GLOBAL UNIQUE SOLUTION TO THE KELLER–SEGEL–NAVIER–STOKES SYSTEM WITH NONLINEAR BOUNDARY CONDITIONS

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1. Introduction

This article gives a summary of [9]. We consider the Keller–Segel–Navier–Stokes system with *nonlinear* boundary conditions in a bounded smooth domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$;

$$
(1.1) \qquad \begin{cases} \partial_t n = \Delta n - \nabla \cdot (nS(t,x)\nabla c) - \boldsymbol{u} \cdot \nabla n, & t > 0,\ x \in \Omega, \\ \partial_t c = \Delta c - c + n - \boldsymbol{u} \cdot \nabla c, & t > 0,\ x \in \Omega, \\ \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \Delta \boldsymbol{u} - \nabla p + n\nabla\varphi + \boldsymbol{f}, & t > 0,\ x \in \Omega, \\ \nabla \cdot \boldsymbol{u} = 0, & t > 0,\ x \in \Omega, \\ \nabla n \cdot \boldsymbol{\nu} = nS(t,x)\nabla c \cdot \boldsymbol{\nu}, \quad \nabla c \cdot \boldsymbol{\nu} = \boldsymbol{u} = 0, & t > 0,\ x \in \partial\Omega, \\ (n, c, \boldsymbol{u})(0, x) = (n_0, c_0, \boldsymbol{u}_0)(x), & x \in \Omega. \end{cases}
$$

Here, $n = n(t,x)$, $c = c(t,x)$, $\boldsymbol{u} = \boldsymbol{u}(t,x)$, and $p = p(t,x)$ stand for the unknown functions that describe for the density of the cell, the concentration of the chemo-attractant, the velocity of the fluid, and the pressure, respectively, whereas the potential function $\varphi = \varphi(x)$ and the vector-valued function $\boldsymbol{f} = \boldsymbol{f}(t,x)$ are assumed to be given and $(n_0, c_0, \boldsymbol{u}_0)(x)$ denotes the given initial data. The *tensor-valued* function $S = S(t,x)$ means the chemotactic sensitivity, which is also assumed to be given. A unit outer normal to $\partial\Omega$ is denoted by $\boldsymbol{\nu} = \boldsymbol{\nu}(x)$. From the biological viewpoint, it is reasonable to expect that n_0 and c_0 are non-negative in Ω and investigate that n and c are non-negative in $(0,\infty) \times \Omega$.

System (1.1) may be regarded as a generalization of the standard Keller–Segel–Navier–Stokes system in the case that the evolution of the chemoattractant is (essentially) governed by production through cells, which is motivated by a recent modeling result [11]. In fact, this study suggests that the motion of bacteria near surfaces has rotational components in the cross-diffusion flux. To the best of my knowledge, there are quite few results on the global existence and uniqueness result for (1.1) with a general S and the asymptotic behavior of the global solutions, in particular, for the *general dimensions* $N \geq 2$. Notice that, in the case that $S \equiv I$ (the identity matrix) or S is replaced by a scalar function, there are abundant results for (1.1), e.g., on the local/global existence, large-time behavior, and blow-up of solutions. However, there are few studies in the case that S is a general matrix. Indeed, the introduction of general tensor-valued sensitivities S induces difficulty due to the destruction of the natural energy structure arising from (1.1), and hence the proof of the existence of (weak) solutions to (1.1) requires technical efforts. For instance, the nonlinear boundary condition $\nabla n \cdot \boldsymbol{\nu} = nS(t,x)\nabla c \cdot \boldsymbol{\nu}$ on $(0,\infty) \times \Omega$ is harmful when we use integration by parts, i.e., when we establish the energy estimates. Notice that this nonlinear boundary condition also prevents us from formulating (1.1) as an abstract Cauchy problem so that we may *not* rewrite System (1.1) in the integral form. To overcome this difficulty, an approximation argument was

2020 *Mathematics Subject Classification.* Primary; 35Q92, Secondary; 35K05, 35K61, 35Q35, 76D03, 92C17.

Key words and phrases. Keller–Segel–Navier–Stokes system; Nonlinear boundary conditions; Maximal regularity; Global well-posedness.

often used. Precisely speaking, by introducing the family of cut-off functions that vanish near the boundary, one may regularize (1.1) as the corresponding system with the homogeneous boundary conditions, i.e., n_η (the regularization of n) satisfies $\nabla n_\eta \cdot \boldsymbol{\nu} = 0$ on $\partial\Omega$. However, it should be emphasized that the uniqueness of the global classical or weak solutions was not discussed in *all* the existing studies even for the case $N = 2$.

