MENGER 空間における ψ-CONTRACTIVE MAPPINGS のいくつかの一般化について

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Abstract. In this paper we consider α-φ contractive mappings by assuming asymptotically regularity in complete meanger spaces and we also consider its generalization. We also prove fixed point theorems and common fixed point theorem for those mappings.

1. Introduction

Probabilistic metric spaces were introduced in 1942 by Menger [14]. The notion of distance between two points x and y is replaced by a distribution function $F_{x,y}$. Sehgal, in his Ph.D. Thesis [19], extended the notion of a contraction mapping to the setting of the Menger probabilistic metric spaces. The probabilistic version of the classical Banach Contraction Principle was first studied in 1972 by Sehgal and Bharucha-Reid [20]. Since then, many authors have obtained fixed point theorems for probabilistic φ-contractions under the assumption that φ is nondecreasing and such that $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for any $t > 0$ (see, e.g., [6] and the references in [5]). Ćirić [5] consider the more weak conditions and Jachymski [13] correctly defined the conditions and give tha following Theorem.

Theorem 1. *(See Jachymski [13].) Let (X, F, Δ) be a complete Menger probabilistic metric space with a continuous t-norm Δ of H-type, and let $\varphi : R_+ \to R_+$ be a function satisfying conditions:*

$$0 < \varphi(t) < t \text{ and } \lim_{n\to\infty} \varphi(t) = 0 \text{ for all } t > 0.$$

If $T : X \to X$ is a probabilistic φ-contraction, then T has a unique fixed point $x^* \in X$, and $\{T^n(x_0)\}$ converges to x^* for each $x_0 \in X$.

A mapping $T : X \to X$ is called a probabilistic φ-contraction (or a φ-contraction in probabilistic metric space) if it satisfies $F_{Tx,Ty}(\varphi(t)) \geq F_{x,y}(t)$ for all $x, y \in X$ and $t > 0$, where $\varphi : R_+ \to R_+$ is a gauge function satisfying certain conditions.

2. Preliminaries

Let R denote the real number and $R_+ = \{x \in R \mid x > 0\}$. A mapping $F : R \to R_+$ is called a distribution if it is non-decreasing left-continuous with $\sup_{t\in R} F(t) = 1$ and $\inf_{t\in R} F(t) = 0$. The set of all distribution functions is denoted by $\mathcal{D}$, and $\mathcal{D}_+ = \{F | F \in \mathcal{D}, F(0) = 0\}$. A special element H of $\mathcal{D}_+$ is defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases}$$

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A mapping $\Delta : [0,1] \times [0,1] \to [0,1]$ is called a triangular norm (for short, a t-norm) if the following conditions are satisfied:

(i) $\Delta(a,1) = a$;
(ii) $\Delta(a,b) = \Delta(b,a)$;
(iii) $a \geq b$, $c \geq d$ implies $\Delta(a,c) \leq \Delta(b,d)$;
(iv) $\Delta(a, \Delta(b,c)) = \Delta(\Delta(a,b),c)$.

Definition 2. *(Menger [14]., Schweizer and Sklar [21]). A triplet (X,F,Δ) is called a Menger probabilistic metric space (for short, a Menger space) if X is a non-empty set, Δ is a t-norm and F is a mapping from $X \times X$ into $\mathcal{D}$ satisfying the following conditions (for $x,y \in X$, we denote $\mathcal{F}(x,y)$ by $F_{x,y}$):*

(1) *$F_{x,y}(t) = H(t)$ for all $t \in R$ if and only if $x = y$;*
(2) *$F_{x,y}(t) = F_{y,x}(t)$ for all $t \in R$;*
(3) *$F_{x,y}(t+s) \geq \Delta(F_{x,z}(t), F_{z,y}(s))$ for all $x,y,z \in X$ and $s,t > 0$.*

Schweizer et al. [17, 18] point out that if the t-norm Δ of a Menger PM-space (X,F,Δ) satisfies the condition

$$\sup_{0<t<1} \Delta(t,t) = 1,$$

then (X,F,Δ) is a Hausdorff topological space in the (ε,λ)-topology τ, i.e., the family of sets

$$\{U_x(\varepsilon,\lambda) : \varepsilon > 0, \lambda \in (0,1]\}(x \in X)$$

is a basis of neighborhoods of point x for τ, where $U_x(\varepsilon,\lambda) = \{y \in X : F_{x,y}(\varepsilon) > 1 - \lambda\}$.

