ON A CONSTRAINED HAMILTON–JACOBI PROBLEM

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Abstract. We discuss existence and regularity of the sublevels of viscosity solutions to the constrained Hamilton–Jacobi problem $H(x,u,\nabla u)=0$ in $\Omega\setminus K$ under the condition $u=0$ on K.

1. Introduction

This note adresses the Hamilton–Jacobi problem

$$H(x,u(x),\nabla u)=0 \quad \text{in } \Omega\setminus K, \tag{1.1}$$

$$u=0 \quad \text{on } K. \tag{1.2}$$

Here $\Omega\subset\mathbb{R}^n$ is a convex open set, possibly being unbounded, $K\subset\Omega$ is compact, and the *Hamiltonian* $H:\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ is continuous.

Problem (1.1)–(1.2) arises in different contexts, from front propagation and geometric optics [4, 10, 27], to optimal control [2], differential games [6, 13], and image reconstruction [1]. Solutions $u\in C(\overline{\Omega})$ to problem (1.1)–(1.2) are considered in the viscosity sense [7].

The purpose of this note is to discuss existence of viscosity solutions to problem (1.1)–(1.2), their variational representation, and the regularity of their sublevels. Under different assumptions on the ingredients H, Ω, and K, a variety of results on the stationary Hamilton–Jacobi equation (1.1) are already available. The reader is referred to [3, 18, 14] for some classical references, as well as to [29] for a comprehensive collection of results. A nice introduction to the topic is in [17]. Standard references for numerical treatment are [22, 23], see also [11]. In the setting of constrained problems $\Omega\neq\mathbb{R}^n$, pioneering papers have been [25, 26] and [5]. In this note, the Finsler metric induced by (1.1) on $\Omega\setminus K$ will be particularly relevant, see [12, 24]. See also [15, 16, 19, 30] for a selection of recent results.

Although the literature on equation (1.1) is extensive, the specific setting considered in this work, i.e., (1.3)–(1.8), appears to be new. Before going on, let us recall that $u\in C(\overline{\Omega})$ is said to be a *viscosity solution* to problem (1.1)–(1.2) if $u=0$ on K and, for all $x\in\Omega\setminus K$ and all $\varphi\in C^1(\mathbb{R}^n)$ with $\varphi(x)-u(x)=\min(\varphi-u)$ relative to Ω ($\varphi(x)-u(x)=\max(\varphi-u)$), we have that $H(x,u(x),\nabla\varphi(x))\le 0$ ($H(x,u(x),\nabla\varphi(x))\ge 0$, respectively).

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We specify our assumptions as follows:

$$\Omega \subset \mathbb{R}^n \text{ nonempty, open, and convex,} \tag{1.3}$$

$$K \subset \Omega \text{ nonempty and compact,} \tag{1.4}$$

$$H : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \text{ is continuous,} \tag{1.5}$$

$$C_{xu} := \{p \in \mathbb{R}^n \mid H(x,u,p) \le 0\} \text{ is convex} \quad \forall (x,u) \in \overline{\Omega} \times \mathbb{R}, \tag{1.6}$$

$$\forall (x,u,p) \in C := \{(x,u,p) \in \overline{\Omega} \times \mathbb{R}^n \times \mathbb{R}^n \mid H(x,u,p) \le 0\}, \ \forall \delta > 0$$
$$\quad \exists p_\delta \in \mathbb{R}^n \text{ such that } |p - p_\delta| \le \delta \text{ and } (x,u,p_\delta) \in C^\circ, \tag{1.7}$$

$$\exists 0 < \sigma_* \le \sigma^* : \quad B_{\sigma_*} \subset C_{xu} \subset B_{\sigma_*} \quad \forall (x,u) \in \overline{\Omega} \times \mathbb{R}. \tag{1.8}$$

Above, we use the notation $B_r(x) = \{y \in \mathbb{R}^n \mid |x - y| < r\}$ for $x \in \mathbb{R}^n$ and $r > 0$, and $B_r = B_r(0)$. Moreover, we indicate by $\overline{A}$ and A° the closure and the interior of a set $A \subset \mathbb{R}^k$, respectively.

We define L_x to be the set of Lipschitz curves in $\overline{\Omega}$ connecting K to $x \in \overline{\Omega}$, namely,

$$L_x := \{\gamma \in W^{1,\infty}(0,1;\mathbb{R}^n) \mid \gamma(\cdot) \in \overline{\Omega}, \ \gamma(0) \in K, \ \gamma(1) = x\} \quad \forall x \in \overline{\Omega}.$$

Note that, for all $x \in \overline{\Omega}$ the set L_x is obviously not empty, as Ω is convex. The notation $d(x,A) = \inf_{a \in A} |x - a|$ denotes the distance between $x \in \mathbb{R}^n$ and the nonempty set $A \subset \mathbb{R}^n$.

For all $(x,u) \in \overline{\Omega} \times \mathbb{R}$ we set $\sigma(x,u,\cdot) : \mathbb{R}^n \to [0,\infty)$ to be the *support function* of the nonempty, convex, and closed set C_{xu}, see (1.5) and (1.6), namely,

$$\sigma(x,u,v) := \sup\{v \cdot p \mid H(x,u,p) \le 0\} \quad \forall (x,u,v) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n.$$

Note that $\sigma(x,u,\cdot)$ is positively 1-homogeneous, in particular, $0 = \sigma(x,u,0) = \min_v \sigma(x,u,v)$.

The existence and variational representation result reads as follows.

Theorem 1.1 (Existence and representation)**.** *Under assumptions* (1.3)–(1.8) *the function* $u \in C(\overline{\Omega})$ *given by*

$$u(x) = \min_{\gamma \in L_x} \int_0^1 \sigma\big(\gamma(s), u(\gamma(s)), \gamma'(s)\big) \, \mathrm{d}s \quad \forall x \in \overline{\Omega} \tag{1.9}$$

is a viscosity solution to problem (1.1)–(1.2).

Theorem 1.1 is proved in Section 2 below. Note that u appears also in the right-hand side of formula (1.9), so that no uniqueness for viscosity solutions to problem (1.1)–(1.2) is implied. In fact, comparison and uniqueness are not expected in this setting, as the elementary one-dimensional example $H(x,p) = |p| - 1$ in $\Omega = \mathbb{R}$ with $K = [0,1]$ shows.

