SCATTERING PROBLEM FOR THE GENERALIZED KORTEWEG-DE VRIES EQUATION IN THE WEIGHTED SOBOLEV SPACE

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1. Introduction

This is a survey of the paper [16] which is a joint work with Satoshi Masaki (Hokkaido University) on the scattering problem for the generalized Korteweg-de Vries equation

$$\begin{cases} \partial_t u + \partial_x^3 u = \mu \partial_x (|u|^{2\alpha} u), & t, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is an unknown function, $u_0 : \mathbb{R} \to \mathbb{R}$ is a given function, and $\mu \in \mathbb{R}\backslash\{0\}$ and $\alpha > 0$ are constants. We call that (1.1) is defocusing if $\mu > 0$ and focusing if $\mu < 0$. Equation (1.1) is a generalization of the Korteweg-de Vries equation which models long waves propagating in a channel [11] and the modified Korteweg-de Vries equation which describes a time evolution for the curvature of certain types of helical space curves [12].

Equation (1.1) has the following conservation laws: If $u(t)$ is a solution to (1.1) on the time interval I with $0 \in I$, then, $u(t)$ has conservation of the mass

$$M[u(t)] := \frac{1}{2} \|u(t,\cdot)\|_{L^2}^2 = M[u_0] \tag{1.2}$$

and conservation of the energy

$$E[u(t)] := \frac{1}{2} \|\partial_x u(t,\cdot)\|_{L^2}^2 + \frac{\mu}{2\alpha+2} \|u(t,\cdot)\|_{L^{2\alpha+2}}^{2\alpha+2} = E[u_0] \tag{1.3}$$

for any $t \in I$.

We take the initial data u_0 from the weighted Sobolev space $H^1 \cap H^{0,1}$, where H^1 is the usual Sobolev space and $H^{0,1}$ is the weighted L^2 space defined by

$$H^{0,1} = \{f \in L^2 \ ; \ \|f\|_{H^{0,1}} = \|f\|_{L^2} + \|xf\|_{L^2} < \infty\}.$$

The purpose of this note is to achieve two primary goals. Firstly, we show small data scattering for (1.1) in the weighted Sobolev space $H^1 \cap H^{0,1}$, ensuring the initial and the asymptotic states belong to the same class. Secondly, we introduce two equivalent characterizations of scattering in the weighted Sobolev space $H^1 \cap H^{0,1}$. In particular, this involves the so-called conditional scattering in the weighted Sobolev space.

We now give the definition of the solution to (1.1) in $H^1 \cap H^{0,1}$. Let $\{V(t)\}_{t\in\mathbb{R}}$ be a unitary group generated by the $-\partial_x^3$, i.e.,

$$[V(t)f](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi+it\xi^3} \hat{f}(\xi) d\xi.$$

Definition 1.1 (a solution to (1.1) in $H^1 \cap H^{0,1}$)**.** *For an interval $I \subset \mathbb{R}$, we say a function $u : I \times \mathbb{R} \to \mathbb{R}$ is a $H^1 \cap H^{0,1}$-solution to (1.1) on I if $V(-t)u(t) \in C(I; H^1 \cap H^{0,1})$, $\|u\|_{L_x^{\frac{5}{2}\alpha}(\mathbb{R}; L_t^{5\alpha}(J))} < \infty$ for any compact $J \subset I$, and the identity*

$$V(-t_2)u(t_2) = V(-t_1)u(t_1) + \mu \int_{t_1}^{t_2} V(-\tau)\partial_x(|u|^{2\alpha}u)(\tau) d\tau \tag{1.4}$$

holds for any $t_1, t_2 \in I$.