Definition 3. *Let (X,F,Δ) be a menger space such that $\sup_{0<t<1} \Delta(t,t) = 1$,*

(1) *A sequence $\{x_n\}$ in (X,F,Δ) is said to be τ-convergent (simply convergent) to $x \in X$ (we write $x_n \to x$ or $\lim_{n\to\infty} F_{x_n,x}(t) = 1$ for any $t > 0$) if for any given $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer $M_{\varepsilon,\lambda}$ such that $F_{x_n,p^*}(\lambda) > 1$ whenever $n \geq M_{\varepsilon,\lambda}$.*
(2) *$\{x_n\}$ is called a τ-Cauchy (simply Cauchy) sequence in (X,F,Δ) if for any given $\varepsilon > 0$ and $\lambda \in (0,1]$, there exists a positive integer $N = N(\varepsilon,\lambda)$ such that $F_{x_n,x_m}(\varepsilon) \geq 1 - \lambda$, whenever $n,m \geq N$;*
(3) *(X,F,Δ) is said to be τ-complete (simply comp), if each τ-Cauchy sequence in X is τ-convergent to some point in X. In what follows, we will always assume that (X,F,Δ) is a Menger space with the (ε,Δ)-topology*

In what follows, we will always assume that (X,F,Δ) is a Menger space with the (ε,Δ)-topology

Lemma 4. *(Sehgal and Bharucha-Reid [12]). Let (X,d) be a metric space. Define a mapping $F : X \times X \to \mathcal{D}$ by*

$$F_{x,y}(t) = H(t - d(x,y)) \text{ for any } x,y \in X \text{ and } t > 0. \tag{1}$$

Then $(X,F,\min)$ is a Menger space. and it is called the induced Menger space by (X,d), and it is complete if (X,d) is complete.

Definition 5. *(Hadžíc [9], Hadžíc and Pap [12]). A t-norm Δ is said to be of H-type (Hadžič̃ type) if the family of functions $\{\Delta^m(t)\}_{m=1}^{\infty}$ is equicontinuous at $t = 1$, where $\Delta^1(t) = \Delta(t,t)$,*

$$\Delta^m(t) = \Delta(t, \Delta^{m-1}(t)), m = 1,2,...,t \in [0,1].$$

for each $\lambda \in (0,1]$ *there exists* $\delta \in (0,\lambda]$ *such that* $\Delta^n(1-\delta) > 1-\lambda$ *for all* $n \in Z^+$.

The t-norm $\Delta_M = min$ *is a trivial example of t-norm of H-type, but there are t-norms* Δ *of H-type with* $\Delta \neq \Delta_M$ *(see, e.g.,* [11]*).*

Definition 6. *Let* (X, F, Δ) *be a menger metric space. A mapping* $T : X \to X$ *is called asymptotic regular if for every* $\varepsilon > 0$ *and every* $\lambda > 0$*, there exists an integer* $M_{\varepsilon,\lambda}$ *such that*

$$F_{T^n x, T^{n+1}x}(\varepsilon) > 1-\lambda$$

whenever $n > M_{\varepsilon,\lambda}$. *In this case we write* $\lim_{n\to\infty} F_{x_n,x_{n+1}}(\varepsilon) = 1$.

Next we define the φ-K contractions and φ_n-K contractions in Menger sapces.