Note that the assumptions of Theorem 1.1 may be generalized, at the expense of a more involved theory. In particular $\overline{\Omega}$ could be allowed to be nonconvex but homeomorphic to a closed convex set $G \subset \mathbb{R}^n$ by setting $\pi := \phi \circ p \circ \phi^{-1}$ where $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism with $\phi(G) = \overline{\Omega}$ and $p : \mathbb{R}^n \to G$ is the projection. The continuity assumption (1.5), as well as the topological condition (1.7), could be relaxed. However, for the sake of clarity, we choose to work within the framework of (1.3)–(1.8), deferring the treatment of more general settings to [28].

The viscosity solution u to problem (1.1)–(1.2) given by formula (1.9) is nonnegative, hence the sublevels

$$U(t) := \{x \in \overline{\Omega} \mid u(x) < t\}$$

are nondecreasing with $t > 0$ with respect to set inclusion. Our second results concerns the regularity of such sublevels. The sets $U(t)$ play a distinguished role in control theory, where they relate to reachable states of controlled ODE systems [2]. In addition, the sets $U(t)$ may arise in connection with free boundary problems where they may serve as the evolving domain for an additional partial differential equation [8, 28].

To start, let us recall that a bounded set $J \subset \mathbb{R}^n$ is said to be a *John domain* [20] with respect to a fixed point $x_0 \in J$ (called *John center*) and a given *John constant* $\kappa \in (0, 1]$ if it satisfies an *internal twisted cone condition*: for all points $x \in J$ one can find an arc-length parametrized curve $\gamma : [0, \ell_\gamma] \to J$ such that $\gamma(0) = x$, $\gamma(\ell_\gamma) = x_0$, and $d(\gamma(s), \partial J) \geq \kappa\, s$ for all $s \in [0, \ell_\gamma]$. In this case, we say that J is a (x_0, κ)-*John domain*. Note that John domains are connected.

In order to assess the regularity of the sets $U(t)$ we additionally pose the following assumptions on Ω and K:

$$\begin{aligned}&\exists \overline{\varepsilon} > 0 \text{ such that the projection } \pi : \{x \in \overline{\Omega} \mid d(x, \partial\Omega) < \overline{\varepsilon}\} \to \partial\Omega \text{ is well defined and}\\ &\mu := \nu \circ \pi : \{x \in \overline{\Omega} \mid d(x, \partial\Omega) < \overline{\varepsilon}\} \to \mathbb{S}^{n-1} \text{ is Lipschitz continuous, where}\\ &\nu : \partial\Omega \to \mathbb{S}^{n-1} \text{ is the interior-normal field,}\end{aligned} \tag{1.10}$$

$$K \text{ is a } (x_0, \kappa_0)\text{-John domain with } d(x_0, \partial K) > 0. \tag{1.11}$$

Property (1.10) corresponds to an internal-ball condition and holds for $C^{1,1}$ domains. We have the following.

Theorem 1.2 (Regularity of $U(t)$)**.** *Under assumptions* (1.3)–(1.8) *and* (1.10)–(1.11) *the sublevels* $U(t) = \{x \in \overline{\Omega} \mid u(x) < t\}$ *of* u *given by* (1.9) *are* (x_0, κ)*-John domains for all* $t \in (0, T)$*, where* $\kappa \in (0, 1]$ *is a function of* κ_0*,* σ_**,* σ^**,* $d(x_0, \partial K)$*,* $\overline{\varepsilon}$*,* $\|\nabla\mu\|_{L^\infty}$*, and* T *only.*

In the unconstrained case $\Omega = \mathbb{R}^n$, Theorem 1.2 follows from [9, Thm. 1.1]. In particular, the John regularity of $\partial U(t)$ is shown to be sharp in [9]. The present case $\Omega \neq \mathbb{R}^n$ is slightly more involved, due to the presence of the boundary $\partial\Omega$. The proof of Theorem 1.2 is given in Section 3.

2. Existence and representation: Proof of Theorem 1.1

In order to check Theorem 1.1 we implement an iterative procedure. Given $u_{i-1} \in C(\overline{\Omega})$ we find a solution $u_i \in C(\overline{\Omega})$ to $H(x, u_{i-1}(x), \nabla u_i) = 0$ which is variationally represented in the spirit of (1.9). Then, we prove that the sequence $(u_i)_i$ admits a locally uniformly converging subsequence, which solves problem (1.1)–(1.2), as well as (1.9).

To start, let us remark that the nondegeneracy assumption (1.8) implies that

$$\sigma_*|v| \leq \sigma(x, u, v) \leq \sigma^*|v| \quad \forall (x, u, p) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n. \tag{2.1}$$

The iteration step is based on the following.

Proposition 2.1 (Iteration step)**.** *Assume* (1.3)–(1.4) *and that*

$$\widehat{H} : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R} \text{ is continuous,} \tag{2.2}$$

$$\widehat{C}_x := \{p \in \mathbb{R}^n \mid \widehat{H}(x,p) \le 0\} \quad \text{is convex} \quad \forall x \in \overline{\Omega}, \tag{2.3}$$

$$\forall (x,p) \in \widehat{C} := \{(x,p) \in \overline{\Omega} \times \mathbb{R}^n \mid \widehat{H}(x,p) \le 0\},\ \forall \delta > 0$$

$$\exists p_\delta \in \mathbb{R}^n \text{ such that } |p - p_\delta| \le \delta \text{ and } (x, p_\delta) \in \widehat{C}^\circ, \tag{2.4}$$

$$\exists 0 < \sigma_* \le \sigma^* : \quad B_{\sigma_*} \subset \widehat{C}_x \subset B_{\sigma_*} \quad \forall x \in \overline{\Omega}. \tag{2.5}$$

