A critical blow-up exponent in a three-dimensional chemotaxis-May–Nowak model

Yuya Tanaka

Department of Mathematical Sciences,
Kwansei Gakuin University

1. Introduction

This paper is based on a joint work with Mario Fuest (Leibniz University Hannover). The May–Nowak model

$$\begin{cases} u_t = -u - Kuw + \kappa, & t > 0, \\ v_t = -v + Kuw, & t > 0, \\ w_t = -w + v, & t > 0 \end{cases}$$

was introduced as one of epidemic models (e.g. HIV infection) by Nowak and May [**14**], where $K > 0$ and $\kappa \geq 0$ are constants, and the functions u and v model the number of uninfected and infected cells, respectively, and w denotes the concentration of virus particles. Global properties of solutions for this model have been obtained in [**11**].

By biological experiments it was confirmed that the cells and the virus have random spatial movements, and the uninfected cells are attracted by high concentrations of the infected cells ([**8, 12, 13**]). In line with this reason, the chemotaxis-May–Nowak model

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) - u - Kuw + \kappa, & x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v - v + Kuw, & x \in \Omega,\ t > 0, \\ w_t = \Delta w - w + v, & x \in \Omega,\ t > 0 \end{cases} \tag{1.1}$$

was proposed in [**15**], where $\Omega \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) is a smooth bounded domain, and $\chi > 0$ and $\tau \in \{0, 1\}$ are constants. The term $-\chi \nabla \cdot (u \nabla v)$ represents the chemotactic attraction of the uninfected cells. Systems with such a term, that is, so-called cheomtaxis systems have been introduced by Keller and Segel [**10**], and also, there are a lot of works on boundedness and finite-time blow-up of solutions in chemotaxis systems (see the survey [**1**]). Similarly, as to the model (1.1) results not only on boundedness but also on finite-time blow-up were obtained; in the case that $n = 1$ and $\tau = 1$ solutions are always global and bounded ([**20**]); in the case that $n \geq 2$ and $\tau = 1$ global boundedness of solutions were established under a smallness condition for χ ([**2**]); on the other hand, if χ is large, then finite-time blow-up occurs in the radial setting in the case that $n \in \{2, 3\}$ and $\tau = 0$ ([**19**]).

Moreover, to consider more realistic situations, some modified models were proposed and investigated in [**4, 9**]. In particular, in [**4**] the model (1.1) with $\tau = 1$ and with Kuw replaced by $Ku^\alpha w$ ($\alpha > 0$) was considered and boundedness of solutions was shown under the condition $\alpha < \frac{2}{n}$. For this result we naturally have the following question:

Does the solution remain bounded in the case $\alpha \geq \frac{2}{n}$?

In this paper we give an answer for this question in the three-dimensional case.

Problem and main result. We consider the parabolic–elliptic–parabolic version of the model in [**4**],

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\nabla v) - u - Ku^\alpha w, & x \in B_R,\ t > 0, \\ 0 = \Delta v - v + Ku^\alpha w, & x \in B_R,\ t > 0, \\ w_t = \Delta w - w + v, & x \in B_R,\ t > 0, \\ \nabla u \cdot \nu = \nabla v \cdot \nu = \nabla w \cdot \nu = 0, & x \in \partial B_R,\ t > 0, \\ u(x,0) = u_0(x), \quad w(x,0) = w_0(x), & x \in B_R, \end{cases} \tag{1.2}$$

where $B_R \subset \mathbb{R}^3$ is a ball with some $R > 0$; $K, \alpha > 0$ are constants; ν is the outward normal vector to ∂B_R; $u_0, w_0 \in C^0(\overline{B_R})$ are nonnegative initial data. The main result reads as follows.