With the definition of the solution to (1.1) in $H^1 \cap H^{0,1}$, we define a maximal lifespan of a $H^1 \cap H^{0,1}$-solution on an interval I. Let

$$\begin{aligned} T_{\max} &:= \sup\{T \in \mathbb{R}; \exists u : H^1 \cap H^{0,1}\text{-solution to (1.1) on } [t_0, T]\}, \\ T_{\min} &:= \inf\{T \in \mathbb{R}; \exists u : H^1 \cap H^{0,1}\text{-solution to (1.1) on } [T, t_0]\} \end{aligned}$$

with a picked $t_0 \in I$. Note that these quantities are independent of the choice of $t_0 \in I$. Further, we refer $I_{\max} = (T_{\min}, T_{\max})$ to as the maximal lifespan of a solution u. A solution u on $I_{\max}$ is referred to as a maximal-lifespan solution. We say a solution u is global for positive time direction (resp. negative time direction) if $T_{\max} = \infty$ (resp. $T_{\min} = -\infty$).

We give the definition of scattering in $H^1 \cap H^{0,1}$.

Definition 1.2. *We say a $H^1 \cap H^{0,1}$-solution $u(t)$ scatters in $H^1 \cap H^{0,1}$ for positive time direction if $T_{\max} = +\infty$ and there exists a unique function $u_+ \in H^1 \cap H^{0,1}$ such that*

$$\lim_{t\to+\infty} \|V(-t)u(t) - u_+\|_{H^1\cap H^{0,1}} = 0. \tag{1.5}$$

The scattering for negative time direction is defined by a similar fashion.

The following notation will be used throughout this note: $|D_x|^s = (-\partial_x^2)^{s/2}$ and $\langle D_x \rangle^s = (I - \partial_x^2)^{s/2}$ denote the Riesz and Bessel potentials of order $-s$, respectively.

The rest of this note is organized as follows. In Section 2, we consider the scattering problem for (1.1) with the small initial data. In Section 3, we consider the scattering criterion for (1.1).

2. Small data scattering for (1.1).

In this section, we consider the small data scattering for (1.1).

There are several results on the small data scattering for (1.1) where the initial and the asymptotic states belong to the same class. Kenig-Ponce-Vega [8] proved the small data scattering of (1.1) in the scaling critical Sobolev space $\dot{H}^{s_\alpha}$ for $\alpha \geqslant 2$, where $s_\alpha := 1/2 - 1/\alpha$. Since the scaling critical exponent s_α is negative in the mass-subcritical case $\alpha < 2$, the scattering of (1.1) in the scaling critical space $\dot{H}^{s_\alpha}$ becomes rather a difficult problem. Tao [17] proved global well-posedness and scattering for small data for (1.1) with the quartic nonlinearity $\mu\partial_x(u^4)$ in the scaling critical space $\dot{H}^{-1/6}$.

In [13], we proved small data scattering for (1.1) in the framework of the scaling critical Fourier-Lebesgue space $\hat{L}^\alpha$ for $8/5 < \alpha < 10/3$, where the Fourier-Lebesgue space is defined by

$$\hat{L}^r = \{f \in \mathcal{S}'(\mathbb{R}) \ ; \ \|f\|_{\hat{L}^r} = \|\hat{f}\|_{L^{r'}} < \infty\} \tag{2.1}$$

and r' denotes the Hölder conjugate of r.

We note that there are several results on the small data scattering of (1.1) for $\alpha \geqslant 1$ when the classes of the initial states and the asymptotic states are different (See Hayashi-Naumkin [3, 4, 5] for instance).

To state the our result on the small data scattering, we introduce a scattering norm for the solution to (1.1): For an interval $I \subset \mathbb{R}$, we define

$$\begin{aligned} S(I) &:= \{u : I \times \mathbb{R} \to \mathbb{R} \ ; \ \|u\|_{S(I)} < \infty\}, \\ \|u\|_{S(I)} &:= \|u\|_{L_x^{\frac{5}{2}\alpha}(\mathbb{R};L_t^{5\alpha}(I))}. \end{aligned} \tag{2.2}$$

We call $\| \cdot \|_{S(I)}$ "scattering norm".

Our first result is as follows.

