Spherical nonspreadingness and perturbations for resolvents of convex functions on geodesic spaces
測地距離空間における凸関数のリゾルベントの球面的非伸長性と摂動関数

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Abstract

In this paper, we consider resolvent operators of convex functions with various perturbation functions. We prove that there are many resolvent operators which are spherically nonspreading of sum-type in a geodesic space with curvature bounded above by one.

1 Introduction and preliminaries

For an admissible CAT(1) space X, a mapping T from X into itself is said to be *spherically nonspreading of sum-type* [3] if an inequality

$$2\cos d(Tx, Ty) \geq \cos d(x, Ty) + \cos d(y, Tx)$$

holds for any two points x and y in X. This concept is a special case of vicinal mappings with ψ defined by Kajimura and Kimura on an admissible CAT(κ) space.

Definition 1.1 (Kajimura–Kimura [2])

Let X be an admissible CAT(1) space and $\psi\colon [0, \pi/2[\to]0, \infty[$ a right-continuous function at 0. Then a mapping T from X into itself is said to be *vicinal with* ψ (for $\kappa = 1$) if

$$\begin{aligned}(\psi(d(x, Tx)) + \psi(d(y, Ty))) \cos d(Tx, Ty)&\\ \geq \psi(d(x, Tx)) \cos d(x, Ty) &+ \psi(d(y, Ty)) \cos d(y, Tx)\end{aligned}$$

for any $x, y \in X$.

Note that its original definition in [2] is in an admissible CAT(κ) space for every $\kappa \in \mathbb{R}$; however, this paper only considers the case where $\kappa = 1$. Let $\mathbb{1}\colon [0, \pi/2[\to \mathbb{R}$ be a constant function defined by $\mathbb{1}(t) = 1$ for any $t \in [0, \pi/2[$. From Definition 1.1, we know that a mapping T is spherically nonspreading of sum-type if and only if it is vicinal with the constant function $\mathbb{1}$, or equivalently, it is vicinal with any positive constant function.

We say that a sequence $\{y_n\}$ on an admissible CAT(1) space X is *spherically bounded* if there exists $u \in X$ such that $\sup_{n\in\mathbb{N}} d(y_n, u) < \pi/2$. Now we introduce the following theorem.

Theorem 1.2 (Kajimura [1, Theorem 5.15])

Let X be an admissible complete CAT(1) *space and $T\colon X \to X$ a vicinal mapping with ψ. Then T has a fixed point if and only if there exists $x \in X$ such that $\{T^n x\}_{n=1}^{\infty}$ is spherically bounded and $\sup_{n\in\mathbb{N}} \psi(d(T_{n+1}x, T_n x)) < \infty$.*

In Theorem 1.2, the condition $\sup_{n\in\mathbb{N}} \psi(d(T_{n+1}x, T_n x)) < \infty$ follows automatically whenever T is vicinal with $\mathbb{1}$. Hence, the following holds:

Corollary 1.3

Let X be an admissible complete CAT(1) *space. Then a spherically nonspreading mapping T of sum-type has a fixed point if and only if $\{T^n x\}_{n=1}^{\infty}$ is spherically bounded for some $x \in X$.*

Consequently, to check whether a spherically nonspreading mapping T of sum-type has a fixed point, we only need to examine the spherical boundedness of $\{T^n x\}$ for $x \in X$.

Let X be an admissible CAT(1) space, and f a proper convex function from X into $]-\infty, \infty]$. We call a mapping J_f a resolvent operator of f if it is a signle-valued mapping from X into itself and its fixed point set $F(J_f)$ coincides with the set $\operatorname{argmin} f$ of all minimizers of f. From previous studies, such as [3, 5, 6, 7], we know that a set-valued mapping $J_f\colon X \to 2^X$ defined by the formula

$$\begin{aligned} J_f x &= \operatorname*{argmin}_{y\in X} \big(f(y) + \Phi(d(x,y))\big) \\ &= \Big\{ z \in X \;\Big|\; \inf_{y\in X} \big(f(y) + \Phi(d(x,y))\big) \geq f(z) + \Phi(d(x,z)) \Big\} \end{aligned}$$

for $x \in X$ becomes a resolvent operator of f on X by using a perturbation function $\Phi\colon [0, \pi/2[\to \mathbb{R}$ under certain assumptions. Now we consider set-valued mappings Q_f, R_f, and S_f from X into 2^X defined by