Then, setting $\widehat{\sigma}(x,v) := \sup\{v \cdot p \mid \widehat{H}(x,p) \le 0\}$ *for all* $(x,v) \in \overline{\Omega} \times \mathbb{R}^n$*, the representation formula*

$$u(x) = \min_{\gamma \in L_x} \int_0^1 \widehat{\sigma}\big(\gamma(s), \gamma'(s)\big)\ \mathrm{d}s \quad \forall x \in \overline{\Omega} \tag{2.6}$$

gives a viscosity solution to problem

$$\widehat{H}(x, \nabla u) = 0 \quad \text{in } \Omega \setminus K, \tag{2.7}$$

$$u = 0 \quad \text{on } K. \tag{2.8}$$

Proof. The proof relies on a penalization procedure and is divided into steps. At first, we extend $\widehat{H}$ to $\widehat{H}_\varepsilon$ defined on the whole $\mathbb{R}^n$, depending on a small parameter $\varepsilon > 0$. For all such $\widehat{H}_\varepsilon$ we solve $\widehat{H}_\varepsilon(x, \nabla u_\varepsilon) = 0$ in $\mathbb{R}^n \setminus K$ and obtain a representation formula for $u_\varepsilon \in C(\mathbb{R}^n)$ (Step 1). By taking $\varepsilon \to 0$, we prove that $u_\varepsilon \to u$ locally uniformly in $\overline{\Omega}$, that $u \in C(\overline{\Omega})$ is a viscosity solution to (2.7)–(2.8) (Step 2), and that the representation formula (2.6) holds (Step 3).

Step 1: Extension. Denote by $\pi : \mathbb{R}^n \to \overline{\Omega}$ the projection onto the closed convex set $\overline{\Omega}$. For all $\varepsilon \in (0,1)$, choose $\alpha_\varepsilon : \mathbb{R} \to [\varepsilon, 1]$ continuous and nonincreasing with $\alpha_\varepsilon(r) = 1$ for $r \le 0$ and $\alpha_\varepsilon(r) = \varepsilon$ for $r \ge 1$. We define $\beta_\varepsilon : \mathbb{R}^n \to [\varepsilon, 1]$ and $\widehat{H}_\varepsilon : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as

$$\beta_\varepsilon(x) := \alpha_\varepsilon(d(x, \overline{\Omega})/\varepsilon^2) \quad \forall x \in \mathbb{R}^n,$$

$$\widehat{H}_\varepsilon(x,p) := \widehat{H}(\pi(x), \beta_\varepsilon(x) p) \quad \forall (x,p) \in \mathbb{R}^n \times \mathbb{R}^n.$$

As α_ε, $x \mapsto d(x, \overline{\Omega})$, and π are continuous, we have that

$$\widehat{H}_\varepsilon : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \text{ is continuous.} \tag{2.9}$$

Moreover, for all $x \in \mathbb{R}^n$ we readily check that

$$\widehat{C}_{x\varepsilon} := \{p \in \mathbb{R}^n \mid \widehat{H}_\varepsilon(x,p) \le 0\} = \{p \in \mathbb{R}^n \mid \beta_\varepsilon(x) p \in \widehat{C}_{\pi(x)}\} = \frac{1}{\beta_\varepsilon(x)} \widehat{C}_{\pi(x)}.$$

This in particular proves that

$$\widehat{C}_{x\varepsilon} \text{ is convex} \quad \forall x \in \mathbb{R}^n. \tag{2.10}$$

Define $\widehat{C}_\varepsilon := \{p \in \mathbb{R}^n \mid \widehat{H}_\varepsilon(x,p) \le 0\}$. We aim at proving that

$$\forall (x,p) \in \widehat{C}_\varepsilon\ \forall \delta > 0\ \exists p_\delta \in \mathbb{R}^n \text{ with } |p - p_\delta| \le \delta \text{ and } (x, p_\delta) \in \widehat{C}_\varepsilon^\circ, \tag{2.11}$$

see (2.4). Fix $(x,p)\in\widehat{C}_\varepsilon$. As $\beta_\varepsilon(x)p\in\widehat{C}_{\pi(x)}$, by using (2.4), for all $\delta>0$ we can find q_δ such that $(\pi(x),q_\delta)\in\widehat{C}^\circ$ and $|\beta_\varepsilon(x)p-q_\delta|\le\beta_\varepsilon(x)\delta$. By setting $p_\delta=q_\delta/\beta_\varepsilon(x)$ we have that $|p-p_\delta|=(\beta_\varepsilon(x))^{-1}|\beta_\varepsilon(x)p_\delta-q_\delta|\le\delta$. Moreover, for all $(\widetilde{x},\widetilde{p})$ sufficiently close to (x,p_δ), from the continuity of π and β_ε we have that $(\pi(\widetilde{x}),\beta_\varepsilon(\widetilde{x})\widetilde{p})$ is close to $(\pi(x),\beta_\varepsilon(x)p_\delta)$. Hence $\widehat{H}_\varepsilon(\widetilde{x},\widetilde{p})=\widehat{H}(\pi(\widetilde{x}),\beta_\varepsilon(\widetilde{x})\widetilde{p})\le 0$ as $(\pi(x),\beta_\varepsilon(x)p_\delta)\in\widehat{C}^\circ$. This proves that (x,p_δ) is internal to $\widehat{C}_\varepsilon$, implying (2.11).

Moving from (2.11), we can check that

$$\widehat{C}_x=\widehat{D}_x:=\overline{\{p\in\mathbb{R}^n\mid (x,p)\in\widehat{C}^\circ_\varepsilon\}}\quad\forall x\in\mathbb{R}^n. \tag{2.12}$$

The inclusion $\widehat{D}_x\subset\widehat{C}_x$ is immediate: for all $p\in\widehat{D}_x$ we find $p_n\to p$ with $\widehat{H}_\varepsilon(x,p_n)\le 0$. The continuity of $\widehat{H}_\varepsilon$ entails that $\widehat{H}_\varepsilon(x,p)\le 0$, namely, $p\in\widehat{C}_x$. Let now $p\in\widehat{C}_x$. Condition (2.11) implies that we can find a sequence $(p_n)_n$ with $(x,p_n)\in\widehat{C}^\circ_\varepsilon$ and $p_n\to p\in\widehat{D}_x$. This proves that $\widehat{C}_x\subset\widehat{D}_x$ and (2.12) follows.

