Weak and strong convergence theorems for monotone nonexpansive-type mappings in Banach spaces

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Abstract

In this paper, we prove weak and strong convergence theorems for finite monotone nonexpansive mappings in uniformly convex Banach spaces endowed with a partial order.

1 Introduction

Let E be a real Banach space, let C be a nonempty subset of E. A mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. For a mapping $T : C \to C$, we denote by $F(T)$ the set of *fixed points* of T, i.e., $F(T) = \{z \in C : Tz = z\}$.

Ran and Reurings [16] proved an analogue of the classical Banach contraction principle [5] in metric spaces endowed with a partial order. In particular, they show how this extension is useful when dealing with some special matrix equations (see also [11, 15, 22, 23]).

Mann [14] introduced an iteration process for approximation of fixed points of a mapping T in a Hilbert space as follows:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n$$

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for all $n \geq 1$, where $\{\alpha_n\}$ are sequences in $[0,1]$. Later, Reich [13] discussed this iteration process in a uniformly convex Banach space whose norm is Frechet differentiable. Bin Dehaish and Khamsi [8] proved a weak convergence theorem of Mann's type [14] iteration for monotone nonexpansive mappings in Banach spaces endowed with a partial order (see also [1, 18]).

In this paper, we prove weak convergence theorems for finite monotone nonexpansive mappings in uniformly convex Banach spaces endowed with a partial order.

2 Preliminaries and notations

Throughout this paper, we assume that E is a real Banach space with norm $\|\cdot\|$ and endowed with a *partial order* $\preceq$ compatible with the linear structure of E, that is,

$$x \preceq y \text{ implies } x+z \preceq y+z,$$

$$x \preceq y \text{ implies } \lambda x \preceq \lambda y$$

for every $x, y, z \in E$ and $\lambda \geq 0$. As usual we adopt the convention $x \succeq y$ if and only if $y \preceq x$. It follows that all *order intervals* $[x, \rightarrow) = \{z \in E : x \preceq z\}$ and $(\leftarrow, y] = \{z \in E : z \in E : z \preceq y\}$ are convex. Moreover, we assume that each order intervals $[x, \rightarrow)$ and $(\leftarrow, y]$ are closed. Recall that an order interval is any of the subsets $[a, \rightarrow) = \{x \in X; a \preceq x\}$ or $(\leftarrow, a] = \{x \in X; x \preceq a\}$. for any $a \in E$. As a direct consequence of this, the subset

$$[a,b] = \{x \in X; a \preceq x \preceq b\} = [a, \rightarrow) \cap (\leftarrow, b]$$

is also closed and convex for each $a, b \in E$.

Let E be a real Banach space with norm $\|\cdot\|$ and endowed with a partial order $\preceq$ compatible with the linear structure of E. We will say that this Banach space $(E, \|\cdot\|, \preceq)$ is an *ordered Banach space*. Let C be a nonempty subset of E. A mapping $T : C \to C$ is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. A mapping $T : C \to C$ is called *monotone* if

$$Tx \preceq Ty$$

for each $x, y \in C$ such that $x \preceq y$ (see also [8]). For a mapping $T : C \to C$, we denote by $F(T)$ the set of *fixed points* of T, i.e., $F(T) = \{z \in C : Tz = z\}$. For a mapping $T : C \to C$ and $\varepsilon > 0$, we define the set $F_\varepsilon(T)$ to be

$$F_\varepsilon(T) = \{x \in C : \|Tx - x\| \leq \varepsilon\}.$$

We denote by E^* the topological dual space of E. We denote by $\mathbb{N}$ and $\mathbb{Z}^+$ the set of all positive integers and the set of all nonnegative integers, respectively. We also denote by $\mathbb{R}$ and $\mathbb{R}^+$ the set of all real numbers and the set of all nonnegative real numbers, respectively.

We write $x_n \to x$ (or $\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in E converges strongly to x. We also write $x_n \rightharpoonup x$ (or w-$\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in E converges weakly to x. We also denote by $\langle y, x^* \rangle$ the value of $x^* \in E^*$ at $y \in E$. For a subset A of E, co$A$ and $\overline{\text{co}}A$ mean the convex hull of A and the closure of convex hull of A, respectively.