$$Q_f x = \operatorname*{argmin}_{y\in X} \big(f(y) - \log\cos d(x,y)\big); \quad R_f x = \operatorname*{argmin}_{y\in X} \big(f(y) + 1 - \cos d(x,y)\big);$$
$$S_f x = \operatorname*{argmin}_{y\in X} \big(f(y) + \tan d(x,y) \sin d(x,y)\big)$$

for $x \in X$, respectively. If X is complete and f is lower semicontinuous, then Q_f and S_f can be defined as a single-valued mapping on X; see [3, Theorem 3.2] and [5, Theorem 4.2], respectively. Moreover, if X is complete, f is lower semicontinuous, and f has at least one minimizer, then R_f becomes a single-valued mapping, see [7, Lemma 5.2.1]. Furthermore, we know that Q_f and R_f are spherically nonspreading of sum-type; however, we do not know whether S_f is so or not.

In this paper, we show that there exist resolvent operators other than Q_f and R_f that satisfy the spherical nonspreadingness of sum-type. Specifically, we prove that the following resolvents $T_f^{\lambda,n}$ and U_f are spherically nonspreading of sum-type:

$$T_f^{\lambda,n}x = \operatorname*{argmin}_{y\in X}\left(f(y) + \int_0^{1-\cos d(x,y)}\left(1+\lambda\cdot\frac{s^n}{1-s}\right)ds\right);$$
$$U_f x = \operatorname*{argmin}_{y\in X}\left(f(y) + \sqrt{(1-\cos d(x,y))(-\log\cos d(x,y))}\right)$$

for $x \in X$, $\lambda \in [0,1]$, and $n \in [1,\infty[$.

The following is a crucial notion that describes a behavior of resolvent operators:

Definition 1.4 (Kajimura–Kimura [2])

Let X be an admissible CAT(1) space and T a mapping from X into itself. Let $\psi\colon [0,\pi/2[\to]0,\infty[$ be a function which is right-continuous at 0. Then, T is said to be *firmly vicinal with* ψ (for $\kappa = 1$) if

$$\begin{aligned}\big(\psi(d(x,Tx))\cos d(x,Tx) + \psi(d(y,Ty))\cos d(y,Ty)\big)\cos d(Tx,Ty)\\ \geq \psi(d(x,Tx))\cos d(x,Ty) + \psi(d(y,Ty))\cos d(y,Tx)\end{aligned}$$

for any $x, y \in X$.

In Definition 1.4, since X is admissible, we know that every firmly vicinal mapping with ψ is vicinal with the same ψ.

Lemma 1.5 (Kajimura–Kimura [2, Theorem 4.1])

Let X be an admissible CAT(1) *space and f a proper convex function from X into $]-\infty,\infty]$. Let $\overline{\varphi}\colon]0,1] \to [0,\infty[$ be a nonincreasing and differentiable function such that $\overline{\varphi}'$ is continuous on $]0,1]$. Suppose that a set-valued mapping $J_f\colon X \to 2^X$ defined by*

$$J_f x = \operatorname*{argmin}_{y\in X}\big(f(y) + \overline{\varphi}(\cos(d(x,y)))\big)$$

for $x \in X$ is well-defined as a single-valued mapping on X. Then, J_f is firmly vicinal with $-\overline{\varphi}' \circ \cos$.

Lemma 1.6 ([6, Theorem 6.5])

The mapping J_f defined in Lemma 1.5 satisfies $F(J_f) = \operatorname{argmin} f$.

Lemma 1.7 ([6, Theorem 6.8, Theorem 5.30])

Let X be an admissible complete CAT(1) *space and $f\colon X \to]-\infty,\infty]$ a proper lower* *semicontinuous convex function. Let* $\overline{\varphi}\colon]0,1] \to [0,\infty[$ *be a strictly decreasing and differentiable function such that* $\overline{\varphi}'$ *is nondecreasing and continuous on* $]0,1]$. *Suppose that* $\lim_{t\searrow 0}\overline{\varphi}(t) = \infty$. *Define a set-valued mapping* $J_f\colon X \to 2^X$ *by*

$$J_f x = \operatorname*{argmin}_{y\in X}\bigl(f(y) + \overline{\varphi}(\cos d(x,y))\bigr)$$

for $x \in X$. *Then the following hold:*

(i) J_f *is well-defined as a single-valued mapping on* X*;*
(ii) J_f *is firmly vicinal with* $-\overline{\varphi}' \circ \cos\colon [0,\pi/2[\to]0,\infty[$ *(by Lemma 1.5);*
(iii) $F(J_f) = \operatorname{argmin} f$ *(by Lemma 1.6).*

