(t)) = |s - t|$ for all $s, t \in [0, l]$. We denote its image by $[x, y]$, which is called a geodesic segment with endpoints x and y. X is said to be a D-geodesic space if there exists a geodesic joining all $x, y \in X$ with $d(x, y) < D$. In this paper, for a D-geodesic metric space X, a geodesic joining any two points of X whose distance is less than D, is always assumed to be unique.

Let X be a D-geodesic space. For all $x, y \in X$ with $d(x, y) < D$ and $\alpha \in [0, 1]$, there exists a unique point $z \in X$ such that $d(x, z) = (1 - \alpha) d(x, y)$ and $d(z, y) = \alpha d(x, y)$. This point is called a convex combination of x and y, denoted by $\alpha x \oplus (1 - \alpha) y$. Assume that $d(u, v) < D$ for all $u, v \in X$. f is said to be convex if

$$f(tx \oplus (1-t)y) \le t f(x) + (1-t) f(y)$$

for any $x, y \in X$ and $t \in \,]0, 1[$. A subset $C \subset X$ is said to be convex if $[x, y] \subset X$ for all $x, y \in C$.

We define a CAT(0) space by using the notions of geodesic triangles and comparison triangle on a model space. In this paper, we use the following definition which is equivalent to the original one.

Definition 2.1. (Bačák [1]) A uniquely geodesic space X is called CAT(0) space if the inequality

$$d(\alpha x \oplus (1-\alpha)y, z)^2 \le \alpha d(x,z)^2 + (1-\alpha)d(y,z)^2 - \alpha(1-\alpha)d(x,y)^2$$

holds for any $x, y, z \in X$ and $\alpha \in [0,1]$.

An Hadamard space is defined as a complete CAT(0) space.

Let X be a metric space. A mapping $T\colon X \to X$ is said to be nonexpansive if $d(Tx, Ty) \le d(x, y)$ for all $x, y \in X$. A mapping $T\colon X \to X$ is said to be metrically nonspreading [5] if

$$2d(Tx, Ty)^2 \le d(x, Ty)^2 + d(Tx, y)^2$$

for all $x, y \in X$. A mapping $T\colon X \to X$ is said to be quasinonexpansive if $\mathcal{F}(T)$ is nonempty and $d(Tx, z) \le d(x, z)$ for all $x \in X$ and $z \in \mathcal{F}(T)$. A mapping T is said to be closed if $x \in \mathcal{F}(T)$ whenever $\{x_n\} \subset X$ satisfies $x_n \to x$ and $d(x_n, Tx_n) \to 0$. If T is a nonexpansive mapping with $\mathcal{F}(T) \ne \varnothing$, then T is quasinonexpansive and closed. Moreover, if T is a metrically nonspreading mapping with $\mathcal{F}(T) \ne \varnothing$, then T is quasinonexpansive and closed.

Let X be an Hadamard space and $C \subset X$ be a nonempty closed convex subset of X. It is known that for $x \in X$, there exists a unique $y_x \in C$ such that

$$d(x, y_x) = \inf_{y \in C} d(x, y).$$

Using this point, we define the metric projection $P_C\colon X \to C$ by $P_C x = y_x$.

3 Haltable shrinking projection method on an Hadamard space

In this section, we modify the result by [3] to strong convergence theorem. Moreover, we generalized the assumptions of the results.

Lemma 3.1. *Let X be an Hadamard space and $\{C_n\}$ a sequence of nonempty closed convex sets which is decreasing with respect to inclusion, that is, $C_{n+1} \subset C_n$ for any $n \in \mathbb{N}$. Suppose that $C = \bigcap_{n=1}^{\infty} C_n$ is nonempty. Then, $\{P_{C_n} x\}$ converges to $P_C x \in X$ for any $x \in X$.*

This lemma is a corollary of the result of [2]. For the sake of completeness, we give a proof.