Definition 7. *Let* $(X, \mathcal{F}, \Delta)$ *be a complete Menger space with continuous t-norm* Δ *and non-decreasing mapping* $\varphi : R_+ \to R_+$ *satisfying for any* $t > 0$ *there exists* $t_1, t_2 > 0$, $0 \leq K < \infty$ *and* $r \geq t$ *such that* $0 \leq \varphi(r) + K(t_1+t_2) < t$. *Then* $T : X \to X$ *is a probabilistic* φ-K *contraction if* T *satisfy the following inequality:*
If $K > 0$, *then*

$$F_{Tx,Ty}(\varphi(r) + K(t_1+t_2)) \geq \Delta\left(F_{x,y}(t),\, F_{x,Tx}(Kt_1),\, F_{y,Ty}(Kt_2)\right), \tag{2}$$

if $K = 0$, *then*

$$F_{Tx,Ty}(\varphi(r)) \geq F_{x,y}(t). \tag{3}$$

Definition 8. *Let* $(X, \mathcal{F}, \Delta)$ *be a complete Menger space with continuous t-norm* Δ *and sequence of non-decreasing mapping* $\{\varphi_n\}$ *with* $\varphi_n : R_+ \to R_+$ *satisfying for any* $t > 0$ *there exists* $t_1, t_2 > 0$, $0 \leq K < \infty$ *and* $r \geq t$ *such that* $0 \leq \sum_{n=1}^{\infty} \varphi_n(r) + K(t_1+t_2) < t$. *Then* $T : X \to X$ *is a probabilistic* φ_n-K *contraction if* T *satisfy the following inequality:*
If $K > 0$, *then*

$$F_{T^n x,T^n y}(\varphi_n(r) + K(t_1+t_2)) \geq \Delta\left(F_{x,y}(t),\, F_{T^{n-1}x,T^n x}(Kt_1),\, F_{T^{n-1}y,T^n y}(Kt_2)\right), \tag{4}$$

if $K = 0$, *then*

$$F_{T^n x,T^n y}(\varphi_n(r)) \geq F_{x,y}(t). \tag{5}$$

Next we also define the orbit set with respect to the mapping $T : X \to X$ at x with

$$O(x,T) = \{T^n x \in X, n = 0, 1, \ldots\}.$$

Definition 9. [4] *Let* (X, d) *be a metric space and let* T *be mapping from* X *into itself. Then* T *is called orbitally continuous if for any* $x \in X$ *and for any sequence* $\{x_n\}$ *in* $O(x,T)$, *for any* $t > 0$ $\lim_{n\to\infty} E_{x_n,p}(t) = 1$ *implies* $\lim_{n\to\infty} E_{Tx_n,Tp}(t) = 1$.

Definition 10. [4] *Let* (X, d) *be a metric space and let* T *be mapping from* X *into itself. Then* T *is called* k*-continuous,* $k = 1, 2, 3, \ldots$, *if for any* $t > 0$ $\lim_{n\to\infty} E_{T^k x_n,Tp}(t) = 1$ *whenever* $\{x_n\}$ *is a sequence in* (X, d) *such that* $\lim_{n\to\infty} E_{T^{k-1}x_n,p}(t) = 1$.

Definition 11. [2, 3, 22] *Let* $x \in X$. *A sequence* $\{x_n\}$ *of points in* X *is called an iterative sequence of* T *at* x *if* $Tx_n = T^n x$, $n = 1, 2, \ldots$. *Note that define sequence* $\{x_n\}$ *by* $x_{n+1} = T^n x$, *it is naturally iterative.*

3. Main result

We give the following Theorem.