2 Main results

In this section, we investigate the nature of several resolvents. Let X be an admissible CAT(1) space, $f\colon X \to]-\infty,\infty]$ a proper convex function, $\lambda \in [0,1]$, and $n \in [1,\infty[$. Let us consider mappings Q_f, R_f, S_f, $T_f^{\lambda,n}$, and U_f defined in Section 1. Define functions $\overline{\varphi}_1^{\lambda,n}, \overline{\varphi}_2\colon]0,1] \to [0,\infty[$ by

$$\overline{\varphi}_1^{\lambda,n}(t) = \int_0^{1-t}\Bigl(1 + \lambda\cdot\frac{s^n}{1-s}\Bigr)ds \quad\text{and}\quad \overline{\varphi}_2(t) = \sqrt{(1-t)(-\log t)}$$

for $t \in]0,1]$, respectively. Then, $T_f^{\lambda,n}x$ and $U_f x$ are figured by

$$\operatorname*{argmin}_{y\in X}\bigl(f(y) + \overline{\varphi}_1^{\lambda,n}(\cos d(x,y))\bigr) \quad\text{and}\quad \operatorname*{argmin}_{y\in X}\bigl(f(y) + \overline{\varphi}_2(\cos d(x,y))\bigr)$$

for $x \in X$, respectively. Consider the case where X is complete, f is lower semicontinuous, and $\lambda \neq 0$. Then Lemma 1.7 ensures that each mapping $Q_f, S_f, T_f^{\lambda,n}, U_f\colon X \to 2^X$ becomes a single-valued mapping on X. In contrast, Lemma 1.7 does not guarantee the well-definedness of R_f as a single-valued mapping. However, as mentioned before, Sudo [7, Lemma 5.2.1] showed that R_f is well-defined if $\operatorname{argmin} f \neq \varnothing$.

Now we prove the following result, which provides a sufficient condition for a firmly vicinal mapping T with ψ to be spherically nonspreading of sum-type.

Theorem 2.1

Let $\chi\colon]0,1] \to]0,\infty[$ *be a left-continuous function at* 1. *Let* X *be an admissible* CAT(1) *space and* $T\colon X \to X$ *a firmly vicinal mapping with* $\psi = \chi\circ\cos$. *Suppose that there exists* $k > 0$ *such that*

$$1 \le \frac{1}{k}\chi(t) \le \frac{1}{t} \tag{2.1}$$

for any $t \in]0,1[$. *Then,* T *is spherically nonspreading of sum-type.*

Proof. Suppose that $T\colon X \to X$ is firmly vicinal with $\psi = \chi\circ\cos$ and that (2.1) holds for any $t \in]0,1]$. Then we have $\psi(d)\cos d \le k \le \psi(d)$ for any $d \in [0,\pi/2[$. Hence, putting $K_x = d(x, Tx)$ and $K_y = d(y, Ty)$, we get

$$\begin{aligned} 2k \cos d(Tx, Ty) &\geq (\psi(K_x)\cos d(x, Tx) + \psi(K_y)\cos d(y, Ty))\cos d(Tx, Ty) \\ &\geq \psi(K_x)\cos d(x, Ty) + \psi(K_y)\cos d(y, Tx) \\ &\geq k\cos d(x, Ty) + k\cos d(y, Tx) \end{aligned}$$

for any $x, y \in X$, which is the desired result. □

From the following result, we can construct many functions $\overline{\varphi}$ which make a resolvent defined by the perturbation $\overline{\varphi} \circ \cos$ spherically nonspreading of sum-type.

Theorem 2.2

Let X be an admissible CAT(1) *space and f a proper convex function from X into $]-\infty, \infty]$. Let $\chi_1 \colon]0,1] \to \mathbb{R}$ be a continuous function such that*

$$1 \leq \chi_1(t) \leq \frac{1}{t}$$

for any $t \in]0,1]$. For arbitrarily fixed $k > 0$, define $\overline{\varphi} \colon]0,1] \to [0, \infty[$ by

$$\overline{\varphi}(t) = \int_t^1 k\chi_1(s)\, ds \tag{2.2}$$

for $t \in]0,1]$, and $J_f \colon X \to 2^X$ by

$$J_f x = \operatorname*{argmin}_{y \in X} \big(f(y) + \overline{\varphi}(\cos d(x, y))\big) \tag{2.3}$$

for $x \in X$. Suppose that J_f is well-defined as a single-valued mapping on X. Then the following hold:

(i) *J_f is firmly vicinal with $\chi_1 \circ \cos \colon [0, \pi/2[\to]0, \infty[$;*
(ii) *$F(J_f) = \operatorname{argmin} f$;*
(iii) *J_f is spherically nonspreading of sum-type.*

Theorem 2.3

Let X, f, χ_1, k, and $\overline{\varphi}$ be the same as in Theorem 2.2. We additionally suppose that the following conditions hold:

- *X is complete;*
- *f is lower semicontinuous;*
- *χ_1 is nondecreasing;*
- $\displaystyle\int_0^1 \chi_1(s)\, ds = \infty$.

Then, $J_f \colon X \to 2^X$ defined by (2.3) *becomes a single-valued mapping on X.*

Proof of Theorem 2.2. Let $\chi := -\overline{\varphi}' = k\chi_1$. Then Theorem 1.5 implies that J_f is firmly vicinal with $\chi \circ \cos$, and hence (i) holds. Moreover, (ii) is proved by Lemma 1.6.

We also deduce that $1 \le (1/k)\chi(t) \le 1/t$ for any $t \in \,]0,1[$, and therefore Theorem 2.1 implies (iii). □

Proof of Theorem 2.3. We can see that $\overline{\varphi}$ is strictly increasing, $\overline{\varphi}'$ is nondecreasing, and $\lim_{t \searrow 0} \overline{\varphi}(t) = \infty$. Therefore, from Lemma 1.7, we obtain the conclusion. □

We also obtain the following:

Corollary 2.4

Let X and f be the same as in Theorem 2.2. Let $M\colon]0,1] \to [0,1]$ be a continuous function, and define $\chi_1\colon]0,1] \to \mathbb{R}$ by

$$\chi_1(t) = 1 + \frac{1-t}{t} M(t)$$

for $t \in \,]0,1]$. For arbitrarily fixed $k > 0$, define $\overline{\varphi}$ and J_f by formulas (2.2) and (2.3). Then the following hold:

(I) *If J_f is well-defined as a single-valued mapping on X, then the same statements* (i)–(iii) *as in Theorem 2.2 hold;*
(II) *if the same four conditions as in Theorem 2.3 hold, then J_f becomes a single-valued mapping on X.*

Proof. Since $1 \le \chi_1(t) \le 1/t$ for any $t \in \,]0,1[$, we obtain the conclusion. □

In what follows, we always assume that X is an admissible CAT(1) space and $f\colon X \to \,]-\infty, \infty]$ is a proper convex function. Consider the resolvents $T_f^{\lambda,n}$, Q_f, and R_f defined in Section 1, and we assume that these are single-valued mappings on X. Using Corollary 2.4, we can show that these resolvents are spherically nonspreading of sum-type.

Theorem 2.5

For fixed real numbers $\lambda \in [0,1]$ and $n \in [1,\infty[$, the resolvent $T_f^{\lambda,n}$ is spherically nonspreading of sum-type.

Proof. Put $M(t) = \lambda(1-t)^{n-1}$ if $n > 1$, and $M(t) = \lambda$ if $n = 1$. Moreover, let

$$\chi(t) = 1 + \frac{1-t}{t} M(t) = 1 + \lambda \cdot \frac{(1-t)^n}{t}$$

for $t \in \,]0,1]$. Then we can see that $M(t) \in [0,1]$ for any $t \in \,]0,1]$, and

$$\overline{\varphi}_1^{\lambda,n}(t) = \int_0^{1-t} \Big(1 + \lambda \cdot \frac{s^n}{1-s}\Big) ds = \int_t^1 \chi(s)\, ds$$

for any $t \in \,]0,1]$. Hence, Corollary 2.4 implies the desired result. □

Theorem 2.6 (Kajimura–Kimura [3, Theorem 3.5])

The resolvent Q_f is spherically nonspreading of sum-type.

Proof. Consider the case where $k = 1$ and $M(t) = 1$ for $t \in \,]0,1]$ in Corollary 2.4. □

Theorem 2.7 (Sudo [7, Remark 11, Remark 16])

The resolvent R_f is spherically nonspreading of sum-type.