Theorem 1.1 ([**7**]). *Let $B_R \subset \mathbb{R}^3$ be a ball with some $R > 0$, and let $K > 0$. Assume that*

$$\frac{2}{3} < \alpha \le 1.$$

Then for all radially symmetric positive function $w_0 \in C^0(\overline{B_R})$ and all $m > 0$, $m_1 \in (0, m)$ there exist $r_1 \in (0, R)$ and $\eta > 0$ with the following property: Whenever $u_0 \in C^0(\overline{B_R})$ is radially symmetric and nonnegative and satisfies

$$\int_{B_R} u_0(x)\,dx = m \quad \text{and} \quad \int_{B_{r_1}} u_0(x)\,dx \ge m_1$$

as well as

$$u_0(x) \le m|x|^{-3(3\alpha-1)-\eta} \quad \text{for all } x \in B_R, \tag{1.3}$$

there exist $T_{\max} \in (0, \infty)$ and a unique nonnegative classical solution (u, v, w) of (1.2) blowing up in finite time in the sense that

$$\lim_{t \nearrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(B_R)} = \infty.$$

Remark 1.1. This theorem gives us the answer such that in the three-dimensional case the value $\alpha = \frac{2}{3}$ is the critical exponent distinguishing between boundedness and finite-time blow-up of solutions.

Our approach is based on an analysis of the moment-type functional ϕ defined by $\phi(t) := \int_0^{s_0} s^{-\gamma}(s_0 - s)\big[\int_0^{s^{1/3}} \rho^2 u(\rho, t)\,d\rho\big]\,ds$ for $t \in (0, T_{\max})$ with some $s_0 \in (0, R^3)$ and $\gamma \in (0, 1)$, which was used in studies of finite-time blow-up for chemotaxis systems with logistic source (see e.g. [**16, 17, 18, 21**]). The goal is to establish the differential inequality

$$\phi'(t) \ge C\phi^2(t) - C'\phi(t) - C' \tag{1.4}$$

with some $C, C' > 0$, which leads to $T_{\max} < \infty$. One of the key to the proof is to show upper and lower estimates for w. This together with (1.3) and [**6**, Theorem 1.4] leads to the estimate $u(x,t) \le C_3|x|^{-3(3\alpha-1)-\eta}$ with some $C_3 > 0$, which plays an important role in controlling the term $Ku^\alpha w$.

2. Preliminaries

We first state a result on local existence, which can be proved as in [**19**, Lemma 2.1].

Lemma 2.1. *Let $m > 0$. Suppose that*

$$u_0 \in C^0(\overline{B_R}) \text{ is nonnegative and radially symmetric with } \int_{B_R} u_0(x)\,dx = m \tag{2.1}$$

and

$$w_0 \in C^0(\overline{B_R}) \text{ is positive and radially symmetric.} \tag{2.2}$$

Then there exist $T_{\max} \in (0,\infty]$ and a classical solution (u,v,w) of (1.2)*, uniquely determined by the inclusions*

$$\begin{aligned} u &\in C^0(\overline{B_R} \times [0,T_{\max})) \cap C^{2,1}(\overline{B_R} \times (0,T_{\max})),\\ v &\in \bigcap_{q>n} C^0([0,T_{\max}); W^{1,q}(B_R)) \cap C^{2,0}(\overline{B_R} \times (0,T_{\max})),\\ w &\in C^0(\overline{B_R} \times [0,T_{\max})) \cap C^{2,1}(\overline{B_R} \times (0,T_{\max})), \end{aligned}$$

such that

$$\text{if} \quad T_{\max} < \infty, \quad \text{then} \quad \lim_{t \nearrow T_{\max}} \left\{ \|u(\cdot,t)\|_{L^\infty(B_R)} + \|w(\cdot,t)\|_{L^\infty(B_R)} \right\} = \infty. \tag{2.3}$$

Moreover, $u, v,$ and w are nonnegative and radially symmetric.

In the model (1.2) an upper bound for the total mass $\int_{B_R} u$ can be immediately obtained. As a preparation we introduce the following statement.