Theorem 2.1 (Small data scattering in $H^1 \cap H^{0,1}$ ([16], Theorem 1.6))**.** *Let $8/5 < \alpha < 2$. Then there exists $\varepsilon_0 > 0$ such that if $u_0 \in H^1 \cap H^{0,1}(\mathbb{R})$ satisfies $\|u_0\|_{H^1 \cap H^{0,1}} \leqslant \varepsilon_0$, then there exists a unique $H^1 \cap H^{0,1}$-solution u to (1.1). Furthermore, u scatters in $H^1 \cap H^{0,1}$ for both time directions. Moreover,*

$$\|V(-t)u\|_{L^\infty(\mathbb{R};H^1\cap H^{0,1})} + \|u\|_{S(\mathbb{R})} + \sup_{t\in\mathbb{R}} \langle t\rangle^{\frac{1}{3}} \|u(t)\|_{L_x^\infty} \lesssim \|u_0\|_{H^1\cap H^{0,1}},$$

where $\langle t\rangle = \sqrt{1+t^2}$ and $\| \cdot \|_{S(\mathbb{R})}$ is defined by (2.2).

We now give the outline of the proof of Theorem 2.1. We focus on the proof of the scattering in $H^1 \cap H^{0,1}$ only, i.e., we prove that there uniquely exists $u_\pm \in H^1 \cap H^{0,1}$ such that

$$\lim_{t\to\pm\infty} \|V(-t)u(t) - u_\pm\|_{H^1\cap H^{0,1}} = 0. \tag{2.3}$$

To prove this, it suffices to show that $V(-t)u(t)$ is a Cauchy sequence in $H^1 \cap H^{0,1}$. We split the proof into two steps.

Step 1. We prove that $V(-t)u(t)$ is a Cauchy sequence in H^1, i.e.,

$$\lim_{s,t\to\pm\infty} \|V(-t)u(t) - V(-s)u(s)\|_{H^1} = 0. \tag{2.4}$$

From (1.4), we have

$$\begin{aligned} &\|V(-t)u(t) - V(-s)u(s)\|_{H_x^1} \\ &\qquad \leqslant C\|\int_s^t V(-\tau)\partial_x(|u|^{2\alpha}u)(\tau)d\tau\|_{H_x^1}. \end{aligned} \tag{2.5}$$

To evaluate the right hand side of (2.5), we employ the space-time estimates for the Airy evolution group $V(t)$.

To state this estimates, we define an admissible pair in L^2 for $V(t)$. Let $2 \leqslant p, q \leqslant \infty$. We call (p, q) is admissible in L^2 if (p, q) satisfies

$$0 \leqslant \frac{1}{p} \leqslant \frac{1}{4}, \qquad \frac{2}{p} + \frac{1}{q} = \frac{1}{2}.$$

Note that (p,q) is admissible in L^2 if and only if $(1/p, 1/q)$ belongs to the segment in Figure 1.

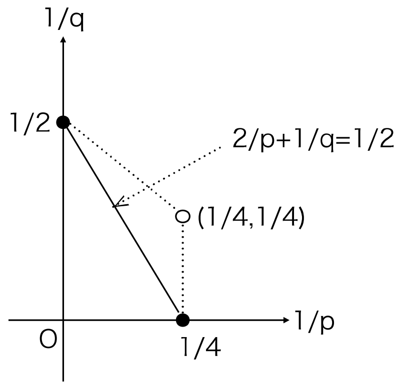

Figure 1

Proposition 2.2 (Space-time estimates ([7], Theorems 2.5 and 4.1))**.**
(i) (homogeneous estimate) Let (p,q) be admissible in L^2 and let I be an interval on $\mathbb{R}$. Then, the inequality

$$\| |D_x|^s V(t) f \|_{L^p_x(\mathbb{R}, L^q_t(I))} \leqslant C \|f\|_{L^2} \tag{2.6}$$

holds for any $f \in L^2$, where

$$s = -\frac{1}{p} + \frac{2}{q},$$

and the constant C depends on s and I.