Proof. Fix $x \in X$ arbitrarily. Since $C = \bigcap_{n=1}^{\infty} C_n \ne \varnothing$, for $p \in C$ and $n \in \mathbb{N}$, we have

$$d(x, P_{C_n} x) \le d(x, p)$$

and hence $\{d(x, P_{C_n} x)\}$ is bounded. Furthermore, since $C_{n+1} \subset C_n$ for any $n \in \mathbb{N}$, we have

$$d(x, P_{C_n} x) \le d(x, P_{C_{n+1}} x)$$

and thus $\{d(x, P_{C_n} x)\}$ is increasing. Therefore, $\{d(x, P_{C_n} x)\}$ has a limit

$$c = \lim_{n\to\infty} d(x, P_{C_n} x).$$

Then, there exists a sequence $\{\alpha_n\}$ of nonnegative real numbers such that

$$|d(x, P_{C_m}x)^2 - d(x, P_{C_n}x)^2| \leq \alpha_n$$

for any $m \geq n$ and $\alpha_n \to 0$. From Definition 2.1, for $m, n \in \mathbb{N}$ with $m \geq n$, we have

$$\begin{aligned} d(x, P_{C_n}x)^2 &\leq d(x, \frac{1}{2}P_{C_m}x \oplus \frac{1}{2}P_{C_n}x)^2 \\ &\leq \frac{1}{2}d(x, P_{C_m}x)^2 + \frac{1}{2}d(x, P_{C_n}x)^2 - \frac{1}{4}d(P_{C_m}x, P_{C_n}x)^2 \end{aligned}$$

and hence

$$\begin{aligned} d(P_{C_m}x, P_{C_n}x)^2 &\leq 2(d(P_{C_m}x, x)^2 - d(P_{C_n}x, x)^2) \\ &\leq 2|d(P_{C_m}x, x)^2 - d(P_{C_n}x, x)^2| \\ &\leq 2\alpha_n \to 0. \end{aligned}$$

Therefore $\{P_{C_n}x\}$ is a Cauchy sequence. Since X is complete, there exists a point $x_0 \in X$ such that $P_{C_n}x \to x_0$.

Next, we show that $x_0 \in C$. For any $k \in \mathbb{N}$, since $\{P_{C_n}x\}_{n \geq k} \subset C_k$ and C_k is closed, we have $x_0 \in C_k$ and thus $x_0 \in \bigcap_{k=1}^{\infty} C_k$.

Finally, we show that $x_0 = P_C x$. For any $n \in \mathbb{N}$, from the property of the metric projection, we have

$$d(x, P_{C_n}x) \leq d(x, P_C x) \leq d(x, x_0).$$

Letting $n \to \infty$, we obtain

$$d(x, P_C x) = d(x, x_0)$$

and hence $x_0 = P_C x$.

Consequently, we have $P_{C_n}x \to P_C x$ for any $x \in X$. □

Using Lemma 3.1, we get the following main result.

Theorem 3.2. *Let X be an Hadamard space and suppose that*

$$\{z \in X \mid d(u, z) \leq d(v, z)\}$$

of X is convex for any $u, v \in X$. Let $\{T_i \colon X \to X \mid i = 1, 2, \ldots, m\}$ be a family of closed mappings such that if there exists $z \in \mathcal{F}(T)$ then $d(Tx, z) \leq d(x, z)$ for all $x \in X$. Generate a sequence $\{x_n\}$ in X with a sequence $\{C_n\}$ of subsets of X by the following steps:

Step 0. *$u, x_1 \in X$, $C_1 = X$, and $n = 1$;*
Step 1. *$C_{n+1} = \bigcap_{i=1}^{m} \{z \in X \mid d(T_i x_n, z) \leq d(x_n, z)\} \cap C_n$;*
Step 2. (1) *if $C_{n+1} \neq \varnothing$, then let $x_{n+1} = P_{C_{n+1}}u$, increment n to 1, and go to Step 1;*
(2) *if $C_{n+1} = \varnothing$, then $C_k = \varnothing$ and leave x_k to be undefined for all $k \geq n+1$, and terminate the generating process.*

Then, the following conditions are equivalent:

(a) *$\bigcap_{i=1}^{m} \mathcal{F}(T_i) \neq \varnothing$;*

(b) $\bigcap_{k=1}^{\infty} C_k \neq \varnothing$.

Further, in this case, $\{x_n\}$ is well defined and convergent to $P_{\bigcap_{i=1}^{m} \mathcal{F}(T_i)} u$.