Lemma 2.2. *Suppose* (2.1) *and* (2.2)*. Then the following holds.*

$$\int_{B_R} u(x,t)\,dx \le \int_{B_R} u_0(x)\,dx \qquad \text{for all } t \in [0,T_{\max}). \tag{2.4}$$

Proof. Integrating the first equation in (1.2), we have $\frac{d}{dt}\int_{B_R} u = -\int_{B_R} u - K\int_{B_R} u^\alpha w \le 0$ for all $t \in (0,T_{\max})$, which proves this lemma. □

In the case $\alpha = 1$, Theorem 1.1 has been established without the condition (1.3) in [**19**]. Therefore, throughout the sequel, we consider only the case

$$\frac{2}{3} < \alpha < 1. \tag{2.5}$$

Moreover, without further explicit mentioning, we always assume $K > 0$, (2.1) and (2.2). Also, introducing $r := |x|$, we denote by $(u,v,w) = (u(r,t), v(r,t))$ the radially symmetric classical solution of (1.2) in $\overline{B_R} \times (0,T_{\max})$ given by Lemma 2.1.

3. Upper and lower estimates for w

In this section we first show an upper estimate for w. The following lemma can be proved as in the proof of [**19**, Lemma 3.4]. However, due to the importance of this lemma, we give a full proof here.

Lemma 3.1. *There exists $t_* \le 1$ such that*

$$\|w(\cdot,t)\|_{L^\infty(B_R)} \le 2\|w_0\|_{L^\infty(B_R)} =: K^* \qquad \text{for all } t \in (0, \min\{t_*, T_{\max}\}).$$

Proof. We define $\widehat{T} > 0$ as

$$\widehat{T} := \sup\left\{\tau \in (0, \min\{t_*, T_{\max}\}) \,\middle|\, \|w\|_{L^\infty(B_R)} < 2\|w_0\|_{L^\infty(B_R)} \quad \text{on } [0,\tau)\right\}, \tag{3.1}$$

where $t_* \le 1$ will be given later. We shall make sure that $\widehat{T} = \min\{t_*, T_{\max}\}$ by a contradiction argument. To this end, let us assume that $\widehat{T} < \min\{t_*, T_{\max}\}$. Then we have $\|w(\cdot,\widehat{T})\|_{L^\infty(B_R)} = 2\|w_0\|_{L^\infty(B_R)}$. Let $p \in \left[\frac{3}{2}, 3\right)$. By virtue of [**3**, Lemma 23] with the Sobolev embedding theorem and the second equation in (1.2) there is $c_1 > 0$ such that

$$\|v\|_{L^p(B_R)} \le c_1\|-\Delta v + v\|_{L^1(B_R)} = c_1 K\|u^\alpha w\|_{L^1(B_R)} \le c_1 K\|u^\alpha\|_{L^1(B_R)}\|w\|_{L^\infty(B_R)},$$

which together with Hölder's inequality, (2.4) and (3.1) yields

$$\|v\|_{L^p(B_R)} \le 2c_1 K|B_R|^{1-\alpha}\|u_0\|^\alpha_{L^1(B_R)}\|w_0\|_{L^\infty(B_R)} =: c_2 \quad \text{on } (0,\widehat{T}).$$

Noting that $t_* \le 1$, from the third equation in (1.2) and semigroup estimates ([**19**, Lemma 3.3]) we obtain $c_3 > 0$ such that

$$\begin{aligned}\|w(\cdot,t)\|_{L^\infty(B_R)} &\le \|w_0\|_{L^\infty(B_R)} + c_3\int_0^t (t-s)^{-\frac{3}{2}\cdot\frac{1}{p}}\|v(\cdot,s)\|_{L^p(B_R)}\,ds \\ &\le \|w_0\|_{L^\infty(B_R)} + c_2c_3\cdot\frac{t^{1-\frac{3}{2}\cdot\frac{1}{p}}}{1-\frac{3}{2}\cdot\frac{1}{p}} \quad \text{for all } t \in [0,\widehat{T}].\end{aligned}$$