Proof. Consider the case where $k = 1$ and $M(t) = 0$ for $t \in \,]0, 1]$ in Corollary 2.4. $\square$

For $\lambda \in [0, 1]$ and $n \in [1, \infty[$, consider the perturbation $\Phi^{\lambda,n} := \overline{\varphi}_1^{\lambda,n} \circ \cos$ of $T_f^{\lambda,n}$, which is represented by

$$\Phi^{\lambda,n}(d) = \int_0^{1-\cos d} \Big(1 + \lambda \cdot \frac{s^n}{1-s}\Big) ds$$

for each $d \in [0, \pi/2[$. Then we see that $\Phi^{1,1}(d) = -\log \cos d$ for any $d \in [0, \pi/2[$. Moreover, for fixed $n \in [1, \infty[$, we obtain $\Phi^{0,n}(d) = 1 - \cos d$. Furthermore, for fixed $\lambda \in [0, 1]$ and $d \in [0, \pi/2[$, it follows that $\lim_{n\to\infty} \Phi^{\lambda,n}(d) = 1 - \cos d$. These mean that the perturbation of the resolvent $T_f^{\lambda,n}$ connects smoothly those of Q_f and R_f.

Recently, Kajimura et al. [4] showed that the resolvent $V_f \colon X \to X$ defined by

$$V_f x = \operatorname*{argmin}_{y\in X} \big(f(y) + 1 - \cos d(x, y) - \log \cos d(x, y)\big)$$

for $x \in X$ is spherically nonspreading of sum-type. It is obtained by considering the case where $k = 2$ and $M(t) = 1/2$ for $t \in \,]0, 1]$ in Corollary 2.4. Moreover, we also know that V_f is ideltical to $T_{(1/2)f}^{1/2,1}$.

Finally, we show the resolvent operator U_f is spherically nonspreading of sum-type. Before that, we introduce the following lemma.

Lemma 2.8

Let $f, g \colon \,]0, 1[\,\to \mathbb{R}$ be differentiable functions having the following properties:

- $0 < f(t) \le g(t)$ *for any* $t \in \,]0, 1[$;
- $f'(t) \le 0$, $g'(t) \le 0$ *for any* $t \in \,]0, 1[$;
- $f'(t)g(t) - f(t)g'(t) \ge 0$ *for any* $t \in \,]0, 1[$.

Let $h(t) = \sqrt{f(t)g(t)}$ for any $t \in \,]0, 1[$. Then, $g'(t) \le h'(t) \le f'(t)$ for any $t \in \,]0, 1[$.

Proof. Fix $t \in \,]0, 1[$. Then we have $0 < f(t) \le h(t) \le g(t)$, which yields

$$2h(t)g'(t) \le 2f(t)g'(t) \le f'(t)g(t) + f(t)g'(t) = \big(f(t)g(t)\big)'.$$

Similarly, it follows that

$$2h(t)f'(t) \ge 2g(t)f'(t) \ge f'(t)g(t) + f(t)g'(t) = \big(f(t)g(t)\big)'.$$

Therefore we obtain

$$g'(t) \le \frac{(f(t)g(t))'}{2h(t)} \le f'(t).$$

Since

$$\frac{(f(t)g(t))'}{2h(t)} = \frac{(f(t)g(t))'}{2\sqrt{f(t)g(t)}} = h'(t),$$

we get the conclusion. □

Assume that U_f is well-defined as a single-valued mapping on X.

Theorem 2.9

The resolvent U_f is spherically nonspreading of sum-type.

Proof. Define $\chi\colon]0,1] \to [1,\infty[$ by

$$\chi(t) = -\overline{\varphi}_2'(t) = \frac{1-t-t\log t}{2t\sqrt{(1-t)(-\log t)}}$$

for $t \in]0,1[$, and $\chi(1) = \lim_{t\nearrow 1} \chi(t) = 1$. Then U_f is firmly vicinal with $\chi \circ \cos$. Let $f(t) = 1-t$ and $g(t) = -\log t$ for $t \in]0,1]$. Then $\overline{\varphi}_2(t) = \sqrt{f(t)g(t)}$ for any $t \in]0,1]$. Therefore, from Lemma 2.8, we conclude that $g'(t) \leq \overline{\varphi}_2'(t) \leq f'(t)$ for any $t \in]0,1[$, from which it follows that

$$1 \leq \chi(t) \leq \frac{1}{t}$$

for any $t \in]0,1[$. Consequently, Theorem 2.1 implies the desired result. □

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