A Banach space E is said to be strictly convex if

$$\frac{\|x+y\|}{2} < 1$$

for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. In a strictly convex Banach space, we have that if

$$\|x\| = \|y\| = \|(1-\lambda)x + \lambda y\|$$

for $x, y \in E$ and $\lambda \in (0,1)$, then $x = y$. For every ε with $0 \leq \varepsilon \leq 2$, we define the modulus $\delta(\varepsilon)$ of convexity of E by

$$\delta(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} \ : \ \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \varepsilon\right\}.$$

A Banach space E is said to be uniformly convex if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. If E is uniformly convex, then for r, ε with $r \geq \varepsilon > 0$, we have $\delta\left(\frac{\varepsilon}{r}\right) > 0$ and

$$\left\|\frac{x+y}{2}\right\| \leq r\left(1 - \delta\left(\frac{\varepsilon}{r}\right)\right)$$

for every $x, y \in E$ with $\|x\| \leq r$, $\|y\| \leq r$ and $\|x-y\| \geq \varepsilon$. It is well-known that a uniformly convex Banach space is reflexive and strictly convex.

The following lemmas were proved in [10].

Lemma 2.1 ([10])**.** *Let E be a uniformly convex Banach space and let C be a nonempty bounded closed convex subset of E. Then, for each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\overline{\text{co}}F_\delta(T) \subset F_\varepsilon(T)$$

for every nonexpansive mapping T of C into itself.

Lemma 2.2 ([10])**.** *Let E be a uniformly convex Banach space and let C be a nonempty bounded closed convex subset of E. Then,*

$$\lim_{n\to\infty} \sup_{\substack{x\in C \\ T\in N(C)}} \left\|\frac{1}{n}\sum_{i=0}^{n-1} T^i x - T\left(\frac{1}{n}\sum_{i=0}^{n-1} T^i x\right)\right\| = 0$$

where $N(C)$ denotes the set of all nonexpansive mappings of C into itself.

The following theorem was proved in [9].

Theorem 2.3 ([9]). *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and let T be a nonexpansive mapping of C into itself. Let $\{x_n\}$ be a sequence in C such that it converges weakly to an element u in C and $\{x_n - Tx_n\}$ converges strongly to* 0. *Then, u is a fixed point of T.*

The following lemma was proved in [24].

Lemma 2.4 ([24]). *Let $p > 1$ and $r > 0$ be two fixed real numbers. Let E be a uniformly convex Banach space. Then, there is a continuous, strictly increasing and convex function g : $[0,\infty) \to [0,\infty)$ such that $g(0) = 0$ and*

$$\begin{aligned} &\|\lambda x + (1-\lambda)y\|^p \\ &\leq \lambda\|x\|^p + (1-\lambda)\|y\|^p - \{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}g(\|x-y\|) \end{aligned}$$

for all $x, y \in B_r$ and λ with $0\lambda < 1$ where $B_r = \{x \in E : \|x\| \leq r\}$

3 Lemmas

In this paper, we consider the following iteration process for approximation of common fixed points of finite nonexpansive mappings in a Banach space: Let C be a nonempty closed convex subset of an ordered Banach space E. Let $T_1, T_2, \ldots, T_r$ be finite nonexpansive mappings of C into itself such that $T_iT_j = T_jT_i$ for every $i, j = 1, 2, \ldots, r$ and $\bigcap_{i=1}^r F(T_i)$ is nonempty. For $x_1 \in C$, $\{x_n\}$ is defined by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) \cdot \frac{1}{(n+1)^r} \sum_{i_1=0}^{n} \cdots \sum_{i_r=0}^{n} T_1{}^{i_1} \cdots T_r{}^{i_r} x_n$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0,1]$ (see also [14]).

Let C be a nonempty subset of an ordered Banach space E and let T be a mapping of C into itself. A sequence $\{x_n\}$ in C is said to be an *approximate fixed point sequence* of a mapping T of C into itself if

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0$$

(see also [13, 19]). Let $T_1, T_2, \ldots, T_r$ be mappings of C into itself. A sequence $\{x_n\}$ in C is said to be an *approximate common fixed point sequence* of mappings $T_1, T_2, \ldots, T_r$ of C if for every $k = 1, 2, \ldots, r$,

$$\lim_{n\to\infty} \|x_n - T_k x_n\| = 0.$$

A sequence $\{x_n\}$ in E is said to be *monotone increasing* if

$$x_1 \preceq x_2 \preceq x_3 \preceq \cdots$$

(see also [8]).