Now we choose

$$t_* := \min\left\{1, \left(\frac{\left[1-\frac{3}{2}\cdot\frac{1}{p}\right]\|w_0\|_{L^\infty(B_R)}}{2c_2c_3}\right)^{\frac{1}{1-\frac{3}{2}\cdot\frac{1}{p}}}\right\}.$$

Then $\|w(\cdot,t)\|_{L^\infty(B_R)} \le \|w_0\|_{L^\infty(B_R)} + \frac{1}{2}\|w_0\|_{L^\infty(B_R)}$ for all $t \in [0,\widehat{T}]$, which contradicts $\|w(\cdot,\widehat{T})\|_{L^\infty(B_R)} = 2\|w_0\|_{L^\infty(B_R)}$. Thus we can see that $\widehat{T} = \min\{t_*, T_{\max}\}$, and moreover, (3.1) leads to the conclusion. □

Next we prove a lower bound for w.

Lemma 3.2. *There exists $t_{**} > 0$ such that*

$$w(x,t) \ge \frac{1}{2}\inf_{x\in B_R} w_0(x) =: K_* \qquad \text{for all } (x,t) \in B_R \times (0, \min\{t_{**}, T_{\max}\}).$$

Proof. We put

$$t_{**} := \min\left\{t_*, \frac{1}{2K^*}\inf_{x\in B_R} w_0(x)\right\},$$

where $t_* > 0$ and $K^* > 0$ are given by Lemma 3.1. This together with the third equation in (1.2) and a simple comparison argument yields this lemma (see [**19**, Lemma 3.5]). □

4. Differential inequality for a moment-type functional

We put $T := \min\{t_{**}, T_{\max}\}$, where t_{**} is given by Lemma 3.2. Following [**19**], we set

$$U(s,t) = \int_0^{s^{\frac{1}{3}}} \rho^2 u(\rho,t)\,d\rho \quad \text{and} \quad V(s,t) = \int_0^{s^{\frac{1}{3}}} \rho^2 v(\rho,t)\,d\rho$$

for $(s,t) \in [0,R^3] \times [0,T)$, which transform (1.2) to a scalar equation by a straightforward calculations; more precisely, from the second and first equations in (1.2) we have

$$s^{\frac{2}{3}} v_r(s^{\frac{1}{3}},t) = V(s,t) - 3^{\alpha-1}\int_0^s U_s^\alpha(\sigma,t)w(\sigma^{\frac{1}{3}},t)\,d\sigma$$

and

$$\begin{aligned} U_t(s,t) &= 9s^{\frac{4}{3}}U_{ss}(s,t) + 3^\alpha K U_s(s,t)\int_0^s U_s^\alpha(\sigma,t)w(\sigma^{\frac{1}{3}},t)\,d\sigma - 3V(s,t)U_s(s,t) \\ &\quad - U(s,t) - 3^{\alpha-1}K\int_0^s U_s^\alpha(\sigma,t)w(\sigma^{\frac{1}{3}},t)\,d\sigma \end{aligned} \tag{4.1}$$

for all $(s,t) \in (0,R^3) \times (0,T)$. Moreover, as in [**21**] (also, e.g. [**16, 17, 18**]), introducing the moment-type functional

$$\phi(s_0,t) := \int_0^{s_0} s^{-\gamma}(s_0-s)U(s,t)\,ds \qquad \text{for } (s_0,t) \in [0,R^3] \times [0,T), \tag{4.2}$$

where $\gamma \in (0,1)$ and ϕ belongs to $C^0([0,T)) \cap C^1((0,T))$ for each $s_0 \in (0,R^3)$, we will derive the desired superlinear differential inequality (1.4). To achieve this, the following pointwise estimate for u plays an important role.