Proof. First, we suppose $C = \bigcap_{k=1}^{\infty} C_k \neq \varnothing$ and show $\bigcap_{i=1}^{m} \mathcal{F}(T_i) \neq \varnothing$. By Lemma 3.1, $x_n = P_{C_n} u \to P_C u$. Since $P_C u \in C$, we have

$$\begin{aligned} 0 &\leq d(x_n, T_i x_n) \\ &\leq d(x_n, P_C u) + d(P_C u, T_i x_n) \\ &\leq 2d(x_n, P_C u) \to d(P_C u, P_C u) = 0. \end{aligned}$$

Thus $d(x_n, T_i x_n) \to 0$. Since T_i is closed, we have $P_C u \in \mathcal{F}(T_i)$ for every $i = 1, 2, \dots, m$ and hence $P_C u \in \bigcap_{i=1}^{m} \mathcal{F}(T_i) \neq \varnothing$.

Next, we suppose that $\bigcap_{i=1}^{m} \mathcal{F}(T_i) \neq \varnothing$ and prove $\bigcap_{k=1}^{\infty} C_k \neq \varnothing$. It is sufficient to show that $\bigcap_{i=1}^{m} \mathcal{F}(T_i) \subset C_k$ for every $k \in \mathbb{N}$. We prove this inclusion by induction. It is obvious for the case $k = 1$. Suppose $\bigcap_{i=1}^{m} \mathcal{F}(T_i) \subset C_k$ and we consider the case $k + 1$. Notice that, in this case, x_k is defined. Let $z \in \bigcap_{i=1}^{m} \mathcal{F}(T_i)$. Then, we have

$$d(T_i x_k, z) \leq d(x_k, z)$$

for each $i = 1, 2, \dots, m$. This fact and the assumption of induction imply $z \in C_{k+1}$. Consequently, we obtain

$$\bigcap_{k=1}^{\infty} C_k \supset \bigcap_{i=1}^{m} \mathcal{F}(T_i) \neq \varnothing,$$

and this is the desired result.

We prove the latter part of the theorem. Since $\bigcap_{k=1}^{\infty} C_k \supset \bigcap_{i=1}^{m} \mathcal{F}(T_i)$ and $P_C u \in \bigcap_{i=1}^{m} \mathcal{F}(T_i)$, we have $P_C u = P_{\bigcap_{i=1}^{m} \mathcal{F}(T_i)} u$. Therefore $x_n \to P_{\bigcap_{i=1}^{m} \mathcal{F}(T_i)} u$. □

A nonexpansive mapping and a metrically nonspreading mapping satisfy assumptions of the previous theorem. As a corollary of Theorem 3.2, we obtain the following theorem.

Theorem 3.3. *Let X be an Hadamard space and suppose that*

$$\{z \in X \mid d(u, z) \leq d(v, z)\}$$

of X is convex for any $u, v \in X$. Let $\{T_i \colon X \to X \mid i = 1, 2, \dots, m + l\}$ be a family of mappings. Suppose that T_i is nonexpansive for $i = 1, \dots, m$ and is metrically nonspreading for $i = m + 1, \dots, m + l$. Generate a sequence $\{x_n\}$ in X with a sequence $\{C_n\}$ of subsets of X by the following steps:

Step 0. *$u, x_1 \in X$, $C_1 = X$, and $n = 1$;*
Step 1. *$C_{n+1} = \bigcap_{i=1}^{m+l} \{z \in X \mid d(T_i x_n, z) \leq d(x_n, z)\} \cap C_n$;*
Step 2. (1) *if $C_{n+1} \neq \varnothing$, then let $x_{n+1} = P_{C_{n+1}} u$, increment n to 1, and go to Step 1;*
(2) *if $C_{n+1} = \varnothing$, then $C_k = \varnothing$ and leave x_k to be undefined for all $k \geq n + 1$, and terminate the generating process.*

Then, the following conditions are equivalent:

(a) *$\bigcap_{i=1}^{m+l} \mathcal{F}(T_i) \neq \varnothing$;*

(b) $\bigcap_{k=1}^{\infty} C_k \neq \varnothing$.

Further, in this case, $\{x_n\}$ is well defined and convergent to $P_{\bigcap_{i=1}^{m+l} \mathcal{F}(T_i)} u$.

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