Eventually, we have that

$$B_{\sigma_*}\overset{(2.5)}{\subset}\widehat{C}_{\pi(x)}\overset{\beta_\varepsilon\le 1}{\subset}\frac{1}{\beta_\varepsilon(x)}\widehat{C}_{\pi(x)}=\widehat{C}_{x\varepsilon}\overset{(2.5)}{\subset}\frac{1}{\beta_\varepsilon(x)}B_{\sigma^*}\overset{\varepsilon\le\beta_\varepsilon}{\subset}B_{\sigma^*/\varepsilon}$$

so that $\widehat{H}_\varepsilon$ fulfills

$$B_{\sigma_*}\subset\widehat{C}_{x\varepsilon}\subset B_{\sigma^*/\varepsilon}\quad\forall x\in\mathbb{R}^n. \tag{2.13}$$

Set now

$$\widehat{L}_x:=\{\gamma\in W^{1,\infty}(0,1;\mathbb{R}^n)\mid\gamma(0)\in K,\ \gamma(1)=x\}.$$

By virtue of properties (2.9)–(2.13) we can now apply [21, Thm. 3.15] and find that

$$u_\varepsilon(x)=\min_{\gamma\in\widehat{L}_x}\int_0^1\widehat{\sigma}_\varepsilon\big(\gamma(s),\gamma'(s)\big)\,\mathrm{d}s\quad\forall x\in\mathbb{R}^n \tag{2.14}$$

is the unique nonnegative viscosity solution to

$$\widehat{H}_\varepsilon(x,\nabla u_\varepsilon)=0\quad\text{in }\mathbb{R}^n\setminus K, \tag{2.15}$$

$$u_\varepsilon=0\quad\text{on }K. \tag{2.16}$$

In (2.14), the support function $\widehat{\sigma}_\varepsilon$ of $\widehat{C}_{x\varepsilon}$ is defined as

$$\widehat{\sigma}_\varepsilon(x,v):=\sup\{v\cdot p\mid\widehat{H}_\varepsilon(x,p)\le 0\}=\frac{1}{\beta_\varepsilon(x)}\widehat{\sigma}(\pi(x),v)\quad\forall(x,v)\in\mathbb{R}^n\times\mathbb{R}^n.$$

For all $x\in\mathbb{R}^n$ the map $v\mapsto\widehat{\sigma}(x,v)$ is convex and positively 1-homogeneous. In particular, $0=\widehat{\sigma}(x,0)=\min\sigma(x,\cdot)$.

Owing to (2.5) and (2.13) we readily check that

$$\sigma_*|v|\le\widehat{\sigma}(\pi(x),v)\le\widehat{\sigma}_\varepsilon(x,v)\le\frac{\sigma^*}{\varepsilon}|v|\quad\forall(x,v)\in\mathbb{R}^n\times\mathbb{R}^n, \tag{2.17}$$

$$\widehat{\sigma}_\varepsilon(x,v)=\widehat{\sigma}(x,v)\le\sigma^*|v|\quad\forall(x,v)\in\overline{\Omega}\times\mathbb{R}^n. \tag{2.18}$$

Moreover, we can prove that

$$\widehat{\sigma}:\mathbb{R}^n\times\mathbb{R}^n\to[0,\infty)\ \text{ is continuous.} \tag{2.19}$$

To this aim, let $(x_k,v_k)\to(x,v)$ and, for all $\eta>0$ let $p_\eta\in\widehat C_x$ such that $\widehat\sigma(x,v)\le v\cdot p_\eta+\eta$. Using (2.4) we find $p_{\eta\delta}$ such that $(x,p_{\eta\delta})\in\widehat C^\circ$ and $v\cdot p_\eta\le v\cdot p_{\eta\delta}+\delta$. Hence, we have that $(x_k,p_{\eta\delta})\in\widehat C$ for k large enough, so that $p_{\eta\delta}\in\widehat C_{x_k}$. We conclude that

$$\begin{aligned}\widehat\sigma(x,v)&\le v\cdot p_\eta+\eta\le v\cdot p_{\eta\delta}+\eta+\delta\\&=\lim_{k\to\infty}v_k\cdot p_{\eta\delta}+\eta+\delta\overset{p_{\eta\delta}\in\widehat C_{x_k}}{\le}\liminf_{k\to\infty}\widehat\sigma(x_k,v_k)+\eta+\delta.\end{aligned}$$

As η and δ are arbitrary, this proves that $\widehat\sigma$ is lower semicontinuous.

Let us now check the upper semicontinuity. For all $k\in\mathbb N$ we can find $p_k\in C_{x_k}$ such that $\widehat\sigma(x_k,v_k)\le v_k\cdot p_k+1/k$. Let the subsequence $(k_j)_j$ be such that $\lim_{j\to\infty}\sigma(x_{k_j},v_{k_j})=\limsup_{k\to\infty}\sigma(x_k,v_k)$. As $p_{k_j}\in\widehat C_{x_{k_j}}\subset B_{\sigma^*}$ by (2.5), we can extract without relabeling in such a way that $p_{k_j}\to p$. As $\widehat C$ is closed by (2.2), we have that $(x,p)\in\widehat C$, namely, $p\in\widehat C_x$, and

$$\limsup_{k\to\infty}\widehat\sigma(x_k,v_k)=\lim_{j\to\infty}\widehat\sigma(x_{k_j},v_{k_j})\le\lim_{j\to\infty}\left(v_{k_j}\cdot p_{k_j}+\frac{1}{k_j}\right)=v\cdot p\overset{p\in\widehat C_x}{\le}\widehat\sigma(x,v)$$

and $\widehat\sigma$ is upper semicontinuous, so that we have proved (2.19).