Lemma 4.1. *Let $\eta > 0$. Then there is $L > 0$ such that for each u_0 with* (2.1) *and* (1.3),

$$u(x,t) \le L|x|^{-3(3\alpha-1)-\eta} \quad \textit{for all } (x,t) \in B_R \times (0,T). \tag{4.3}$$

Proof. By applying Lemma 3.1 to the last term in the second equation in (1.2), we have $Ku^\alpha w \le K^* K u^\alpha$ on $(0,T)$. Also, it follows from the first equation in (1.2) that $u_t \le \Delta u - \nabla\cdot(u\nabla v)$. Noting (2.4) holds, we employ [**6**, Theorem 1.4] to obtain (4.3). □

This lemma implies that $u^{\alpha-1}(r,t) \ge L^{\alpha-1}r^{3(1-\alpha)(3\alpha-1+\frac{\eta}{3})}$ for all $(r,t) \in (0,R) \times (0,T)$, that is, $U_s^{\alpha-1}(s,t) \le \left(\frac{L}{3}\right)^{\alpha-1} s^{(1-\alpha)(3\alpha-1+\frac{\eta}{3})}$ for all $(s,t) \in (0,R^3) \times (0,T)$. By utilizing this estimate and Lemma 3.2 to the second term in the right of (4.1) and integrating by parts, we can derive the following estimate.

Lemma 4.2. *Let $\eta > 0$. Then there are $C_1, C_2 > 0$ such that for each u_0 fulfilling* (2.1) *and* (1.3),

$$\begin{aligned} \phi_t(s_0,t) &\ge 9\int_0^{s_0} s^{\frac{4}{3}-\gamma}(s_0-s)U_{ss}\,ds + C_1\int_0^{s_0} s^{\lambda-\gamma}(s_0-s)UU_s\,ds \\ &\quad - C_2\int_0^{s_0} s^{-\gamma}(s_0-s)\int_0^s \sigma^{\lambda-1}U(\sigma,t)\,d\sigma\,ds - 3\int_0^{s_0} s^{-\gamma}(s_0-s)VU_s\,ds \\ &\quad - \phi(s_0,t) - 3^{\alpha-1}K\int_0^{s_0} s^{-\gamma}(s_0-s)\int_0^s U_s^\alpha(\sigma,t)w(\sigma^{\frac{1}{3}},t)\,d\sigma\,ds \\ &=: I_1(s_0,t) + I_2(s_0,t) + I_3(s_0,t) + I_4(s_0,t) + I_5(s_0,t) + I_6(s_0,t) \end{aligned} \tag{4.4}$$

for all $(s_0,t) \in (0,R^3) \times (0,T)$, where $\lambda = (1-\alpha)(3\alpha-1+\frac{\eta}{3})$.

The choice $\gamma = \frac{1}{3}$ in the above lemma makes treating I_1 particularly simple.

Lemma 4.3. *Let $\gamma = \frac{1}{3}$. Then there is $C_3 > 0$ such that for each u_0 fulfilling* (2.1) *and with I_1 as in* (4.4),

$$I_1(s_0,t) \geq -C_3 s_0 \qquad \text{for all } (s_0,t) \in (0,R^3) \times (0,T).$$

Proof. We integrate by parts and rely on (2.4) to conclude that

$$\begin{aligned}\frac{I_1(s_0,t)}{9} &= \int_0^{s_0} s(s_0-s)U_{ss}(s,t)\,ds = -\int_0^{s_0}(s_0-2s)U_s(s,t)\,ds \\ &\geq -s_0\int_0^{s_0} U_s(s,t)\,ds = -s_0 U(s_0,t) \geq -\frac{m}{\omega_3}s_0\end{aligned}$$

for all $(s_0,t) \in (0,R^3)\times(0,T)$, where $\omega_3 := 3|B_R|$. □

In order to show (1.4) we will estimate I_3, I_4 and I_6 by I_2. Moreover, we need the relation between I_2 and ϕ. The following lemma enables us to derive such estimates, which is established as in [**21**, Lemma 4.2].