Our aim is to show the global well-posedness of (1.1) in the sense of Hadamard, i.e., we also verify that the solution depends continuously on the data. In particular, we show that (1.1) admits a *unique* global strong solution provided that the initial data are small, and if given functions are suitably regular, we also investigate the smoothness and non-negativity issues for the global solution. In contrast to the previous works, we rely on the maximal regularity technique so that we do *not* need to approximate (1.1) by the corresponding system with the homogeneous boundary conditions. Namely, our approach enables one to deal with the system more directly.

2. Main results

Before giving our main results, we shall introduce the abstract form of System (1.1). To this end, we shall introduce several notations. For $1 \le r \le \infty$ and $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, denote the usual Lebesgue spaces and Sobolev spaces defined on Ω by $L^r(\Omega)$ and $W^{k,r}(\Omega)$, respectively. We agree that $W^{0,r}(\Omega) := L^r(\Omega)$ in the case of $k = 0$. Moreover, we also set

$$\begin{aligned} L^r_0(\Omega) &:= \left\{ \psi \in L^r(\Omega) \;\middle|\; \int_\Omega \psi(x)\,dx = 0 \right\}, \\ W^{k,r}_\nu(\Omega) &:= \left\{ \psi \in W^{k,r}(\Omega) \;\middle|\; \nabla\psi \cdot \boldsymbol{\nu} = 0 \text{ on } \partial\Omega \right\} \quad \text{for } k \ge 2 \end{aligned}$$

and introduce the Neumann–Laplacian $-\Delta_{\mathrm{N},1} := -\Delta$ in $W^{1,r}(\Omega)$ with the domain $D(-\Delta_{\mathrm{N},1}) := W^{3,r}_\nu(\Omega)$, where the boundary condition $\nabla\psi \cdot \boldsymbol{\nu} = 0$ on $\partial\Omega$ is regarded as $(\gamma\nabla\psi)(x) \cdot \boldsymbol{\nu}(x) = 0$ a.e. $x \in \partial\Omega$ with a suitable trace operator γ (see Remark 2.2 (ii) below). Let $C^\infty_0(\Omega)$ be the set of smooth functions which have a compact support in Ω and let $C^\infty_{0,\sigma}(\Omega)$ be the set of $\boldsymbol{\psi} \in C^\infty_0(\Omega)^N$ such that $\nabla \cdot \boldsymbol{\psi} = 0$ on Ω. Then $W^{1,r}_0(\Omega)$ and $L^r_\sigma(\Omega)$ denote the $W^{1,r}(\Omega)$-closures of $C^\infty_0(\Omega)$ and the $L^r(\Omega)$-closures of $C^\infty_{0,\sigma}(\Omega)$ for $1 < r < \infty$, respectively. With the aid of the Helmholtz projection operator $P : L^r(\Omega)^N \to L^r_\sigma(\Omega)$ for $1 < r < \infty$, we define the Stokes operator in $L^r_\sigma(\Omega)$ by $A := -P\Delta : D(A) \to L^r_\sigma(\Omega)$ with the domain $D(A) := L^r_\sigma(\Omega) \cap W^{1,r}_0(\Omega)^N \cap W^{2,r}(\Omega)^N$. Then, we may rewrite System (1.1) to the following form:

$$\begin{cases} \partial_t n - \Delta n = -\nabla \cdot (nS(t,x)\nabla c) - \boldsymbol{u} \cdot \nabla n, & t > 0,\ x \in \Omega, \\ \partial_t c + (1 - \Delta_{\mathrm{N},1})c = n - \boldsymbol{u} \cdot \nabla c, & t > 0,\ x \in \Omega, \\ \partial_t \boldsymbol{u} + A\boldsymbol{u} = -P(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + P(n\nabla\varphi) + P\boldsymbol{f}, & t > 0,\ x \in \Omega, \\ \nabla n \cdot \boldsymbol{\nu} = nS(t,x)\nabla c \cdot \boldsymbol{\nu}, & t > 0,\ x \in \partial\Omega, \\ (n, c, \boldsymbol{u})(0,x) = (n_0, c_0, \boldsymbol{u}_0)(x), & x \in \Omega. \end{cases} \tag{2.1}$$

Our first result is on the unique existence of global strong solutions of (2.1) for small given data. To simplify the notation, for $1 \le q \le \infty$ and a Banach space X, we abbreviate $\|\cdot\|_{L^q(X)} := \|\cdot\|_{L^q(0,\infty;X)}$ and $\|\cdot\|_{W^{1,q}(X)} := \|\cdot\|_{W^{1,q}(0,\infty;X)}$. We also set the mean value $\overline{n}_0$ of the mass of the initial density n_0 as $\overline{n}_0 := |\Omega|^{-1}\int_\Omega n_0(x)\,dx$. Our result on the global well-posedness reads as follows.