(ii) (inhomogeneous estimate) Let (p_j, q_j) $(j = 1, 2)$ be admissible in L^2 and let I be an interval on $\mathbb{R}$. Then, the inequality

$$\left\| |D_x|^{s_1} \int_0^t V(t-\tau) F(\tau) d\tau \right\|_{L^{p_1}_x(\mathbb{R}, L^{q_1}_t(I))} \leqslant C \| |D_x|^{-s_2} F \|_{L^{p_2'}_x(\mathbb{R}, L^{q_2'}_t(I))} \tag{2.7}$$

holds for any F satisfying $|D_x|^{-s_2} F \in L^{p_2'}_x(\mathbb{R}, L^{q_2'}_t(I))$, where

$$s_j = -\frac{1}{p_j} + \frac{2}{q_j}, \qquad j = 1, 2,$$

and the constant C depends on s_1, s_2 and I.

Applying the inhomogeneous version of the space-time estimates for $V(t)$ (Proposition 2.2 (2.7)) to (2.5), we have

$$\begin{aligned}
&\|V(-t)u(t) - V(-s)u(s)\|_{H^1_x} \\
&\quad \lesssim \|\partial_x(|u|^{2\alpha}u)\|_{L^{\frac{5}{4}}_x(\mathbb{R}, L^{\frac{10}{9}}_t(s,t))} + \|\partial_x^2(|u|^{2\alpha}u)\|_{L^{\frac{5}{4}}_x(\mathbb{R}, L^{\frac{10}{9}}_t(s,t))} \\
&\quad \lesssim \|u\|^{2\alpha}_{S(s,t)} \|\partial_x u\|_{L^\infty_x(\mathbb{R}, L^2_t(\mathbb{R}))} + \|u\|^{2\alpha-1}_{S(s,t)} \|\partial_x u\|^2_{L^{5\alpha}_x(\mathbb{R}, L^{\frac{20\alpha}{5\alpha+2}}_t(s,t))} \\
&\quad \lesssim \|u\|^{2\alpha}_{S(s,t)} (\|\partial_x u\|_{L^\infty_x(\mathbb{R}, L^2_t(\mathbb{R}))} + \|\partial_x^2 u\|_{L^\infty_x(\mathbb{R}, L^2_t(\mathbb{R}))}),
\end{aligned}$$

where $\| \cdot \|_{S(s,t)}$ is defined by (2.2). Therefore to prove (2.4), we need

$$\|u\|_{S(\mathbb{R})} < +\infty, \tag{2.8}$$

$$\|\partial_x u\|_{L_x^\infty(\mathbb{R},L_t^2(\mathbb{R}))} + \|\partial_x^2 u\|_{L_x^\infty(\mathbb{R},L_t^2(\mathbb{R}))} < +\infty. \tag{2.9}$$

The difficulty to prove (2.8) is that $(5\alpha/2, 5\alpha)$ is not admissible in L^2. To overcome this difficulty, we employ the result on global well-posedness of (1.1) in the Fourier-Lebesgue space $\hat{L}^\alpha$.

To state this result, we define an admissible pair in $\hat{L}^\alpha$ for the Airy evolution group. Let $2 \leqslant p, q \leqslant \infty$. We call (p,q) is admissible in $\hat{L}^\alpha$ if (p,q) satisfies

$$0 \leqslant \frac{1}{p} < \frac{1}{4}, \quad 0 \leqslant \frac{1}{q} < \frac{1}{2} - \frac{1}{p}, \quad \text{and} \quad \frac{2}{p} + \frac{1}{q} = \frac{1}{\alpha}.$$

Note that (p,q) is admissible in $\hat{L}^\alpha$ if and only if $(1/p, 1/q)$ belongs to the segment in Figure 2.