Lemma 4.4. *Let $s_0 > 0$ and $\gamma, \lambda \in \mathbb{R}$ with $\gamma > \lambda$, and suppose that $\varphi \in C^1([0,s_0];[0,\infty))$ fulfills $\varphi(0) = 0$ and $\varphi' \geq 0$ in $(0,s_0)$. Then*

$$\varphi(s) \leq \sqrt{2}s^{\frac{\gamma-\lambda}{2}}(s_0-s)^{-\frac{1}{2}}\left(\int_0^{s_0}\sigma^{\lambda-\gamma}(s_0-\sigma)\varphi(\sigma)\varphi'(\sigma)\,d\sigma\right)^{\frac{1}{2}} \qquad \text{for all } s \in (0,s_0).$$

By making use of this lemma, we can provide some estimates for I_2, I_3 and I_6.

Lemma 4.5. *Let $\gamma = \frac{1}{3}$. Then there exists $\eta > 0$ such that $\gamma > \lambda$, where λ is as given by Lemma 4.2. Moreover, there exist $C_4, C_5, C_6 > 0$ such that for each u_0 fulfilling* (2.1) *and with ϕ, I_2, I_3 and I_6 as in* (4.2) *and* (4.4),

$$I_2(s_0,t) \geq C_4 s_0^{\lambda+\gamma-3}\phi^2(s_0,t) \tag{4.5}$$

as well as

$$I_3(s_0,t) \geq -C_5 s_0^{\frac{3}{2}+\frac{\lambda-\gamma}{2}} I_2^{\frac{1}{2}}(s_0,t) \quad \text{and} \quad I_6(s_0,t) \geq -C_6 s_0^{3-\frac{3\alpha}{2}-\gamma+\frac{(\gamma-\lambda)\alpha}{2}} I_2^{\frac{\alpha}{2}}(s_0,t) \tag{4.6}$$

for all $(s_0,t) \in (0,R^3)\times(0,T)$.

Proof. It follows from (2.5) that $(1-\alpha)(3\alpha-1) < \frac{1}{3}$, and thus we can take $\eta > 0$ small enough such that $(1-\alpha)(3\alpha-1+\frac{\eta}{3}) < \frac{1}{3}$, that is, $\lambda < \gamma$.

By Lemma 4.4 and as $-\frac{\lambda+\gamma}{2} > -\frac{1}{3} > -1$, we have

$$\begin{aligned}\phi(s_0,t) &\leq \sqrt{2}\left(\int_0^{s_0} s^{-\frac{\lambda+\gamma}{2}}(s_0-s)^{\frac{1}{2}}\,ds\right)\left(\int_0^{s_0} s^{\lambda-\gamma}(s_0-s)UU_s\,ds\right)^{\frac{1}{2}} \\ &= \sqrt{2}B\left(1-\frac{\lambda+\gamma}{2},\frac{3}{2}\right)s_0^{\frac{3-(\lambda+\gamma)}{2}}C_1^{-\frac{1}{2}}I_2^{\frac{1}{2}}(s_0,t)\end{aligned}$$

for all $(s_0,t) \in (0,R^n) \times (0,T)$, where B denotes Euler's Beta function and where C_1 is as in Lemma 4.2. Squaring yields (4.5). As to I_3, by Fubini's theorem we may estimate

$$\begin{aligned} -\frac{I_3(s_0,t)}{C_2} &= \int_0^{s_0} \int_\sigma^{s_0} s^{-\gamma}(s_0-s)\,ds\,\sigma^{\lambda-1} U(\sigma,t)\,d\sigma \\ &\le \int_0^{s_0} \int_\sigma^{s_0} s^{-\gamma}\,ds\,\sigma^{\lambda-1}(s_0-\sigma) U(\sigma,t)\,d\sigma \\ &\le \frac{s_0^{1-\gamma}}{1-\gamma} \int_0^{s_0} \sigma^{\lambda-1}(s_0-\sigma) U(\sigma,t)\,d\sigma \end{aligned}$$