Theorem 2.1 ([9, Thm. 1.1]). *Let Ω be a bounded smooth domain in $\mathbb{R}^N$, $N \ge 2$. In addition, let $N < r < \infty$ and $2 < q < \infty$ satisfy $1/r + 2/q \ne 1$ and let $0 < \lambda_1 < \min\{1, \lambda_{\mathrm{N}}(\Omega)/q\}$ and*

$\lambda_2 \in [0, \lambda_1] \cap [0, \lambda_{\mathrm{D}}(\Omega)/q)$, *where* $\lambda_{\mathrm{N}}(\Omega), \lambda_{\mathrm{D}}(\Omega) \in (0, \infty)$ *are given by*

$$\lambda_{\mathrm{N}}(\Omega) \coloneqq \inf_{\psi \in (W^{1,2} \cap L^2_0)(\Omega) \setminus \{0\}} \frac{\|\nabla \psi\|^2_{L^2(\Omega)}}{\|\psi\|^2_{L^2(\Omega)}}, \quad \lambda_{\mathrm{D}}(\Omega) \coloneqq \inf_{\psi \in W^{1,2}_0(\Omega) \setminus \{0\}} \frac{\|\nabla \psi\|^2_{L^2(\Omega)}}{\|\psi\|^2_{L^2(\Omega)}}. \tag{2.2}$$

Assume that the initial data $n_0 \in B^{2-2/q}_{r,q}(\Omega)$, $c_0 \in B^{3-2/q}_{r,q}(\Omega)$, *and* $\boldsymbol{u}_0 \in L^r_\sigma(\Omega) \cap B^{2-2/q}_{r,q}(\Omega)^N$ *satisfy the conditions* $\nabla c_0 \cdot \boldsymbol{\nu} = \boldsymbol{u}_0 = 0$ *on* $\partial\Omega$ *and the given functions* φ, $\boldsymbol{f}$, *and* S *satisfy*

$$\nabla \varphi \in L^r(\Omega)^N, \qquad e^{\lambda_2 t} \boldsymbol{f} \in L^q(0, \infty; L^r(\Omega)^N),$$
$$S \in L^\infty(0, \infty; W^{1,r}(\Omega)^{N^2}), \qquad \partial_t S \in L^q(0, \infty; L^r(\Omega)^{N^2}).$$

Furthermore, in the case $1/r + 2/q < 1$, *suppose that* n_0, c_0, *and* S *fulfill the additional condition*

$$\nabla n_0 \cdot \boldsymbol{\nu} = n_0 S(0, \cdot) \nabla c_0 \cdot \boldsymbol{\nu} \quad \text{on } \partial\Omega.$$

Then there exists a constant $\varepsilon > 0$ *independent of* n_0, c_0, $\boldsymbol{u}_0$, φ, $\boldsymbol{f}$, *and* S *such that if*

$$\begin{aligned} [\![(n_0, c_0, \boldsymbol{u}_0, \boldsymbol{f})]\!] &\coloneqq \|n_0\|_{B^{2-2/q}_{r,q}(\Omega)} + \|c_0\|_{B^{3-2/q}_{r,q}(\Omega)} + \|\boldsymbol{u}_0\|_{B^{2-2/q}_{r,q}(\Omega)} + \|e^{\lambda_2 t} \boldsymbol{f}\|_{L^q(L^r(\Omega))} \\ &\le \varepsilon (1 + \|S\|_{L^\infty(W^{1,r}(\Omega))} + \|\partial_t S\|_{L^q(L^r(\Omega))})^{-1} (1 + \|\nabla \varphi\|_{L^r(\Omega)})^{-2}, \end{aligned}$$

then System (2.1) *has a unique global strong solution* $(n, c, \boldsymbol{u})$ *satisfying*

$$\begin{cases} e^{\lambda_1 t}(n - \overline{n}_0) \in L^q(0, \infty; (W^{2,r} \cap L^r_0)(\Omega)) \cap W^{1,q}(0, \infty; L^r_0(\Omega)), \\ e^{\lambda_1 t}\{c - (1 - e^{-t})\overline{n}_0\} \in L^q(0, \infty; W^{3,r}_\nu(\Omega)) \cap W^{1,q}(0, \infty; W^{1,r}(\Omega)), \\ e^{\lambda_2 t} \boldsymbol{u} \in L^q(0, \infty; D(A)) \cap W^{1,q}(0, \infty; L^r_\sigma(\Omega)). \end{cases} \tag{2.3}$$