The following lemma plays an important role in this paper (see also [3]).

Lemma 3.1. *Let E be a uniformly convex Banach space, let C be a nonempty bounded closed convex subset of E. Let $T_1, T_2, \dots, T_r$ be nonexpansive mappings of C into itself such that $T_iT_j = T_jT_i$ for every $i, j = 1, 2, \dots, r$. For any $n \in \mathbb{N}$, put*

$$S_n^{l_1,\dots,l_r}x = \frac{1}{n^r}\sum_{i_1=0}^{n-1}\cdots\sum_{i_r=0}^{n-1} {T_1}^{i_1+l_1}\cdots {T_r}^{i_r+l_r}x \quad \text{for every} \quad x \in C.$$

Then, for every $k = 1, 2, \dots, r$,

$$\lim_{n\to\infty} \sup_{\substack{x\in C \\ l_1,\dots,l_r\in\mathbb{Z}^+}} \left\| S_n^{l_1,\dots,l_r}x - T_k\left(S_n^{l_1,\dots,l_r}x\right)\right\| = 0. \tag{3.1}$$

Lemma 3.2 ([2])**.** *Let E be a Banach space and let C be a nonempty closed convex subset of E. Let $T_1, T_2, \dots, T_r$ be nonexpansive mappings of C into itself such that $T_iT_j = T_jT_i$ for every $i, j = 1, 2, \dots, r$ and $\bigcap_{i=1}^r F(T_i)$ is nonempty. Suppose $x_1 \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\frac{1}{(n+1)^r}\sum_{i_1=0}^{n}\cdots\sum_{i_r=0}^{n} {T_1}^{i_1}\cdots {T_r}^{i_r}x_n \quad \text{for every} \quad n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. Let w be a common fixed point of $T_1, T_2, \dots, T_r$. Then, $\lim_{n\to\infty} \|x_n - w\|$ exists.

Lemma 3.3 ([2])**.** *Let C be a nonempty closed convex subset of an ordered uniformly convex Banach space E. Let $T_1, T_2, \dots, T_r$ be monotone nonexpansive mappings of C into itself such that $T_iT_j = T_jT_i$ for every $i, j = 1, 2, \dots, r$, and $\bigcap_{i=1}^r F(T_i)$ is nonempty. Suppose $x_1 \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\frac{1}{(n+1)^r}\sum_{i_1=0}^{n}\cdots\sum_{i_r=0}^{n} {T_1}^{i_1}\cdots {T_r}^{i_r}x_n \quad \text{for every} \quad n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0,a]$ for some $0 < a < 1$. Then, $\{x_n\}$ is an approximate common fixed point sequence of mappings $T_1, T_2, \dots, T_r$ of C, i.e., for every $k = 1, 2, \dots, r$,

$$\lim_{n\to\infty} \|x_n - T_k x_n\| = 0.$$

Lemma 3.4 ([2])**.** *Let C be a nonempty closed convex subset of an ordered Banach space E. Let $T_1, T_2, \dots, T_r$ be monotone nonexpansive mappings of C into itself such that $T_iT_j = T_jT_i$ for every $i, j = 1, 2, \dots, r$, and $\bigcap_{i=1}^r F(T_i)$ is nonempty. Assume that $x \preceq T_k x$ for every $k = 1, 2, \dots, r$ and $x \in C$. Suppose $x_1 \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\frac{1}{(n+1)^r}\sum_{i_1=0}^{n}\cdots\sum_{i_r=0}^{n} {T_1}^{i_1}\cdots {T_r}^{i_r}x_n \quad \text{for every} \quad n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. Then, $\{x_n\}$ is monotone increasing.

4 Convergence theorems of Mann's type iteration

In this section, we provide the approximation of common fixed points of families of monotone nonexpansive mappings in ordered Banach spaces. It is connected with the theory of differential equations (see [12]). We assume that E is an ordered Banach space in the sense that

$$x \preceq y \text{ implies } x+z \preceq y+z,$$
$$x \preceq y \text{ implies } \lambda x \preceq \lambda y$$

for every $x, y, z \in E$ and $\lambda \geq 0$. Note that this is a very typical situation.