for all $(s_0,t) \in (0,R^3) \times (0,T)$, which together with Lemma 4.4 leads to the first estimate in (4.6). Finally we derive the estimate for I_6. We see from Lemma 3.1 and Hölder's inequality that

$$-\frac{I_6(s_0,t)}{3^{\alpha-1}K} \le K^* \int_0^{s_0} s^{-\gamma}(s_0-s) \int_0^s U_s^\alpha(\sigma,t)\,d\sigma\,ds \le K^* \int_0^{s_0} s^{1-\alpha-\gamma}(s_0-s) U^\alpha\,ds$$

for all $(s_0,t) \in (0,R^3) \times (0,T)$, and Lemma 4.4 implies the second estimate in (4.6). □

We next give an estimate for I_4. The idea of the proof is based on [**21**, Section 4.2], so that we only state the result here.

Lemma 4.6. *Let $\gamma = \frac{1}{3}$ and $\lambda > 0$ with*

$$\frac{1}{\alpha} \cdot \frac{4}{3} - 2 + \gamma < \lambda < \gamma. \tag{4.7}$$

Then there exists $C_7 > 0$ such that for each u_0 fulfilling (2.1) and with I_2, I_4 as in (4.4),

$$I_4(s_0,t) \ge -C_7 s_0^{2-\alpha-\frac{\lambda}{2}} I_2^{\frac{1}{2}}(s_0,t) - C_7 s_0^{\frac{11}{6}-\frac{3\alpha}{2}+\frac{(\gamma-\lambda)(\alpha+1)}{2}} I_2^{\frac{\alpha+1}{2}}(s_0,t)$$

for all $(s_0,t) \in (0,R^3) \times (0,T)$.

As a consequence of these lemmas, we can derive the desired inequality.

Lemma 4.7. *Let $\gamma = \frac{1}{3}$. Then there exists $\eta > 0$ small enough such that λ satisfies (4.7), where λ is as given by Lemma 4.2. Moreovrer, there exist $C_8, C_9 > 0$ such that for each $s_0 > 0$ and u_0 fulfilling (2.1) and (1.3),*

$$\phi_t(s_0,t) \ge C_8 s_0^{\lambda+\gamma-3} \phi^2(s_0,t) - \phi(s_0,t) - C_9 s_0 \qquad \text{for all } t \in (0,T).$$

Proof. Since $\alpha > \frac{2}{3}$, we have

$$\begin{aligned} (1-\alpha)(3\alpha-1) - \left(\frac{1}{\alpha}\cdot\frac{4}{3} - 2 + \gamma\right) &= \frac{1}{3\alpha}\big[3\alpha(1-\alpha)(3\alpha-1) - (4-5\alpha)\big] \\ &> \frac{1}{3\alpha}[3(1-\alpha) - (4-5\alpha)] = \frac{1}{3\alpha}(-1+2\alpha) > 0, \end{aligned}$$

and moreover we know that $(1-\alpha)(3\alpha-1) < \frac{1}{3}$. Therefore we can find $\eta > 0$ small enough such that (4.7) holds. From Lemmas 4.2, 4.3 and 4.6 as well as (4.6) we infer that

$$\begin{aligned} \phi_t(s_0,t) \ge I_2(s_0,t) - \phi(s_0,t) \\ - C_3 s_0 - C_5 s_0^{\frac{3}{2}+\frac{\lambda-\gamma}{2}} I_2^{\frac{1}{2}}(s_0,t) - C_6 s_0^{3-\frac{3\alpha}{2}-\gamma+\frac{(\gamma-\lambda)\alpha}{2}} I_2^{\frac{\alpha}{2}}(s_0,t) \\ - C_7 s_0^{2-\alpha-\frac{\lambda}{2}} I_2^{\frac{1}{2}}(s_0,t) - C_7 s_0^{\frac{11}{6}-\frac{3\alpha}{2}+\frac{(\gamma-\lambda)(\alpha+1)}{2}} I_2^{\frac{\alpha+1}{2}}(s_0,t) \end{aligned}$$