Moreover, it holds that

$$\|(n - \overline{n}_0, c - (1 - e^{-t})\overline{n}_0, \boldsymbol{u})\|_{\mathbb{E}} \le C(1 + \|\nabla \varphi\|_{L^r(\Omega)}) [\![(n_0, c_0, \boldsymbol{u}_0, \boldsymbol{f})]\!],$$

where

$$\begin{aligned} \|(\widehat{n}, \widehat{c}, \widehat{\boldsymbol{u}})\|_{\mathbb{E}} &\coloneqq \|e^{\lambda_1 t} \widehat{n}\|_{L^q(W^{2,r}(\Omega)) \cap W^{1,q}(L^r(\Omega))} + \|e^{\lambda_1 t} \widehat{c}\|_{L^q(W^{3,r}(\Omega)) \cap W^{1,q}(W^{1,r}(\Omega))} \\ &\quad + \|e^{\lambda_2 t} \widehat{\boldsymbol{u}}\|_{L^q(W^{2,r}(\Omega)) \cap W^{1,q}(L^r(\Omega))} \end{aligned}$$

and $C > 0$ *is a constant independent of* ε, n_0, c_0, $\boldsymbol{u}_0$, φ, $\boldsymbol{f}$, S, n, c, *and* $\boldsymbol{u}$. *Suppose additionally that* $(n^*, c^*, \boldsymbol{u}^*)$ *is a global solution of* (2.1) *with the given data* $(n_0, c_0, \boldsymbol{u}_0, \boldsymbol{f})$ *replaced by* $(n^*_0, c^*_0, \boldsymbol{u}^*_0, \boldsymbol{f}^*)$ *such that*

$$[\![(n^*_0, c^*_0, \boldsymbol{u}^*_0, \boldsymbol{f}^*)]\!] \le \varepsilon (1 + \|S\|_{L^\infty(W^{1,r}(\Omega))} + \|\partial_t S\|_{L^q(L^r(\Omega))})^{-1} (1 + \|\nabla \varphi\|_{L^r(\Omega)})^{-2}.$$

Then the Lipschitz continuity of the solution mapping is obtained in the following sense:

$$\begin{aligned} &\|((n - \overline{n}_0) - (n^* - \overline{n}^*_0), c - (1 - e^{-t})\overline{n}_0 - \{c^* - (1 - e^{-t})\overline{n}^*_0\}, \boldsymbol{u} - \boldsymbol{u}^*)\|_{\mathbb{E}} \\ &\le C(1 + \|\nabla \varphi\|_{L^r(\Omega)}) [\![(n_0 - n^*_0, c_0 - c^*_0, \boldsymbol{u}_0 - \boldsymbol{u}^*_0, \boldsymbol{f} - \boldsymbol{f}^*)]\!]. \end{aligned}$$

Remark 2.2. We make some comments on Theorem 2.1.

(i) Since the solution $(n, c, \boldsymbol{u})$ belongs to the maximal regularity class (2.3), we infer from the embeddings $B^{2-2/q}_{r,q}(\Omega) \subset C(\overline{\Omega})$ and $B^{3-2/q}_{r,q}(\Omega) \subset C^1(\overline{\Omega})$ and the usual property of the trace space that

$$\begin{cases} e^{\lambda_1 t}(n - \overline{n}_0) \in BUC([0, \infty); C(\overline{\Omega})), \\ e^{\lambda_1 t}\{c - (1 - e^{-t})\overline{n}_0\} \in BUC([0, \infty); C^1(\overline{\Omega})), \\ e^{\lambda_2 t} \boldsymbol{u} \in BUC([0, \infty); C(\overline{\Omega})^N). \end{cases}$$

Thus, we have the following *exponential decay* for n and c:

$$\|n(t,\,\cdot\,)-\overline{n}_0\|_{L^\infty(\Omega)}+\|c(t,\,\cdot\,)-(1-e^{-t})\overline{n}_0\|_{L^\infty(\Omega)}=O(e^{-\lambda_1 t})\quad \text{as } t\to\infty.$$

Concerning the asymptotic behaviors of $\boldsymbol{u}$, we also obtain $\|\boldsymbol{u}(t,\,\cdot\,)\|_{L^\infty(\Omega)}=O(e^{-\lambda_2 t})$ as $t\to\infty$ under a suitable exponential time-weighted assumption for $\boldsymbol{f}$. We remark that our result includes the case of $\lambda_2=0$; although we may not expect any decay properties of $\boldsymbol{u}$, global solutions would be constructed even under weaker conditions $\boldsymbol{f}\in L^q(0,\infty;L^r(\Omega)^N)$.

(ii) The tensor-valued function S is motivated by the model including bacterial chemotaxis near surfaces that contain rotational components orthogonal to the signal gradient [11], where the typical choice of S is

$$S=a\begin{pmatrix}1&0\\0&1\end{pmatrix}+b\begin{pmatrix}0&-1\\1&0\end{pmatrix},\qquad a>0,\ b\in\mathbb{R}$$

in the case of $N=2$. Clearly, if $S\equiv I$ (the identity matrix), then System (1.1) reduces to the standard Keller–Segel–Navier–Stokes system with chemotactic cross-diffusion being directed to increasing signal concentrations. Moreover, note that we may obtain unique global strong solutions *without* assuming any smallness of φ and S by taking n_0, c_0, $\boldsymbol{u}_0$, and $\boldsymbol{f}$ sufficiently small depending on φ and S. Such a relaxation of the smallness condition of φ relies on the method of [3, 8].