Theorem 12. *Let (X, F, Δ) be a complete probabilistic Menger space such that Δ is a continuous triangular norm of Hadžić type. Let $\varphi : R_+ \to R_+$ be a mapping such that for any $t > 0$, there exist $0 \le K < \infty$, $t_1(t)$, $t_2(t) > 0$ and $r \ge t$ such that $0 \le \varphi(r) + Kt_1 + Kt_2 < t$ and a mapping $T : X \to X$ be asymptotic regular, and $Kt_1(t)$, $Kt_2(t) \to \infty$ as $t \to \infty$. We assume that T is a probabilistic φ-K contraction. Then T is orbitally continuous if and only if T has a unique fixed point x^*. For this fixed point x^*, for any $x_0 \in X$ $\lim_{n\to\infty} T^n x_0 = x^*$.*

Proof. Let $x_0 \in X$ and $x_n := Tx_{n-1}$ for any $n \in N$. Since T is asymptotic regular, we have

$$\lim_{n\to\infty} F_{x_n,Tx_n}(t) = 1 \tag{6}$$

for any $t > 0$. If $K = 0$, then the proof is similar to that of [7, Lemma 3.2.]. Now let $n \in N$ and $t > 0$, then there exist $0 < K < \infty$, t_1, $t_2 > 0$ and $r \ge t$ such that $\varphi(r) + Kt_1 + Kt_2 < t$. Put $\psi(t) = \varphi(t) + Kt_1 + Kt_2$. We show by induction that, for any $k \in N$,

$$\lim_{n\to\infty} F_{x_n,x_{n+k}}(t) \ge \Delta^k(\lim_{n\to\infty} F_{x_n,x_{n+1}}(t - \psi(t)). \tag{7}$$

Since the mapping T is asymptotic regular, we have

$$\lim_{n\to\infty} F_{x_n,x_{n+1}}(Kt_1) = 1,\ \lim_{n\to\infty} F_{x_{n+k},x_{n+k+1}}(Kt_2) = 1.$$

We note that

$$F_{Tx,Ty}(\psi(t)) \ge \Delta(F_{x,y}(t), F_{x,Tx}(Kt_1), F_{y,Ty}(Kt_2)),$$

then we have

$$\begin{aligned}
&F_{x_n,x_{n+k+1}}(t)\\
&= F_{x_n,x_{n+k+1}}(t - \psi(t) + \psi(t))\\
&= \Delta(F_{x_n,x_{n+1}}(t - \psi(t)), F_{x_{n+1},x_{n+k+1}}(\psi(t)))\\
&\ge \Delta\left(F_{x_n,x_{n+1}}(t - \psi(r)), \Delta\left(F_{x_n,x_{n+k}}(t),\ F_{x_n,x_{n+1}}(Kt_1),\ F_{x_{n+kr},x_{n+k+1}}(Kt_2)\right)\right).
\end{aligned}$$

Since Δ is continuous, we have

$$\begin{aligned}
&\Delta\left(\lim_{n\to\infty} F_{x_n,x_{n+k}}(t),\ \lim_{n\to\infty} F_{x_n,x_{n+1}}(Kt_1),\ \lim_{n\to\infty} F_{x_{n+kr},x_{n+k+1}}(Kt_2)\right)\\
&= \Delta\left(\lim_{n\to\infty} F_{x_n,x_{n+k}}(t),\ 1,\ 1\right)\\
&\ge \Delta^k\left(\lim_{n\to\infty} F_{x_n,x_{n+1}}(t - \psi(t))\right).
\end{aligned}$$

Then we have

$$\begin{aligned}
&\lim_{n\to\infty} F_{x_n,x_{n+k+1}}(t)\\
&\ge \Delta\left(\lim_{n\to\infty} F_{x_n,x_{n+k+1}}(t - \psi(t)), \Delta^k\left(\lim_{n\to\infty} F_{x_n,x_{n+1}}(t - \psi(t))\right)\right)\\
&\ge \Delta^{k+1}\left(\lim_{n\to\infty} F_{x_n,x_{n+1}}(t - \psi(t))\right).
\end{aligned}$$

We show that sequence $\{x_n\}$ is Cauchy, that is,

$$\lim_{m,n\to\infty} F_{x_n,x_m}(t) = 1 \text{ for any } t > 0.$$

Let $t > 0$ and $\varepsilon > 0$. By hypothesis, $\{\Delta^n(t) \mid n \in N\}$ is equicontinuous at 1 and $\Delta^n(1) = 1$, so there is $\delta > 0$ such that

$$\text{if } s \in (1-\delta, 1], \text{ then } \Delta^n(s) \geq 1 - \varepsilon \text{ for all } n \in N. \tag{8}$$