We get a weak convergence theorem for a family of monotone nonexpansive mappings in an ordered uniformly convex Banach space.

Theorem 4.1 ([2]). *Let C be a nonempty closed convex subset of an ordered uniformly convex Banach space E. Let $T_1, T_2, \ldots, T_r$ be monotone nonexpansive mappings of C into itself such that $T_iT_j = T_jT_i$ for every $i, j = 1, 2, \ldots, r$ and $\bigcap_{i=1}^r F(T_i)$ is nonempty. Let $x \in C$. Assume that $x \preceq T_k x$ for every $k = 1, 2, \ldots, r$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\frac{1}{(n+1)^r}\sum_{i_1=0}^{n}\cdots\sum_{i_r=0}^{n} {T_1}^{i_1}\cdots {T_r}^{i_r} x_n \quad \textit{for every} \quad n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. If $\{\alpha_n\}$ is chosen so that $\alpha_n \in [0,a]$ for some a with $0 < a < 1$, then the sequence $\{x_n\}$ converges weakly to a point of $\bigcap_{i=1}^r F(T_i)$.

Theorem 4.2. *Let C be a nonempty closed convex subset of an ordered uniformly convex Banach space E. Let $T_1, T_2, \ldots, T_r$ be monotone nonexpansive mappings of C into itself such that $T_iT_j = T_jT_i$ for every $i, j = 1, 2, \ldots, r$ and $\bigcap_{i=1}^r F(T_i)$ is nonempty. Let $x \in C$. Assume that $x \preceq T_k x$ for every $k = 1, 2, \ldots, r$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\frac{1}{(n+1)^r}\sum_{i_1=0}^{n}\cdots\sum_{i_r=0}^{n} {T_1}^{i_1}\cdots {T_r}^{i_r} x_n \quad \textit{for every} \quad n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. Let $\{\alpha_n\}$ be a sequence in $[0,1]$. Then, $\{x_n\}$ converges strongly to a point of $\bigcap_{i=1}^r F(T_i)$ if and only if $\lim_{n\to\infty} d\Big(x_n, \bigcap_{i=1}^{r} F(T_i)\Big) = 0$

Using Theorem 4.1, we get some convergence theorems for monotone nonexpansive mappings in ordered uniformly convex Bxanach spaces (see also [18]).

Theorem 4.3 ([2]). *Let C be a nonempty closed convex subset of an ordered uniformly convex Banach space E. Let S and T be monotone nonexpansive mappings of C into itself such that $ST = TS$ and $F(S) \cap F(T) \neq \emptyset$. Let $x \in C$. Assume that $x \preceq Tx$ and $x \preceq Sx$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\frac{1}{(n+1)^2}\sum_{i=0}^{n}\sum_{j=0}^{n} S^i T^j x_n \quad \textit{for every} \quad n \in \mathbb{N},$$

where $\{\alpha_n\}$ *is a sequence in* $[0,1]$. *If* $\{\alpha_n\}$ *is chosen so that* $\alpha_n \in [0,a]$ *for some* a *with* $0 < a < 1$, $\{x_n\}$ *converges weakly to a point of* $F(T) \cap F(S)$.

Theorem 4.4 ([2]). *Let* C *be a nonempty closed convex subset of an ordered uniformly convex Banach space* E *and let* T *be a monotone nonexpansive mapping of* C *into itself such that* $F(T) \neq \emptyset$. *Let* $x \in C$. *Assume that* $x \preceq Tx$. *Suppose* $x_1 = x \in C$ *and* $\{x_n\}$ *is given by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\frac{1}{n+1}\sum_{i=0}^{n} T^i x_n \quad \text{for every} \quad n \in \mathbb{N},$$

where $\{\alpha_n\}$ *is a sequence in* $[0,1]$. *If* $\{\alpha_n\}$ *is chosen so that* $\alpha_n \in [0,a]$ *for some* a *with* $0 < a < 1$, $\{x_n\}$ *converges weakly to a point of* $F(T)$.

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