Under the same assumption as in Theorem 2.1, we may prove that the unique global strong solution $(n,c,\boldsymbol{u})$ to (2.1) constructed in Theorem 2.1 is indeed a *classical solution* in the pointwise sense provided that the given functions φ, $\boldsymbol{f}$, and S satisfy some additional regularity conditions. As a by-product of such a result, we also obtain the *non-negativity* of solutions n and c, which must be observed from the viewpoint of the biological model since n and c stand for the density and the concentration, respectively.

Theorem 2.3 ([9, Thm. 1.3])**.** *Assume that all assumptions in Theorem 2.1 are fulfilled. Suppose additionally that*

$$\varphi\in C^{1+\theta_0}(\overline{\Omega}),\quad \boldsymbol{f}\in C((0,\infty);C^{\theta_0}(\overline{\Omega})^N),$$
$$S\in C((0,\infty);C^{1+\theta_0}(\overline{\Omega})^{N^2})$$

for some $0<\theta_0<\min\{1,2-2/q-N/r\}$*. Then, the solution* $(n,c,\boldsymbol{u})$ *obtained in Theorem 2.1 also admits the regularities*

$$\begin{cases} n\in BUC([0,\infty);C(\overline{\Omega}))\cap C((0,\infty);C^{2+\theta}(K))\cap C^1((0,\infty);C^{\theta}(K)),\\ c\in BUC([0,\infty);C^1(\overline{\Omega}))\cap C((0,\infty);C^{4+\theta}(K))\cap C^1((0,\infty);C^{2+\theta}(K)),\\ \boldsymbol{u}\in BUC([0,\infty);C(\overline{\Omega})^N)\cap C((0,\infty);C^{2+\theta}(\overline{\Omega})^N)\cap C^1((0,\infty);C^{\theta}(\overline{\Omega})^N)\end{cases}\tag{2.4}$$

for all $0<\theta<\theta_0$*, where* $K\subset\Omega$ *is an arbitrary compact subset. Moreover, there holds*

$$\int_\Omega n(t,x)\,dx=\int_\Omega n_0(x)\,dx,\quad \int_\Omega c(t,x)\,dx=e^{-t}\int_\Omega c_0(x)\,dx+(1-e^{-t})\int_\Omega n_0(x)\,dx$$

for all $0<t<\infty$*. In particular, if* $N/r+2/q<1$ *and the initial data* n_0 *and* c_0 *are non-negative in* Ω*, then* n *and* c *are non-negative in* $(0,\infty)\times\Omega$*.*

Remark 2.4. There are a few comments on Theorem 2.3.

(i) The regularity result (2.4) will be proved by the standard bootstrap argument but it is *not* so simple to show the smoothing effects acting on n due to the nonlinearity of the boundary condition $\nabla n\cdot\boldsymbol{\nu}=nS(t,x)\nabla c\cdot\boldsymbol{\nu}$ on $\partial\Omega$. To overcome this difficulty, we need to introduce a suitable cut-off function to ignore the effects near the boundary $\partial\Omega$. Then, extending the function on Ω to the function on $\mathbb{R}^N$, we may consider the integral form with the aid of the usual heat semigroup $e^{t\Delta}$ defined on $\mathbb{R}^N$ to prove that n is regular in space and time. This is the reason why the compact subset $K \subset \Omega$ appears in (2.4).

(ii) In the last assertion of Theorem 2.3, the condition $N/r + 2/q < 1$ is mainly used to ensure that the nonlinear boundary condition $\nabla n \cdot \boldsymbol{\nu} = nS(t,x)\nabla c \cdot \boldsymbol{\nu}$ makes sense in $C(\partial\Omega)$.

As mentioned before, while the global existence of solutions under smallness assumptions has been established in prior works such as [2, 12], our work not only reproduces such results but also ensures the *uniqueness* of global solutions within an appropriate analytical framework. A further noteworthy aspect of our results lies in the fact that the global solutions that we construct are rigorously subject to the nonlinear boundary condition $\nabla n \cdot \boldsymbol{\nu} = nS(t,x)\nabla c \cdot \boldsymbol{\nu}$ in the $L^r(\partial\Omega)$-sense. This stems directly from the regularity properties articulated in (2.3), which, in turn, ensure the applicability of the trace operator. This achievement attains particular significance when juxtaposed with earlier contributions, such as those in [2, 12]. In fact, the weak solution framework delineated in [2, Def. 1] is devoid of explicit information regarding the nonlinear boundary condition, since it relies exclusively on test functions compactly supported in Ω. Hence, the results in [2] leave unresolved the question of whether the nonlinear boundary condition $\nabla n \cdot \boldsymbol{\nu} = nS(t,x)\nabla c \cdot \boldsymbol{\nu}$ is certainly satisfied. As a trade-off of our framework necessitating more stringent assumptions compared to those in [2, 12], it should be emphasized that these conditions allow us to construct a *unique* regular global solution. Thus, we believe that this will enable us a more robust characterization of the interplay between the nonlinear boundary conditions and the governing dynamics.