Since T is asymtotic regular, we have

$$\lim_{n\to\infty} F_{x_n,x_{n+1}}(t - \psi(r)) = 1.$$

Then there exists $n_0 \in N$ such that, for any $n \geq n_0$,

$$F_{x_n,x_{n+1}}(t - \psi(r)) \in (1-\delta, 1].$$

Hence, by (7) and (8), we get $F_{x_n,x_{n+k}}(t) > 1 - \varepsilon$ for any $k \in N$. This proves the Cauchy condition for $\{x_n\}$. By completeness, $\{x_n\}$ converges to some $p \in X$, that is,

$$\lim_{n\to\infty} F_{x_n,p}(t) = 1.$$

for any $t > 0$. We show that p is a fixed point of T. By asymptotic continuity and order continuity of T, we get

$$F_{p,Tp}(t) \geq \Delta(F_{p,x_n}(t/3), F_{x_n,x_{n+1}}(t/3), F_{Tx_n,Tp}(t/3))) = \Delta(1.1.1) = 1$$

This yields $F_{p,Tp}(t) = 1$ for any $t > 0$, and hence $p = Tp$.

Finally, we show the uniqueness of a fixed point. Let p and q be fixed point of mapping T with $p \neq q$. Then $F_{p,q} < 1$. Since T is asymptotic regular, sequence $\{x_n\}$ in X satisifies $\lim_{n\to\infty} F_{Tx_n,Tx_{n+1}} = 1$. In this case it can be prove that $\{Tx_n\}$ is Cauchy and converges in X. Then for the p, q, any $t > 0$ $\lim_{n\to\infty} F_{Tx_n,p}(t) = 1$ and $\lim_{n\to\infty} F_{Tx_{n+1},q}(t) = 1$. Then

$$1 > F_{p,q}(t) \geq \Delta(F_{p,Tx_n}(t/3), F_{Tx_n,Tx_{n+1}}(t/3), F_{Tx_{n+1},q}(t/3)) \to \Delta(1,1,1) = 1.$$

This is contaradiction. Therefore $p = q$. □

Theorem 13. *Let $(X, \mathcal{F}, \Delta)$ be a complete probabilistic Menger space such that Δ is a continuous triangular norm of Hadžić type. Let $\varphi_j : R_+ \to R_+$ $(j = 1, 2, \ldots)$ be mappings such that for any $t > 0$, $n \in N$, there exist $r \geq t$, $0 \leq K < \infty$, t_1, $t_2 > 0$ such that $0 \leq \sum_{j=1}^{\infty} \varphi_j(r) + Kt_1 + Kt_2 < t$ and a mapping T be asymptotic regular, and $Kt_1(t)$, $Kt_2(t) \to \infty$ as $t \to \infty$. We assume that T is a probabilistic φ_n-K contraction. Then T is orbitally continuous if and only if T has a unique fixed point x^*. For this fixed point x^*, for any $x_0 \in X$ $\lim_{n\to\infty} T^n x_0 = x^*$.*

Proof. Let $x_0 \in X$ and $x_{n+1} := Tx_n$ for any $n \in N$. Since T is asymptotic regular we have

$$\lim_{n\to\infty} F_{x_n,Tx_n}(Kt_1) = \lim_{n\to\infty} E_{T^n x_0, T^{n+1} x_0}(Kt_1) = 1,$$
$$\lim_{n\to\infty} F_{x_n,Tx_n}(Kt_2) = \lim_{n\to\infty} E_{T^n x_0, T^{n+1} x_0}(Kt_2) = 1.$$

Now let $n \in N$, $t > 0$ and $K > 0$. We show that for any k with $k \geq 4$

$$\lim_{m\to\infty} F_{x_n,x_{n+k}}(t) \geq \lim_{m\to\infty} F_{x_n,x_{n+1}}\left(t - Kt_1 - Kt_2 - \sum_{j=1}^{k-3} \varphi_j(t)\right). \tag{9}$$