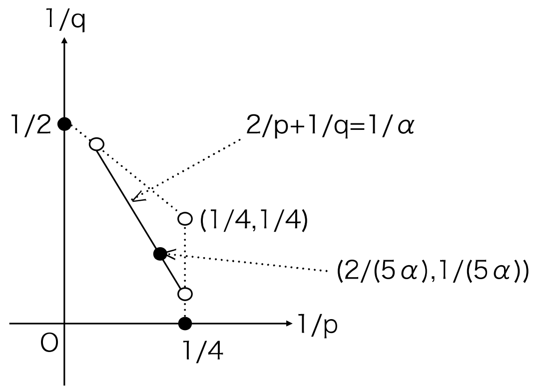

Figure 2

Proposition 2.3 (Global well-posedness in $\hat{L}^\alpha$ ([13], Theorem 1.7))**.** *Assume $8/5 < \alpha < 10/3$. Then there exists $\varepsilon_1 > 0$ such that if $u_0 \in \hat{L}^\alpha$ satisfies $\|u_0\|_{\hat{L}^\alpha} \leqslant \varepsilon_1$, then solution u to (1.1) satisfies $u \in C(\mathbb{R}, \hat{L}^\alpha(\mathbb{R}))$ and for any $\hat{L}^\alpha$-admissible pair (p,q),*

$$\||D_x|^s u\|_{L_x^p(\mathbb{R},L_t^q(\mathbb{R}))} < +\infty, \tag{2.10}$$

where $s = -\dfrac{1}{p} + \dfrac{2}{q}$.

Since $H^1 \cap H^{0,1} \subset \hat{L}^\alpha$ for any $1 \leqslant \alpha \leqslant \infty$, we see that if u_0 is sufficiently small in $H^1 \cap H^{0,1}$, then the corresponding solution u to (1.1) satisfies (2.10). In particular, we have (2.8) because $(5\alpha/2, 5\alpha)$ is admissible in $\hat{L}^\alpha$. Furthermore combining (2.8) with the space-time estimates for the free evolution group $V(t)$ (Proposition 2.2), we obtain (2.9). Collecting the above estimates, we have (2.4).

Step 2. We prove that $V(-t)u(t)$ is a Cauchy sequence in $H^{0,1}$, i.e.,

$$\lim_{s,t\to\pm\infty} \|x(V(-t)u(t) - V(-s)u(s))\|_{L^2} = 0. \tag{2.11}$$

To show this, we note

$$\begin{aligned}&\|x(V(-t)u(t)-V(-s)u(s))\|_{L^2}\\&\quad=\ \|V(-t)J(t)u(t)-V(-s)J(s)u(s)\|_{L^2},\end{aligned}$$

where $J(t):=V(t)xV(-t)=x-3t\partial_x^2$. Hence, to show (2.11), we need to obtain the space-time bound of $J(t)u$.

The difficulty to show (2.11) is that the operator $J(t)$ does not work well for the nonlinear term of (1.1) because J includes the second order derivatives. To overcome this difficulty, as in Hayashi and Naumkin [3, 4, 5], we introduce another variable

$$v:=J(t)u+3\mu t|u|^{2\alpha}u. \tag{2.12}$$

A direct computation shows that v solves a KdV-like equation

$$\partial_t v+\partial_x^3 v=(2\alpha+1)\mu|u|^{2\alpha}\partial_x v-2(\alpha-1)\mu|u|^{2\alpha}u. \tag{2.13}$$

It is noteworthy that the equation is written in the integral form and hence that one can utilize the space-time estimates for the free evolution group $V(t)$ (Proposition 2.2) to obtain various estimates for v. Hence we have (2.11). Collecting (2.4) and (2.11), we obtain (2.3).

3. Scattering Criterion for (1.1).

In this section, we consider the scattering criterion for (1.1). For (1.1) with the large initial data, Dodson [1] has shown the global well-posedness and scattering in L^2 for (1.1) with the defocusing and mass-critical nonlinearity (i.e., $\mu>0$ and $\alpha=2$). His proof is based on the concentration compactness argument by Kenig and Merle [6] and the monotonicity formula for (1.1) by Tao [18] (see also Killip-Kwon-Shao-Vişan [9] for the existence of the minimal non-scattering solution for (1.1) with the focusing, mass-critical nonlinearity). After that Farah-Linares-Pastor-Visciglia [2] proved the global well-posedness and scattering in H^1 for (1.1) with the defocusing and mass-supercritical nonlinearity (i.e., $\mu>0$ and $\alpha>2$) by adapting the the concentration compactness argument into H^1. For the mass-subcritical case $\alpha<2$, in [14, 15] we proved the existence of the minimal non-scattering solution for (1.1) with $5/3<\alpha<2$ by applying the concentration compactness argument in the Fourier-Bourgain-Morrey space $\hat{M}^{\beta}_{2,\delta}$, where