3. Sketch of the proofs

3.1. **Linear theory.** A crucial point in the linear theory is to establish the maximal regularity theorem for the linear heat equation with the *inhomogeneous* Neumann boundary conditions. Roughly speaking, this can be shown by using the argument due to Shibata [7] together with the fact that the Neumann–Laplacian generates an analytic C_0-semigroup of negative exponential type.

Theorem 3.1 ([9, Thm. 3.1]). *Let $r, q \in (1,\infty)$ satisfy $1/r + 2/q \neq 1$ and let $0 \le \lambda < \lambda_{\mathrm{N}}(\Omega)/q$, where $\lambda_{\mathrm{N}}(\Omega) > 0$ is given by (2.2). Assume that F_0, F_{E}, and $\boldsymbol{F}_{\mathrm{B}}$ satisfy*

$$\begin{cases} F_0 \in (B^{2-2/q}_{r,q} \cap L^r_0)(\Omega), \quad e^{\lambda t} F_{\mathrm{E}} \in L^q(0,\infty; L^r_0(\Omega)), \\ e^{\lambda t}\boldsymbol{F}_{\mathrm{B}} \in L^q(0,\infty; W^{1,r}(\Omega)^N) \cap W^{1,q}(0,\infty; L^r(\Omega)^N). \end{cases}$$

Furthermore, if $1/r + 2/q < 1$, assume that F_0 and $\boldsymbol{F}_{\mathrm{B}}$ fulfill the additional condition $\nabla F_0 \cdot \boldsymbol{\nu} = \boldsymbol{F}_{\mathrm{B}}(0,\,\cdot\,)\cdot \boldsymbol{\nu}$ on $\partial\Omega$. Then there exists a unique global strong solution U of the equations

$$\begin{cases} \partial_t U - \Delta U = -\nabla \cdot \boldsymbol{F}_{\mathrm{B}} + F_{\mathrm{E}}, & t > 0,\ x \in \Omega, \\ \nabla U \cdot \boldsymbol{\nu} = \boldsymbol{F}_{\mathrm{B}} \cdot \boldsymbol{\nu}, & t > 0,\ x \in \partial\Omega, \\ U(0,x) = F_0(x), & x \in \Omega \end{cases} \tag{3.1}$$

such that $e^{\lambda t} U \in \mathbb{E}^0_{q,r}(-\Delta_{\mathrm{N}})$. Moreover, it holds that

$$\|e^{\lambda t} U\|_{\mathbb{E}^0_{q,r}} \le C \left(\|F_0\|_{B^{2-2/q}_{r,q}(\Omega)} + \|e^{\lambda t} F_{\mathrm{E}}\|_{L^q(L^r(\Omega))} + \|e^{\lambda t}\boldsymbol{F}_{\mathrm{B}}\|_{L^q(W^{1,r}(\Omega)) \cap W^{1,q}(L^r(\Omega))} \right),$$

where $C > 0$ is a constant independent of F_0, F_{E}, $\boldsymbol{F}_{\mathrm{B}}$, and U.

Notice that Theorem 3.1 is different from the standard result on the maximal regularity theorem for the linear heat equation with an inhomogeneous Neumann boundary condition (cf. Prüss and Simonett [5, Thm. 6.3.2]) since the left-hand side is given by $\partial_t U - \Delta U$ and since the solution to (3.1) decays exponentially as $t \to \infty$ by taking $\lambda > 0$. Indeed, recall that the Neumann–Laplacian defined on $L^r(\Omega)$ with the domain $W^{2,r}_{\nu}(\Omega)$ does *not* admit 0 as its resolvent, but if one replaces $L^r(\Omega)$ with $L^r_0(\Omega)$, then 0 becomes the resolvent of the Neumann–Laplacian. Hence, one may expect to observe that the Neumann–Laplacian admits maximal regularity on the semi-infinite time interval $(0,\infty)$. Although this fact seems to be standard, the proof of Theorem 3.1 is partly hard to find in the common literature. This result may be proved by following the approach due to Shibata [7, Sec. 4] together with the general theory for maximal regularity [5, Thm. 6.3.2]. Notice that the crucial point here is that the domain of the Neumann–Laplacian is $(W^{2,r}_{\nu} \cap L^r_0)(\Omega)$ instead of $W^{2,r}_{\nu}(\Omega)$, which is completely different from the operator $-\Delta_{\mathrm{N},1}$ introduced in Section 1. Indeed, this replacement induces the *invertibility* of the Neumann–Laplacian in $L^r_0(\Omega)$. To show Theorem 3.1, we use the following proposition.