Since the mapping T is asymptotic regular, so we have

$$\lim_{n\to\infty} F_{x_n,x_{n+1}}(Kt_1) = 1,\ \lim_{n\to\infty} F_{x_{n+k},x_{n+k+1}}(Kt_2) = 1.$$

In this case we have

$$\begin{aligned}
&F_{x_n,x_{n+k-3}}(t - Kt_1 - Kt_2 - \varphi_{k-3}(t))\\
&\geq \Delta\left(F_{x_n,x_{n+k-4}}\left(t - Kt_1 - Kt_2 - \varphi_{k-3}(t) - \varphi_{k-4}(t)\right),\, F_{x_{n+k-4},x_{n+k-3}}(\varphi_{k-4}(t))\right)\\
&\geq \Delta\left(F_{x_n,x_{n+k-4}}\left(t - Kt_1 - Kt_2 - \varphi_{k-3}(t) - \varphi_{k-4}(t)\right), F_{x_n,x_{n+1}}(t)\right)\\
&\geq \cdots.
\end{aligned}$$

We continue this, the right side become

$$\Delta\left(\Delta\cdots\left(F_{x_n,x_{n+1}}\left(t - Kt_1 - Kt_2 - \sum_{j=1}^{k-3}\varphi_j(t)\right), F_{x_n,x_{n+1}}(t)\right)\cdots, F_{x_n,x_{n+1}}(t)\right).$$

Siince T is asymptotic regular, we have

$$\lim_{n\to\infty} F_{x_n,x_{n+1}}(t) = 1.$$

Since Δ is continous and $\Delta(a,1) = a$, for any $k \in N$ with $k \geq 4$, we have

$$\begin{aligned}
&\Delta\left(\lim_{n\to\infty} F_{x_n,x_{n+1}}\left(t - Kt_1 - Kt_2 - \sum_{j=1}^{k-3}\varphi_j(t)\right), \lim_{n\to\infty} F_{x_n,x_{n+1}}(t)\right)\\
&= \Delta\left(\lim_{n\to\infty} F_{x_n,x_{n+1}}\left(t - Kt_1 - Kt_2 - \sum_{j=1}^{k-3}\varphi_j(t)\right), 1\right)\\
&= \lim_{n\to\infty} F_{x_n,x_{n+1}}\left(t - Kt_1 - Kt_2 - \sum_{j=1}^{k-3}\varphi_j(t)\right).
\end{aligned}$$

Note that

$$\begin{aligned}
&F_{x_n,x_{n+k}}(t)\\
&= F_{x_n,x_{n+k}}\left(t - Kt_1 - Kt_2 - \varphi_{k-3}(t) + (Kt_1 + Kt_2 + \varphi_{k-3}(t))\right)\\
&\geq \Delta\big(F_{x_n,x_{n+k-3}}(t - Kt_1 - Kt_2 - \varphi_{k-3}(t)),\\
&\qquad F_{x_{n+k-3},x_{n+k}}\left(\varphi_{k-3}(t) + Kt_1 + Kt_2\right)\big)\\
&\geq \Delta(F_{x_n,x_{n+k-3}}(t - Kt_1 - Kt_2 - \varphi_{k-3}(t)),\\
&\qquad \Delta(F_{x_{n+k-2},x_{n+k-1}}(Kt_1),\, F_{x_{n+k-1},x_{n+k}}(Kt_2),\, F_{x_{n+k-3},x_{n+k-2}}\varphi_{k-3}(t)))\\
&\geq \Delta(F_{x_n,x_{n+k-3}}(t - Kt_1 - Kt_2 - \varphi_{k-3}(t)),\\
&\qquad \Delta(F_{x_{n+k-2},x_{n+k-1}}(Kt_1),\, F_{x_{n+k-1},x_{n+k}}(Kt_2),\, F_{x_n,x_{n+1}}(t))),
\end{aligned}$$

we have

$$\lim_{n\to\infty} F_{x_n,x_{n+k}}(t) \geq \lim_{n\to\infty} F_{x_n,x_{n+1}}\left(t - Kt_1 - Kt_2 - \sum_{j=1}^{k-3}\varphi_j(t)\right).$$