$$\begin{aligned}\hat{M}^{\beta}_{2,\delta}\ &=\ \{f\in\mathcal{S}'(\mathbb{R})\ ;\ \|f\|_{\hat{M}^{\beta}_{2,\delta}}<\infty\},\\ \|f\|_{\hat{M}^{\beta}_{2,\delta}}\ &=\ \left\||\tau^j_k|^{\frac12-\frac1\beta}\|\hat{f}\|_{L^2(\tau^j_k)}\right\|_{\ell^\delta_{j,k}}\end{aligned}$$

with $\tau^j_k:=[k2^{-j},(k+1)2^{-j})$, $j,k\in\mathbb{Z}$, $1\leqslant\beta\leqslant2$, and $\beta'<\delta\leqslant\infty$. Furthermore, Kim [10] proved the conditional scattering in the Fourier-Bourgain-Morrey space for (1.1) when the nonlinear term is defocusing and mass-subcritical with $5/3<\alpha<2$.

The second main result is the two equivalent characterizations of the scattering for (1.1) in the weighted Sobolev space.

Theorem 3.1 (Scattering criterion ([16], Theorem 1.7))**.** *Assume* $8/5<\alpha<2$. *Let* $u(t)$ *be a unique maximal-lifespan* $H^1\cap H^{0,1}$*-solution of* (1.1). *The following statements are equivalent:*

(i) $u(t)$ *scatters for positive time direction in* $H^1 \cap H^{0,1}$*;*

(ii) $u(t)$ *is bounded in a weighted norm, i.e., for some* $t_0 \in I_{\max}$*,*

$$\sup_{t\in[t_0,T_{\max})} \|V(-t)u(t)\|_{H^{0,1}_x} < +\infty. \tag{3.1}$$

(iii) *There exist* $\kappa > \frac{\alpha}{3(\alpha-1)(2\alpha+1)}$ *and* $t_0 \in I_{\max}$ *such that*

$$\|u\|_{S([t_0,T_{\max}))} + \sup_{t\in[t_0,T_{\max})} \langle t\rangle^{\kappa} \|u(t)\|_{L^{2(2\alpha+1)}_x} < +\infty,$$

where $S(I)$ *is given by (2.2).*

Further, if one of the above is satisfied then $T_{\max} = \infty$ *and*

$$\|V(-t)u\|_{L^\infty([t_0,\infty);H^1\cap H^{0,1})} + \|u\|_{S([t_0,\infty))} + \sup_{t\in[t_0,\infty)} \langle t\rangle^{\frac{1}{3}} \|u(t)\|_{L^\infty_x} < \infty$$

for any $t_0 \in I_{\max}$*. The similar statements are true for negative time direction.*

Remark 3.2. For a $\hat{L}^\alpha$-solution to (1.1), the boundedness $\|u\|_{S([t_0,T_{\max}))} < \infty$ is a necessary and sufficient condition for scattering in $\hat{L}^\alpha$ for positive time direction (See [13, Theorem 1.7] for the detail). The equivalence of (i) and (iii) in Theorem 3.1 implies that the additional boundedness condition

$$\sup_{t\in[t_0,T_{\max})} \langle t\rangle^{\kappa}\|u(t)\|_{L^{2(2\alpha+1)}_x} < +\infty$$

bridges the gap between scattering in $\hat{L}^\alpha$ and in $H^1 \cap H^{0,1}$. This gap arises due to the weakness of the persistence property of $H^1 \cap H^{0,1}$-solution.