Proposition 3.2 ([9, Prop. 3.2])**.** *Let $1 < r < \infty$ and define the operator $-\Delta_{\mathrm{N},0} := -\Delta$ with the domain*

$$D(-\Delta_{\mathrm{N},0}) := (W^{2,r}_{\nu} \cap L^r_0)(\Omega) = \left\{ \psi \in W^{2,r}(\Omega) \,\middle|\, \nabla\psi \cdot \boldsymbol{\nu} = 0 \text{ on } \partial\Omega, \quad \int_{\Omega} \psi(x)\, dx = 0 \right\}.$$

Then the following statements hold:

(i) *The resolvent set $\rho(\Delta_{\mathrm{N},0}) \subset \mathbb{C}$ of $\Delta_{\mathrm{N},0}$ satisfies*

$$[0,\infty) \subset \mathbb{C} \setminus (-\infty, -\lambda_{\mathrm{N}}(\Omega)] \subset \rho(\Delta_{\mathrm{N},0}),$$

where $\lambda_{\mathrm{N}}(\Omega) > 0$ is given by (2.2).

(ii) *The domain $D(-\Delta_{\mathrm{N},0})$ is dense in $L^r_0(\Omega)$.*

By virtue of Proposition 3.2, we see that the Neumann–Laplacian $-\Delta_{\mathrm{N},0} : D(-\Delta_{\mathrm{N},0}) \to L^r_0(\Omega)$ generates a bounded analytic C_0-semigroup $e^{t\Delta_{\mathrm{N},0}} : L^r_0(\Omega) \to L^r_0(\Omega)$ such that

$$\|e^{t\Delta_{\mathrm{N},0}}\psi\|_{L^r(\Omega)} \le Ce^{-\lambda_{\mathrm{N}}(\Omega)t}\|\psi\|_{L^r(\Omega)} \tag{3.2}$$

for all $0 < t < \infty$ and $\psi \in L^r_0(\Omega)$, where $C > 0$ is a constant independent of t and ψ. It should be noted that although this consequence (3.2) has already been given by Winkler [10, Lem. 1.3 (i)], another approach by Proposition 3.2 is taken here so that the resolvent set $\rho(\Delta_{\mathrm{N},0})$ is characterized in detail. Using $e^{t\Delta_{\mathrm{N},0}}$, we next show the following proposition by simple calculations. Note that the uniqueness assertion will play an important role in the proof of Theorem 3.1 since it provides a *gain* of regularities of the function F. In addition, noting that a similar estimate to (3.2) is valid for the Stokes operator e^{-tA}, we give the corresponding result in the case of e^{-tA} as well.

Proposition 3.3 ([9, Prop. 3.3])**.** *Let $r, q \in (1,\infty)$ and let $\lambda_{\mathrm{N}}(\Omega), \lambda_{\mathrm{D}}(\Omega) \in (0,\infty)$ be given by* (2.2)*. Then the following statements hold:*

(i) *Let $0 < \lambda_* < \lambda_{\mathrm{N}}(\Omega)/q$ and $0 \le \lambda \le \lambda_*$. For $F \in L^q(0,\infty; L^r_0(\Omega))$, define $\mathcal{I}F$ by setting*

$$(\mathcal{I}F)(t,x) := \int_0^t e^{\lambda(t-\tau)} e^{(t-\tau)\Delta_{\mathrm{N},0}} F(\tau,x)\, d\tau, \quad (t,x) \in (0,\infty) \times \Omega.$$

Then it holds that $\mathcal{I}F \in L^q(0,\infty; L^r_0(\Omega))$ with the estimate

$$\|\mathcal{I}F\|_{L^q(L^r(\Omega))} \le C_{\lambda_*}\|F\|_{L^q(L^r(\Omega))},$$

where $C_{\lambda_*} > 0$ is a constant independent of λ and F. Moreover, if there is a constant $\alpha \in \mathbb{R} \setminus \{0\}$ such that $\mathcal{I}F = \alpha F$ in $(0,\infty) \times \Omega$, then it holds that $F \equiv 0$.

(ii) *Let $0 < \lambda_* < \lambda_{\mathrm{D}}(\Omega)/q$ and $0 \le \lambda \le \lambda_*$. For $F \in L^q(0,\infty; L^r_\sigma(\Omega))$, define $\mathcal{J}F$ by setting*

$$(\mathcal{J}F)(t,x) := \int_0^t e^{\lambda(t-\tau)} e^{-(t-\tau)A} F(\tau,x)\, d\tau, \quad (t,x) \in (0,\infty) \times \Omega.$$

Then it holds that $\mathcal{J}F \in L^q(0,\infty; L^r_\sigma(\Omega))$ *with the estimate*

$$\|\mathcal{J}F\|_{L^q(L^r(\Omega))} \le C_{\lambda_*}\|F\|_{L^q(L^r(\Omega))}.$$

Moreover, if there is a constant $\alpha \in \mathbb{R}\setminus\{0\}$ *such that* $\mathcal{J}F = \alpha F$ *in* $(0,\infty)\times\Omega$*, then it holds that* $F \equiv 0$*.*

Recall that the maximal regularity results for a shifted Neumann heat equation and the Stokes system is well-known. Thus, together with Theorem 3.1, we have the maximal regularity result for the linearized system.