For $k = 1$, we have

$$\lim_{n\to\infty} F_{x_n,x_{n+k}}(t) = 1, \tag{10}$$

and for $k = 2$, since

$$\begin{aligned} &F_{x_n,x_{n+2}}(t) \\ &\geq \Delta(F_{x_n,x_{n+1}}(t - \varphi_1(t)), F_{x_{n+1},x_{n+2}}(\varphi_1(t))) \\ &\geq \Delta(F_{x_n,x_{n+1}}(t - \varphi_1(t)), F_{x_{n+1},x_{n+2}}(\varphi_1(t))) \end{aligned}$$

we have

$$\lim_{n\to\infty} F_{x_n,x_{n+2}}(t) \geq \lim_{n\to\infty} F_{x_n,x_{n+1}}(t - \varphi_1(t)), \tag{11}$$

and for $k = 3$, since

$$\begin{aligned} &F_{x_n,x_{n+3}}(t) \\ &\geq \Delta(F_{x_n,x_{n+1}}(t - \varphi_1(t) - \varphi_2(t)), F_{x_{n+1},x_{n+3}}(\varphi_1(t) + \varphi_2(t))) \\ &\geq \Delta(F_{x_n,x_{n+1}}(t - \varphi_1(t) - \varphi_2(t)), F_{x_{n+1},x_{n+2}}(\varphi_1(t)), F_{x_{n+2},x_{n+3}}(\varphi_2(t)))) \end{aligned}$$

we have

$$\lim_{n\to\infty} F_{x_n,x_{n+3}}(t) \geq \lim_{n\to\infty} F_{x_n,x_{n+1}}(t - \varphi_1(t) - \varphi_2(t)). \tag{12}$$

Put $\phi_{j+2}(t) = \varphi_j(t)$ and $\phi_1(t) = Kt_1$, $\phi_2(t) = Kt_2$, then $Kt_1 + Kt_2 + \sum_{j=1}^{k-3} \varphi_j(t) = \psi_1(t) + \psi_1(t) + \cdots + \psi_{k-1}(t)$ and we have

$$\lim_{n\to\infty} F_{x_n,x_{n+k}}(t) \geq \lim_{n\to\infty} F_{x_n,x_{n+1}} \left(t - \sum_{j=1}^{k-1} \phi_j(t) \right)$$

We show that sequence $\{x_n\}$ is Cauchy, that is,

$$\lim_{m,n\to\infty} F_{x_n,x_m}(t) = 1 \text{ for any } t > 0.$$

By assumption $t - \sum_{j=1}^{k-1} \phi_j(t) > 0$ and T is asymtotic regular, we have

$$\lim_{n\to\infty} F_{x_n,x_{n+1}} \left(t - \sum_{j=1}^{k-1} \phi_j(t) \right) = 1. \tag{13}$$

If $K = 0$, repeating the same arugument we have (13) as $\phi_j = \varphi_j$ $(j = 1, 2, \ldots, k - 1)$. In this case for any $k \geq 1$ and $\varepsilon > 0$ there exists $n_0 \in N$ such that, for any $n \geq n_0$ and $k \in N$, we have

$$F_{x_n,x_{n+k}}(t) > 1 - \varepsilon.$$

This proves that Cauchy condition for $\{x_n\}$. By completeness, $\{x_n\}$ converges to some $p \in X$, that is,

$$\lim_{n\to\infty} F_{x_n,p}(t) = 1, \text{ for any } t > 0.$$

The proofs of p is a fixed point of T and the uniqueness of a fixed point are same as that of Theorem 12. □

References

[1] B. S. Choudhury, P. N. Dutta, K. P. Das, A fixed point result in Menger spaces using a real function, Acta Math. Hungar. 122 (2008) 203–216.