Remark 3.3. We remark that the implication "(ii)⇒(i)" in Theorem 3.1 reads as a conditional scattering result. Indeed, it establishes the scattering under the hypothesis of the a priori bound (3.1). As mentioned above, Kim [10] showed a conditional scattering result for $5/3 < \alpha < 2$ under the boundedness in H^1 and in a Fourier-Bourgain-Morrey space:

$$\sup_{t\in[t_0,T_{\max})} \|\langle D_x\rangle^{\sigma} u\|_{\hat{M}^{\beta}_{2,\delta}} < +\infty.$$

Compared with the result, Theorem 3.1 covers a wider range $8/5 < \alpha < 2$ with a stronger boundedness assumption (3.1).

Remark 3.4. Let us compare the conditional scattering result Theorem 3.1 with the similar results for the nonlinear Schrödinger equation:

$$\begin{cases} i\partial_t u + \Delta u = \mu|u|^{2\alpha}u, & t \in \mathbb{R}, x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{3.2}$$

where $u : \mathbb{R}\times\mathbb{R}^d \to \mathbb{C}$ is an unknown function, $u_0 : \mathbb{R}^d \to \mathbb{C}$ is a given function, and $\mu \in \mathbb{R}\backslash\{0\}$ and $\alpha > 0$ are constants. For (3.2) with the defocusing nonlinearity (i.e., $\mu > 0$), by utilizing the pseudo-conformal transform

$$u(t,x) = \frac{1}{(it)^{d/2}} \overline{v(\frac{1}{t},\frac{x}{t})} e^{\frac{i|x|^2}{4t}},$$

Tsutsumi [19] has shown that any $H^{0,1}$-solution scatters in $H^{0,1}$ when $\alpha \geqslant (-d+2+\sqrt{d^2+12d+4})/(4d)$. As far as the authors know, this kind of transform is not known for (1.1).

We give the outline of the proof of Thorem 3.1. We focus on "ii)⇒i)" only.

Suppose that

$$\sup_{t\in[t_0,T_{\max})} \|V(-t)u(t)\|_{H^{0,1}_x} < +\infty \tag{3.3}$$

for some $t_0 \in I_{\max}$. By the local well-posedness of (1.1) in $H^1 \cap H^{0,1}$, one sees that $T_{\max} = \infty$. Hence, by replacing t_0 with a larger one if necessary, one may suppose that $t_0 > 1$. Let us prove the bound

$$\|u\|_{S([\widetilde{t}_0,\infty))} < \infty \tag{3.4}$$

holds for some $\widetilde{t}_0 \geqslant t_0$. By the Klainerman-Sobolev type inequality

$$\|u\|_{L^p} \lesssim |t|^{-\frac{1}{3}+\frac{2}{3p}} \|u\|_{L^2}^{\frac{1}{2}+\frac{1}{p}} \|J(t)u\|_{L^2}^{\frac{1}{2}-\frac{1}{p}}$$

with $p \in [2,\infty]$ (see [16, Lemma 4.5] for instance), the assumption (3.3) and the conservation of mass (1.2), one sees that

$$t^{\frac{2\alpha-1}{3(2\alpha+1)}} \|u(t)\|_{L^{2\alpha+1}} \lesssim \|u_0\|_{L^2}^{\frac{1}{2}+\frac{1}{2\alpha+1}} \|J(t)u\|_{L^\infty L^2}^{\frac{1}{2}-\frac{1}{2\alpha+1}} < \infty.$$

In particular, $\|u(t)\|_{L^{2\alpha+1}}$ is bounded uniformly in t. Then, by the conservation of energy (1.3) and the assumption (3.3) on u, we have

$$\sup_{t\in[t_0,\infty)} \|V(-t)u(t)\|_{H^1\cap H^{0,1}} < \infty. \tag{3.5}$$

Pick a time sequence $\{t_n\}_{n\geqslant 1} \subset [t_0,\infty)$ so that $t_n < t_{n+1} \to \infty$ as $n \to \infty$. Since the embedding $H^1 \cap H^{0,1} \hookrightarrow \hat{L}^\alpha$ is compact, one can choose a subsequence, which we denote again by $\{t_n\}$, so that $V(-t_n)u(t_n)$ converges (strongly) in $\hat{L}^\alpha$. Let $\psi_+ \in \hat{L}^\alpha$ be the limit of the subsequence.