Step 2: Estimate and limit $\varepsilon\to0$. Fix now x, $y\in\overline\Omega$ with $x\ne y$, let $\gamma_y\in\widehat L_y$ realize the minimum for $u_\varepsilon(y)$ in (2.14), and set $t\in[0,1]\mapsto\gamma(t)=tx+(1-t)y\in\Omega$. By considering the concatenated curve $\widetilde\gamma\in\widehat L_x$ given by

$$\widetilde\gamma(s)=\begin{cases}\gamma_y(2s)&\text{for } s\in[0,1/2],\\\gamma(2s-1)&\text{for } s\in(1/2,1],\end{cases}$$

we can bound

$$\begin{aligned}u_\varepsilon(x)-u_\varepsilon(y)&\le\int_0^1\widehat\sigma_\varepsilon\big(\widetilde\gamma(s),\widetilde\gamma'(s)\big)\,\mathrm ds-\int_0^{1/2}\widehat\sigma_\varepsilon\big(\widetilde\gamma(s),\widetilde\gamma'(s)\big)\,\mathrm ds\\&=\int_{1/2}^1\widehat\sigma_\varepsilon\big(\widetilde\gamma(s),\widetilde\gamma'(s)\big)\,\mathrm ds\overset{\widetilde\gamma(\cdot)\in\overline\Omega}{=}\int_{1/2}^1\widehat\sigma\big(\widetilde\gamma(s),\widetilde\gamma'(s)\big)\,\mathrm ds\overset{(2.18)}{\le}\sigma^*\int_0^1|\gamma'(s)|\,\mathrm ds=\sigma^*|x-y|.\end{aligned}$$

by exchanging the roles of x and y, this proves that $|\nabla u_\varepsilon|\le\sigma^*$ almost everywhere in Ω. A diagonal extraction argument allows to find $u\in C(\overline\Omega)$ and a not relabeled subsequence u_ε such that

$$u_\varepsilon\to u\quad\text{locally uniformly in }\overline\Omega.$$

It is a standard matter to prove that u is a viscosity solution to problem (2.7)–(2.8). Indeed, for all $x\in\Omega\backslash K$ and all $\varphi\in C^1(\mathbb R^n)$ with $\varphi(x)-u(x)=\min(\varphi-u)$ we use [7, Prop. 4.3] in order to find $\Omega\setminus K\ni x_\varepsilon\to x$ and $\varphi_\varepsilon\in C^1(\mathbb R^n)$ with $\varphi_\varepsilon(x_\varepsilon)-u_\varepsilon(x_\varepsilon)=\min(\varphi_\varepsilon-u_\varepsilon)$ and $\nabla\varphi(x_\varepsilon)\to\nabla\varphi(x)$. As $\widehat H(\widehat x,\cdot)=\widehat H_\varepsilon(\widehat x,\cdot)$ for $\widehat x\in\Omega\setminus K$ we get $0\ge\lim_{\varepsilon\to0}\widehat H_\varepsilon(x_\varepsilon,\nabla\varphi(x_\varepsilon))=\widehat H(x,\nabla\varphi(x))$, proving that u is a viscosity subsolution to (2.7) in $\Omega\setminus K$. Similarly we prove that u is a viscosity supersolution, hence a viscosity solution.

Step 3: Representation formula. We now check the validity of the representation formula (2.6). Fix $x \in \overline{\Omega}$ and let $\gamma \in L_x$. Then,

$$u(x) = \lim_{\varepsilon \to 0} u_\varepsilon(x) \le \int_0^1 \widehat{\sigma}_\varepsilon\big(\gamma(s), \gamma'(s)\big)\,\mathrm{d}s \overset{\gamma(\cdot)\in\overline{\Omega}}{=} \int_0^1 \widehat{\sigma}\big(\gamma(s), \gamma'(s)\big)\,\mathrm{d}s \tag{2.20}$$

so that the right-hand side of (2.6) is an upper bound for $u(x)$.

We are hence left with proving that the minimum in (2.6) is actually attained, namely, that there exists $\gamma \in L_x$ realizing the equality in (2.20). To this aim, let $\gamma_\varepsilon \in \widehat{L}_x$ be a minimizer in (2.14) and indicate by $\widehat{\gamma}_\varepsilon \in W^{1,\infty}(0, \ell_{\widehat{\gamma}_\varepsilon}; \mathbb{R}^n)$ the corresponding arc-length parametrization. We have that

$$\sigma_* \ell_{\widehat{\gamma}_\varepsilon} \overset{(2.17)}{\le} \int_0^{\ell_{\widehat{\gamma}_\varepsilon}} \widehat{\sigma}_\varepsilon\big(\widehat{\gamma}_\varepsilon(s), \widehat{\gamma}_\varepsilon'(s)\big)\,\mathrm{d}s = \int_0^1 \widehat{\sigma}_\varepsilon\big(\gamma_\varepsilon(s), \gamma_\varepsilon'(s)\big)\,\mathrm{d}s = u_\varepsilon(x) \to u(x).$$

This implies that the lengths $\ell_{\widehat{\gamma}_\varepsilon}$ are bounded independently of ε. Define $\ell_{\max} = \sup_\varepsilon \ell_{\widehat{\gamma}_\varepsilon} < \infty$ and extend $\widehat{\gamma}_\varepsilon$ to $\widetilde{\gamma}_\varepsilon \in W^{1,\infty}(0, \ell_{\max}; \mathbb{R}^n)$ by setting $\widetilde{\gamma}_\varepsilon(s) := \widehat{\gamma}_\varepsilon(\min\{s, \ell_{\widehat{\gamma}_\varepsilon}\})$ for $s \in [0, \ell_{\max}]$. As the curves $\widetilde{\gamma}_\varepsilon$ are uniformly Lipschitz continuous and $\widetilde{\gamma}_\varepsilon(\ell_{\max}) = x$, we can extract not relabeled subsequences in such a way that $\ell_{\widehat{\gamma}_\varepsilon} \to \ell_{\widehat{\gamma}} \le \ell_{\max}$ and $\widetilde{\gamma}_\varepsilon \to \widetilde{\gamma}$ uniformly in $[0, \ell_{\max}]$ and weakly* in $W^{1,\infty}(0, \ell_{\max}; \mathbb{R}^n)$.

We now check that the restriction $\widehat{\gamma}$ of $\widetilde{\gamma}$ to $[0, \ell_{\widehat{\gamma}}]$ is the arc-length parametrization of a curve γ realizing the minimum in (2.6). To start with, note that $|\widehat{\gamma}'| = 1$ a.e. in $(0, \ell_{\widehat{\gamma}})$, $\widehat{\gamma}(\ell_{\widehat{\gamma}}) = \widetilde{\gamma}(\ell_{\max}) = x$, and $K \ni \widehat{\gamma}_\varepsilon(0) \to \widehat{\gamma}(0)$. As K is closed, the latter convergence ensures that $\widehat{\gamma}(0) \in K$.