3.2. **Nonlinear analyses.** Theorem 2.1 may be proved by the standard fixed-point argument since given data (excluded S and $\nabla\varphi$) are assumed to be small. Since our linear estimates do work for the case of inhomogeneous Neumann boundary conditions, we do not need to approximate the system via cut-off functions like previous works. As mentioned before, we especially obtain the uniqueness of the global solutions.

Showing the higher-regularity result, Theorem 2.3, is more involved. In fact, it is not straightforward to apply the standard bootstrap argument for parabolic-type PDEs (such as Angenent's parameter trick [1]; a combination of the scaling argument and implicit function theorem) to verify the additional regularities for the global solutions. This arises from the fact that n should satisfy $\int_\Omega n(t,x)\,dx = \int_\Omega n_0(x)\,dx$, whereas $\boldsymbol{u}$ has to satisfy the divergence-free condition. Namely, adapting the standard bootstrap argument while keeping these conditions in mind is so technical. To avoid the difficulty, we rely on the smoothing effects of the heat semigroup $e^{t\Delta}$ on $\mathbb{R}^N$. For the estimates of the heat semigroup, see, e.g., Kozono–Ogawa–Taniuchi [4, Lem. 2.2 (i)].

Proposition 3.4. *Let* $1 \le r_* \le \infty$*. Let* $0 < s_0 \le s_1 < \infty$*. If* $\psi \in L^{r_*}(\mathbb{R}^N)$*, then it holds that* $e^{t\Delta}\psi \in B^{s_1}_{r_*,\infty}(\mathbb{R}^N)$ *for all* $0 < t < \infty$ *with the estimate*

$$\|e^{t\Delta}\psi\|_{B^{s_1}_{r_*,\infty}(\mathbb{R}^N)} \le C(1+t^{-s_1/2})\|\psi\|_{L^{r_*}(\mathbb{R}^N)},$$

where $C > 0$ *is a constant independent of* t *and* ψ*. In addition, if* $\psi \in B^{s_0}_{r_*,\infty}(\mathbb{R}^N)$*, then it holds that* $e^{t\Delta}\psi \in B^{s_1}_{r_*,\infty}(\mathbb{R}^N)$ *for all* $0 < t < \infty$ *with the estimate*

$$\|e^{t\Delta}\psi\|_{B^{s_1}_{r_*,\infty}(\mathbb{R}^N)} \le C(1+t^{-(s_1-s_0)/2})\|\psi\|_{B^{s_0}_{r_*,\infty}(\mathbb{R}^N)}.$$

We reformulate the equations as integral forms, where we utilize the cut-off technique to ignore the boundary conditions. After verifying that the solution $(\boldsymbol{u}, n, c)$ is a classical solution, we prove the non-negativity of n and c. To this end, we need to introduce the maximum principle of new type since the system includes the nonlinear boundary condition. This is a variant assertion of that given in Prop. 52.8 of Quittner and Souplet [6].

Lemma 3.5 ([9, Lem. 5.6])**.** *Let* Ω *be a smooth (possibly unbounded) domain and let* $\alpha > 0$ *be a constant. Suppose that*

$$U \in BUC([0,\infty); C^1(\overline{\Omega})) \cap C((0,\infty); C^2(\Omega)) \cap C^1((0,\infty); C(\Omega))$$

satisfies

$$\begin{cases} \partial_t U - \Delta U \ge -\alpha(|\nabla U| + |U|), & t > 0,\ x \in \Omega, \\ \nabla U \cdot \boldsymbol{\nu} \ge -\alpha|U|, & t > 0,\ x \in \partial\Omega, \\ U(0,x) \ge 0, & x \in \Omega. \end{cases} \tag{3.3}$$

Then there holds $U \ge 0$ *in* $(0,\infty)\times\Omega$*.*

Notice that the claim and its proof are almost the same as in Proposition 52.8 and Remark 52.9 in [6], but our boundary condition $\nabla U \cdot \boldsymbol{\nu} \geq -\alpha|U|$ in (3.3) is slightly weaker than the assertion recorded in [6] (in particular, the lower bound of $\nabla U \cdot \boldsymbol{\nu}$ on $\partial\Omega$ is given by $-\alpha|U|$ instead of αU).

Acknowledgements

This work was partially supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University. It was also partially supported by JSPS KAKENHI Grant Numbers JP21K13826, JP22J12100, and JP22KJ2930.

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