We let $\tilde{u}(t)$ be a unique $\hat{L}^\alpha$-solution to (1.1) which scatters to ψ_+ in $\hat{L}^\alpha$, i.e.,

$$\|V(-t)\tilde{u}(t) - \psi_+\|_{\hat{L}^\alpha} \to 0$$

as $t \to \infty$. We choose $T \in \mathbb{R}$ so that $\tilde{u}(t)$ exists on $[T,\infty)$. Without loss of generality, we may suppose that $T \geqslant t_0$.

Note that

$$\|\tilde{u}\|_{S([T,\infty))\cap X([T,\infty))} < +\infty,$$

where

$$\begin{aligned} X(I) &= \{u : I\times\mathbb{R}\to\mathbb{R}\ ;\ \|u\|_{X(I)} < \infty\}, \\ \|u\|_{X(I)} &= \||D_x|^s u\|_{L_x^{\frac{20\alpha}{10-3\alpha}}(\mathbb{R};L_t^{\frac{10}{3}}(I))} \end{aligned} \tag{3.6}$$

with $s = \frac{3}{4} - \frac{1}{2\alpha}$.

We now employ the long time perturbation for (1.1).

Proposition 3.5 (Long time perturbation ([16], Proposition 2.4))**.** *Assume $8/5 < \alpha < 2$. For any $M > 0$, there exists $\varepsilon_2 > 0$ such that the following property holds: Let $t_0 \in \mathbb{R}$ and let $I \subset \mathbb{R}$ be an interval such that $t_0 \in \overline{I}$. Let $\widetilde{u} : I\times\mathbb{R}\to\mathbb{R}$ be a function such that $\widetilde{u} \in S(I)\cap X(I)$, where $S(I)$ and $X(I)$ are given by (2.2) and (3.6). Put*

$$\mathcal{E} := (\partial_t + \partial_x^3)\widetilde{u} - \mu\partial_x(|\widetilde{u}|^{2\alpha}\widetilde{u}).$$

Let $u_0 \in \hat{L}^\alpha$*. Suppose that*

$$\|\widetilde{u}\|_{S(I)\cap X(I)} \leqslant M,$$

$$\left\| V(t-t_0)(u(t_0)-\widetilde{u}(t_0)) - \int_{t_0}^{t} V(t-\tau)\mathcal{E}(\tau)d\tau \right\|_{S(I)\cap X(I)} \leqslant \varepsilon_2.$$

Then, there exists a solution $u \in C(I, \hat{L}^\alpha)$ *of (1.1) satisfying* $u(t_0) = u_0$ *and*

$$\|u-\widetilde{u}\|_{S(I)\cap X(I)} \leqslant C\varepsilon_2,$$

where C *depends only on* M*.*

Let $M := \|\tilde{u}\|_{S([T,\infty))\cap X([T,\infty))}$ and let $\varepsilon_2 = \varepsilon_2(M)$ be given in Proposition 3.5. Applying Proposition 3.5 with the above M and ε_2, and the choice $\widetilde{u}(t) := \tilde{u}(t)$, $I := [T,\infty)$, and $t_0 := t_n$ with sufficiently large n, we obtain

$$\|u-\tilde{u}\|_{S([T,\infty))\cap X([T,\infty))} \lesssim \varepsilon_2.$$

Hence, we have

$$\|u\|_{S([T,\infty))\cap X([T,\infty))} \lesssim M+\varepsilon_2.$$

Thus, we obtain (3.4) with the choice $\widetilde{t_0} = T$. Combining (3.4) and (3.5), we are able to show that $u(t)$ scatters in $H^1 \cap H^{0,1}$ for positive time direction. Thus, we have "(ii)⇒(i)". □

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