Let us now check that $\widehat{\gamma}(s) \in \overline{\Omega}$ for all $s \in [0, \ell_{\widehat{\gamma}}]$. To this aim, define

$$A_\varepsilon := \{s \in [0, \ell_{\widehat{\gamma}_\varepsilon}] \mid d(\widetilde{\gamma}_\varepsilon(s), \overline{\Omega}) \ge \varepsilon^2\}.$$

For almost all $s \in A_\varepsilon$ we have that

$$\widehat{\sigma}_\varepsilon\big(\widetilde{\gamma}_\varepsilon(s), \widetilde{\gamma}_\varepsilon'(s)\big) = \frac{1}{\varepsilon}\widehat{\sigma}(\pi(\widetilde{\gamma}_\varepsilon(s)), \widetilde{\gamma}_\varepsilon'(s)) \overset{(2.17)}{\ge} \frac{\sigma_*}{\varepsilon}|\widetilde{\gamma}_\varepsilon'(s)| = \frac{\sigma_*}{\varepsilon}.$$

This allows to estimate $|A_\varepsilon|$ as follows

$$\frac{\sigma_*}{\varepsilon}|A_\varepsilon| \le \int_{A_\varepsilon} \widehat{\sigma}_\varepsilon\big(\widetilde{\gamma}_\varepsilon(s), \widetilde{\gamma}_\varepsilon'(s)\big)\,\mathrm{d}s \le \int_0^{\ell_{\widehat{\gamma}_\varepsilon}} \widehat{\sigma}_\varepsilon\big(\widehat{\gamma}_\varepsilon(s), \widehat{\gamma}_\varepsilon'(s)\big)\,\mathrm{d}s = u_\varepsilon(x) \to u(x).$$

This implies that

$$|A_\varepsilon| \le c\varepsilon \tag{2.21}$$

for some $c > 0$ depending on σ_* and $u(x)$ only. Specifically, the length of the portion of the curve $\widetilde{\gamma}_\varepsilon$ laying in the complement of $\overline{\Omega} + B_{\varepsilon^2}$ is smaller than $c\varepsilon$. This in particular entails that

$$d(\widetilde{\gamma}_\varepsilon(s), \overline{\Omega} + B_{\varepsilon^2}) \le c\varepsilon \quad \forall s \in [0, \ell_{\widehat{\gamma}_\varepsilon}]$$

and by the triangle inequality we get

$$d(\widetilde{\gamma}_\varepsilon(s), \overline{\Omega}) \le d(\widetilde{\gamma}_\varepsilon(s), \overline{\Omega} + B_{\varepsilon^2}) + \varepsilon^2 \le c\varepsilon + \varepsilon^2 \quad \forall s \in [0, \ell_{\widehat{\gamma}_\varepsilon}]. \tag{2.22}$$

By passing to the limit as $\varepsilon \to 0$ we obtain that $\widetilde{\gamma}(s) \in \overline{\Omega}$ for all $s \in [0, \ell_{\widehat{\gamma}}]$.

We also find that

$$\begin{aligned}
u(x) &= \lim_{\varepsilon\to 0} u_\varepsilon(x) = \lim_{\varepsilon\to 0}\int_0^{\ell_{\widehat\gamma_\varepsilon}} \widehat\sigma_\varepsilon\big(\widehat\gamma_\varepsilon(s),\widehat\gamma_\varepsilon'(s)\big)\,\mathrm{d}s\\
&\geq \liminf_{\varepsilon\to 0}\int_{[0,\ell_{\widehat\gamma_\varepsilon}]\setminus A_\varepsilon} \widehat\sigma_\varepsilon\big(\widehat\gamma_\varepsilon(s),\widehat\gamma_\varepsilon'(s)\big)\,\mathrm{d}s \stackrel{(2.17)}{\geq} \liminf_{\varepsilon\to 0}\int_{[0,\ell_{\widehat\gamma_\varepsilon}]\setminus A_\varepsilon} \widehat\sigma\big(\pi(\widehat\gamma_\varepsilon(s)),\widehat\gamma_\varepsilon'(s)\big)\,\mathrm{d}s\\
&= \int_0^{\ell_{\widehat\gamma}} \widehat\sigma\big(\widehat\gamma(s),\widehat\gamma'(s)\big)\,\mathrm{d}s
\end{aligned}$$

where we used bound (2.21), the continuity (2.19) of $\widehat\sigma$, and the convexity of $\widehat\sigma(x,\cdot)$. Together with the upper bound (2.20), this proves that

$$u(x) = \int_0^{\ell_{\widehat\gamma}} \widehat\sigma\big(\widehat\gamma(s),\widehat\gamma'(s)\big)\,\mathrm{d}s.$$

Hence, the curve $s\in[0,1]\mapsto\gamma(s) := \widehat\gamma(\ell_{\widehat\gamma}s)\in L_x$ realizes the minimum in (2.6) and the representation formula holds. □

Let us now proceed with the proof of Theorem 1.1. At first, we remark that

$$\sigma : \overline\Omega\times\mathbb{R}\times\mathbb{R}\to[0,\infty) \text{ is continuous.} \tag{2.23}$$

Indeed, we can repeat the proof of (2.19) by including the dependence on u and using assumption (1.6) instead of (2.3).

Set $x\in\overline\Omega\mapsto u_0(x) := d(x,K)\in C(\overline\Omega)$. Assuming to be given $u_{i-1}\in C(\overline\Omega)$ for some $i\geq 1$, we define $\widehat H(x,p) := H(x,u_{i-1}(x),p)$ for all $(x,p)\in\overline\Omega\times\mathbb{R}^n$. Owing to assumptions (1.5)–(1.8) we readily check that $\widehat H$ fulfills (2.2)–(2.5). Specifically, the continuity (2.2) of $\widehat H$ follows from (1.5) and the continuity of u_{i-1}. The convexity (2.3) of $\widehat C_x = C_{xu_{i-1}(x)}$ and the bounds (2.5) are implied by (1.6) and (1.8), respectively. For all $(x,p)\in\widehat C$ and all $\delta>0$ we can find $p_\delta\in\mathbb{R}^n$ with $|p-p_\delta|\leq\delta$ and $(x,u_{i-1}(x),p_\delta)\in C^\circ$ by (1.7). We readily prove that $(x,p_\delta)\in\widehat C$, as well, so that (2.4) holds.