for all $(s_0,t) \in (0,R^3) \times (0,T)$. Because of the relation $\alpha < 1$, we can employ Young's inequality to obtain $c_1 > 0$ such that

$$\begin{aligned} \phi_t(s_0,t) \ge \frac{1}{2} I_2(s_0,t) - \phi(s_0,t) \\ - c_1 \left(s_0 + s_0^{3+\lambda-\gamma} + s_0^{\left(3-\frac{3\alpha}{2}-\gamma+\frac{(\gamma-\lambda)\alpha}{2}\right)\frac{2}{2-\alpha}} + s_0^{4-2\alpha-\lambda} + s_0^{\left(\frac{11}{6}-\frac{3\alpha}{2}+\frac{(\gamma-\lambda)(\alpha+1)}{2}\right)\frac{2}{1-\alpha}} \right) \end{aligned}$$

for all $(s_0,t) \in (0,R^3) \times (0,T)$. Here, $(3+\lambda-\gamma)-1 = 1+\frac{2}{3}+\lambda > 0$. Also we see from the relations $\alpha < 1$ and $\lambda < \gamma = \frac{1}{3}$ that $(4-2\alpha-\lambda)-1 \ge 2-\frac{1}{3}-1 > 0$, and moreover, $\left(3-\frac{3\alpha}{2}-\gamma+\frac{(\gamma-\lambda)\alpha}{2}\right)\frac{2}{2-\alpha}-1 > 0$ as well as $\left(\frac{11}{6}-\frac{3\alpha}{2}+\frac{(\gamma-\lambda)(\alpha+1)}{2}\right)\frac{2}{1-\alpha}-1 > 0$. Thus, for each $s_0 \in (0,R^n)$,

$$\phi_t(s_0,t) \ge \frac{1}{2} I_2(s_0,t) - \phi(s_0,t) - c_2 s_0 \qquad \text{for all } t \in (0,T)$$

with some $c_2 > 0$, which together with Lemma 4.5 leads to the conclusion. □

5. Outline of the proof

Let $m_1 > 0$, $\varepsilon \in (0,1)$ and $\gamma = \frac{1}{3}$. Moreover, let $t_{**} > 0$ and $\lambda > 0$ be as given by Lemmas 3.2 and 4.7, respectively. From [**21**, estimate (5.5)] there exists $C_{10} > 0$ such that for each $s_0 \in (0,R^3)$ and u_0 fulfilling (2.1) as well as $\int_{B_{r_1}} u_0(x)\,dx \ge m_1$, where $r_1 := \left(\frac{s_0}{4}\right)^{\frac{1}{3}}$, it follows that

$$\phi(s_0,0) \ge C_{10} s_0^{2-\gamma}.$$

Thus, by virtue of Lemmas 4.7 we see that

$$\begin{cases} \phi_t(s_0,t) \ge c_1(s_0)\phi^2(s_0,t) - \phi(s_0,t) - c_2(s_0), & t \in (0,T), \\ \phi(s_0,0) \ge \phi_0(s_0), \end{cases}$$

where $c_1(s_0) := C_9 s_0^{\lambda+\gamma-3}$, $c_2(s_0) := C_{10} s_0$ and $\phi_0(s_0) := C_{10} s_0^{2-\gamma}$. Here, we choose $s_0 > 0$ small enough satisfying

$$\phi_0(s_0) \ge \frac{1+\sqrt{c_1(s_0)c_2(s_0)}}{c_1(s_0)} + \frac{2}{c_1(s_0)t_{**}}.$$

Then we can apply [**5**, Lemma 3.2] to make sure that $T = T_{\max} \le t_{**}$, which together with (2.3) and Lemma 3.1 yields the conclusion of Theorem 1.1. □

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Department of Mathematical Sciences
Kwansei Gakuin University
1 Gakuen Uegahara, Sanda, Hyogo 669-1330
JAPAN
E-mail address: yuya.tns.6308@gmail.com

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