[2] A. H. Alwan, Some fixed point theorems in quasi-metric spaces, AIP Conference Proceedings 2398, 060031 (2022); https://doi.org/10.1063/5.0093471 Published Online: 25 October mo2022

[3] I. Bhaumik, B. S. Choudhury, Iterative sequences of maps in a metric space with some chaotic properties. Bulletin of Mathematical Analysis and Applications ISSN: 1821-1291, URL: http://www.bmathaa.org Volume 3 Issue 4 (2011), Pages 116–122.

[4] R. K. Bisht, A note on the fixed point theorem of Górnicki, Journal of Fixed Point Theory and Applications 21:54 (2019) https://doi.org/10.1007/s11784-019-0695-x.

[5] Lj. B. Ćirić, Solving the Banach fixed point principle for nonlinear contractions in probabilistic metric spaces, Nonlinear Anal. 72 (2010) 2009–2018.

[6] J. -X. Fang, Common fixed point theorems of compatible and weakly compatible maps in Menger spaces, Nonlinear Anal. 71 (2009) 1833–1843.

[7] J. -X. Fang, On φ-contractions in probabilistic and fuzzy metric space, Fuzzy Sets and Systems 267 (2015) 86–99

[8] J. Górnicki, Remarks on asymptotic regularity and fixed points, Journal of Fixed Point Theory and Applications 21–29 (2019) https://doi.org/10.1007/s11784-019-0668-0.

[9] O. Hadzić, A fixed point theorem in Menger spaces, Publ. Inst. Math. (Beograd) T 20 (1979) 107–112.

[10] O. Hadžić, Some theorems on the fixed points in probabilistic metric and random normed spaces, Boll. Unione Mat. Ital. Sez. B (6) 1 (1982) 381–391.

[11] O. Hadzić, Fixed point theorems for multivalued mappings in probabilistic metric spaces, Fuzzy Sets and Systems 88 (1997) 219–226.

[12] O. Hadžić, E. Pap, Fixed Point Theory in Probabilistic Metric Spaces, in: Mathematics and its Applications, vol. 536, Kluwer Academic Publishers, Dordrecht, 2001.

[13] Jachymski. J, On probabilistic ϕ-contractions on Menger spaces, Nonlinear Analysis 73 (2010) 2199–2203.

[14] K. Meneger, Statistical metrics, Proc. Nat. Acad. Sci. USA 28 (1942) 535–537.

[15] H. K. Pathak, Y. J. Cho, S. M. Kang, Remarks on R-weakly commuting mappings and common fixed point theorems, Bulletin Korean Mathematical Society 34 (1997) 247–257.

[16] K. P. R. Sastry, S. V. R. Naidu, I. H. N. Rao, K. P. R. Rao, Common fixed points for asymptotically regular mappings, Indian Journal of Pure and Applied Mathematics 15-8(1984) 849-854.

[17] B. Schweizer, A. Sklar, Statistical metric spaces, Pacific J. Math. 10 (1960) 313–334.

[18] B. Schweizer, A. Sklar, E. Thorp, The metrization of statistical metric spaces, Pacific J. Math. 10 (1960) 673–675.

[19] V. M. Sehgal, Some fixed point theorems in functional analysis and probability, Ph.D. Thesis, Wayne State University, Detroit, MI, 1966.

[20] V. M. Sehgal, A.T. Bharucha-Reid, Fixed points of contraction mappings on PM-spaces, Math. Syst. Theory 6 (1972) 97–102.

[21] B. Schweizer, A. Sklar, Probabilistic Metric Spaces, Elsevier, North-Holland, New York, 1983, North Holland, New-York, Amesterdam, Oxford, 1983.

[22] C. Tian, G. Chen, Chaos of a sequence of maps in a metric space, Chaos, Solitons and Fractals 28 (2006) 1067–1075.

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