By applying Proposition 2.1 we find that $u_i\in C(\overline\Omega)$ given by

$$u_i(x) = \min_{\gamma\in L_x}\int_0^1 \widehat\sigma\big(\gamma(s),\gamma'(s)\big)\,\mathrm{d}s \tag{2.24}$$

$$= \min_{\gamma\in L_x}\int_0^1 \sigma\big(\gamma(s),u_{i-1}(\gamma(s)),\gamma'(s)\big)\,\mathrm{d}s \quad \forall x\in\overline\Omega \tag{2.25}$$

is a viscosity solution to problem

$$H(x,u_{i-1}(x),\nabla u_i) = 0 \quad \text{in } \Omega\setminus K,, \tag{2.26}$$

$$u_i = 0 \quad \text{on } K. \tag{2.27}$$

Here, we have used $\widehat\sigma(x,v) = \sigma(x,u_{i-1}(x),v)$ for all $(x,v)\in\overline\Omega\times\mathbb{R}^n$.

Arguing as in the proof of Proposition 2.1 we can check that u_i is Lipschitz continuous, namely, $|\nabla u_i|\leq\sigma^*$ almost everywhere. A diagonal extraction argument allows to find $u\in C(\overline\Omega)$ and a not relabeled subsequence u_i such that

$$u_i\to u \quad \text{locally uniformly in } \overline\Omega. \tag{2.28}$$

Following closely the argument of Proposition 2.1 we can check that u is a viscosity solution to problem (1.1)–(1.2) and that the representation formula (1.9) holds. Indeed, for all $x \in \overline{\Omega}$ and $\gamma \in L_x$ we have

$$u(x) = \lim_{i\to\infty} u_i(x) \le \lim_{i\to\infty} \int_0^1 \sigma\big(\gamma(s), u_{i-1}(\gamma(s)), \gamma'(s)\big)\,\mathrm{d}s$$
$$= \int_0^1 \sigma\big(\gamma(s), u(\gamma(s)), \gamma'(s)\big)\,\mathrm{d}s, \tag{2.29}$$

by the continuity (2.23) of σ and dominated convergence, so that the right-hand side of (1.9) is an upper bound for $u(x)$.

For all $i \ge 1$ let $\gamma_i \in L_x$ realize the minimum in (2.25) and indicate by $\widehat{\gamma}_i \in W^{1,\infty}(0,\ell_i;\mathbb{R}^n)$ the corresponding arc-length parametrization. Arguing as in Proposition 2.1 we can extract without relabeling and find $\gamma \in L_x$ with arc-length parametrization $\widehat{\gamma}$ such that

$$u(x) = \lim_{i\to\infty} u_i(x) = \lim_{i\to\infty} \int_0^{\ell_i} \sigma\big(\widehat{\gamma}_i(s), u_{i-1}(\widehat{\gamma}_i(s)), \widehat{\gamma}_i'(s)\big)\,\mathrm{d}s$$
$$= \int_0^{\ell} \sigma\big(\widehat{\gamma}(s), u(\widehat{\gamma}(s)), \widehat{\gamma}'(s)\big)\,\mathrm{d}s$$

where we used the continuity (2.23) of σ, the local uniform convergence (2.28) of u_i, and dominated convergence. Together with the upper bound (2.29), this proves that

$$u(x) = \int_0^{\ell} \sigma\big(\widehat{\gamma}(s), u(\widehat{\gamma}(s)), \widehat{\gamma}'(s)\big)\,\mathrm{d}s.$$

Hence, the curve $s \in [0,1] \mapsto \gamma(s) := \widehat{\gamma}_x(\ell s) \in L_x$ realizes the minimum and the representation formula (1.9) holds.

3. Regularity of the sublevels: Proof of Theorem 1.2

We now turn to the proof of the John regularity of the sublevels $U(t)$ of the viscosity solution u to (1.1)–(1.2) given by the representation formula (1.9). Setting

$$\overline{\kappa} = \frac{\sigma_*}{2\sigma^* + \sigma_*}\min\{\kappa_0, 1\},$$

the argument of [9, Thm. 1.1] ensures that for all $0 < t \le T$ and all $x \in U(t)$ there exists an arc-parametrized $\gamma : [0,\ell_\gamma] \to \overline{\Omega}$ with $\gamma(0) = x$ and $\gamma(\ell_\gamma) = x_0$, such that

$$d(\gamma(s), \partial U(t) \setminus \partial\Omega) \ge \overline{\kappa} s \quad \forall s \in [0,\ell_\gamma], \tag{3.1}$$
$$\ell_\gamma \le \ell_{\max} := \frac{d(x_0,\partial K)}{\kappa_0} + \frac{T}{\sigma_*}. \tag{3.2}$$

In order to prove Theorem 1.2, we need to complement (3.1) by bounding from below the distance of $\gamma(s)$ to $\partial U(t) \cap \partial\Omega$ by a linear term in s. This may however require to change the curve γ, as this may a priori be close, and even touch, $\partial U(t)$ at points in $\partial U(t) \cap \partial\Omega$. Moreover, the John constant may also need to be reduced.

Fix $0 < \varepsilon_0 < \min\{\overline{\varepsilon}, d(x_0, \partial\Omega)\}$, set $\widetilde{\kappa} := \min\{\overline{\kappa}/2, \varepsilon_0/\ell_{\max}\}$, and define $\widetilde{\gamma} : [0, \ell_\gamma] \to \Omega$ as

$$\widetilde{\gamma}(s) = \gamma(s) + \mu(\gamma(s))\big(\varepsilon_0 - d(\gamma(s), \partial\Omega)\big)^+\zeta s.$$

for $\zeta := \widetilde{\kappa}/\varepsilon_0$. Note that $\widetilde{\gamma}$ is Lipschitz continuous. In fact, for all s, $t \in (0, \ell_\gamma)$ we have that

$$\begin{aligned}|\widetilde{\gamma}(t) - \widetilde{\gamma}(s)| &\leq |\gamma(t) - \gamma(s)| + |\mu(\gamma(t)) - \mu(\gamma(s))|\big(\varepsilon_0 - d(\gamma(t), \partial\Omega)\big)^+\zeta t \\ &+ |\mu(\gamma(s))|\Big|\big(\varepsilon_0 - d(\gamma(t), \partial\Omega)\big)^+ - \big(\varepsilon_0 - d(\gamma(s), \partial\Omega)\big)^+\Big|\zeta t \\ &+ |\mu(\gamma(s))|\big(\varepsilon_0 - d(\gamma(s), \partial\Omega)\big)^+\zeta|t - s|\end{aligned}$$

ensuring that

$$\|\widetilde{\gamma}'\|_{L^\infty} \leq 1 + \|\nabla\mu\|_{L^\infty}\varepsilon_0\zeta\ell_{\max} + \varepsilon_0\zeta\ell_{\max} + \varepsilon_0\zeta =: M. \tag{3.3}$$

Moreover, $\widetilde{\gamma}(0) = \gamma(0) = x$, and, using also $\gamma(\ell_\gamma) = x_0$,

$$\widetilde{\gamma}(\ell_\gamma) = x_0 + \mu(x_0)\big(\varepsilon_0 - d(x_0, \partial\Omega)\big)^+\zeta\ell_\gamma = x_0.$$

The lower bound (3.1) and the triangle inequality allow us to control

$$\begin{aligned}\overline{\kappa}s &\leq d(\gamma(s), \partial U(t) \setminus \partial\Omega) \leq d(\widetilde{\gamma}(s), \partial U(t) \setminus \partial\Omega) + |\widetilde{\gamma}(s) - \gamma(s)| \\ &= d(\widetilde{\gamma}(s), \partial U(t) \setminus \partial\Omega) + \big(\varepsilon_0 - d(\gamma(s), \partial\Omega)\big)^+\zeta s \leq d(\widetilde{\gamma}(s), \partial U(t) \setminus \partial\Omega) + \varepsilon_0\zeta s\end{aligned}$$

entailing that

$$d(\widetilde{\gamma}(s), \partial U(t) \setminus \partial\Omega) \geq (\overline{\kappa} - \varepsilon_0\zeta)s \geq (2\widetilde{\kappa} - \varepsilon_0\zeta)s \quad \forall s \in [0, \ell_\gamma] \tag{3.4}$$

For all $s \in [0, \ell_\gamma]$, if $d(\gamma(s), \partial\Omega) \geq \varepsilon_0$ we use the fact that $\zeta\ell_{\max} = \widetilde{\kappa}\ell_{\max}/\varepsilon_0 \leq 1$ to find $d(\widetilde{\gamma}(s), \partial\Omega) \geq \varepsilon_0 \geq \varepsilon\zeta\ell_{\max} \geq \varepsilon_0\zeta s$. On the contrary, if $d(\gamma(s), \partial\Omega) < \varepsilon_0$ we have

$$\begin{aligned}d(\widetilde{\gamma}(s), \partial\Omega) &= d(\gamma(s), \partial\Omega) + \big(\varepsilon_0 - d(\gamma(s), \partial\Omega)\big)\zeta s \\ &= (1 - \zeta s)d(\gamma(s), \partial\Omega) + \varepsilon_0\zeta s \geq (1 - \zeta\ell_{\max})d(\gamma(s), \partial\Omega) + \varepsilon_0\zeta s \overset{\zeta\ell_{\max}\leq 1}{\geq} \varepsilon_0\zeta s.\end{aligned}$$

In both cases, we hence have that

$$d(\widetilde{\gamma}(s), \partial\Omega) \geq \varepsilon_0\zeta s \quad \forall s \in [0, \ell_\gamma]. \tag{3.5}$$

Putting (3.4) and (3.5) together we conclude that

$$\begin{aligned}d(\widetilde{\gamma}(s), \partial U(t)) &= \min\big\{d(\widetilde{\gamma}(s), \partial U(t) \setminus \partial\Omega), d(\widetilde{\gamma}(s), \partial U(t) \cap \partial\Omega)\big\} \\ &\geq \min\big\{d(\widetilde{\gamma}(s), \partial U(t) \setminus \partial\Omega), d(\widetilde{\gamma}(s), \partial\Omega)\big\} \\ &\geq \min\{2\widetilde{\kappa} - \varepsilon_0\zeta, \varepsilon_0\zeta\}s = \widetilde{\kappa}s \quad \forall s \in [0, \ell_\gamma].\end{aligned} \tag{3.6}$$

We conclude the proof by parametrizing $\widetilde{\gamma}$ by arc-length. Define $\widehat{\gamma} : [0, \ell_{\widehat{\gamma}}] \to \overline{\Omega}$ as $\widehat{\gamma}(\tau) = \widetilde{\gamma}(s(\tau))$ where $s : [0, \ell_{\widehat{\gamma}}] \to [0, \ell_\gamma]$ is the inverse function of $\tau : s \in [0, \ell_\gamma] \mapsto \int_0^s |\widetilde{\gamma}'(r)|\,\mathrm{d}r$. Using (3.3), for all $\tau \in [0, \ell_{\widehat{\gamma}}]$ we have that

$$s(\tau) = \int_0^\tau s'(t)\,\mathrm{d}t = \int_0^\tau \frac{\mathrm{d}t}{\tau'(s(t))} = \int_0^\tau \frac{\mathrm{d}t}{|\widetilde{\gamma}'(s(t))|} \overset{(3.3)}{\geq} \int_0^\tau \frac{\mathrm{d}t}{M} = \frac{\tau}{M}. \tag{3.7}$$

Hence, by setting $\kappa := \widetilde{\kappa}/M$ we obtain

$$d(\widehat{\gamma}(\tau), \partial U(t)) = d(\widetilde{\gamma}(s(\tau)), \partial U(t)) \overset{(3.6)}{\geq} \widetilde{\kappa}s(\tau) \overset{(3.7)}{\geq} \kappa\tau \quad \forall \tau \in [0, \ell_{\widehat{\gamma}}].$$

This concludes the proof.

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