Some properties of radial solutions to chemotaxis systems with nonlinear sensitivity functions

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We consider radial solutions to the following system;

$$\begin{cases} U_t = \Delta U - \nabla \left(\dfrac{U}{|\nabla V|^{\alpha}} \nabla V \right) & \text{in } \mathbf{R}^n \times (0, \infty), \\ 0 = \Delta V + U & \text{in } \mathbf{R}^n \times (0, \infty), \\ U(\cdot, 0) = U^I & \text{in } \mathbf{R}^n \times (0, \infty). \end{cases} \tag{1}$$

Here, $n \geq 1$, $\alpha \in \mathbf{R}$ and U^I is a positive and radial function on $\mathbf{R}^n$.

Winkler [7] consider the following system

$$\begin{cases} U_t = \Delta U - \nabla \left(\dfrac{U}{(1 + |\nabla V|^2)^{\alpha/2}} \nabla V \right) & \text{in } \Omega \times (0, \infty), \\ 0 = \Delta V - \dfrac{1}{|\Omega|} \displaystyle\int_{\Omega} U(\tilde{x}, \cdot) d\tilde{x} + U & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial U}{\partial \nu} = \dfrac{\partial V}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(\cdot, 0) = U^I & \text{in } \Omega. \end{cases} \tag{2}$$

Here, Ω is a bounded domain in $\mathbf{R}^n (n \geq 1)$ with smooth boundary $\partial\Omega$ and U^I is a continuous function on $\overline{\Omega}$. He find a critical exponent

$$\alpha_c = \begin{cases} -\infty & \text{if } n = 1, \\ \dfrac{n-2}{n-1} & \text{if } n \geq 2 \end{cases} \tag{3}$$

in the following sense;

• If $\alpha > \alpha_c$, then the solution to (2) exist globally in time;
• If $n \geq 2$, $\alpha < \alpha_c$ and Ω is a bounded ball, there exist radial solutions to (2) blowing up in finite time for some initial condition U^I.

Mao and Li [6] find blowup solutions to (2) whose initial function having sufficiently large mass in the case where $\alpha = \alpha_c$. Furthermore, there are literature on blowup solutions and time global solutions to (2) or variants of (2).

In [3], we see that steady states to (1) have an following explicit formula in the case of $\alpha = \alpha_c$.

Proposition 1. *Suppose that $n \geq 2$. Let us put*

$$U_1(x) \quad = \quad \frac{n^{2n-1}}{(n-1)^{n-1}} \frac{1}{(1+|x|^{n/(n-1)})^n},$$

For any $\lambda > 0$, $U_\lambda(\cdot) = \lambda^n U_1(\lambda \cdot)$ is a radial steady state to (1) satisfying

$$M^* = \|U_\lambda\|_{L^1} = \int_{\mathbf{R}^n} U_\lambda(x)dx = |S^{n-1}| \left\{ \frac{n^2}{n-1} \right\}^{n-1}. \tag{4}$$

Sketch of proof of Proposition 1. For a radial steady state (U, V), let us put

$$\begin{aligned} \overline{U}(|x|) &= U(x), \\ M(r) &= \int_0^r \overline{U}(r) r^{n-1} dr. \end{aligned}$$

Since (U, V) satisfies (1), we know that the function M satisfies that

$$\frac{d^2 M}{dr^2}(r) - \frac{n-1}{r} \frac{dM}{dr}(r) + \frac{M(r)^{1/(n-1)}}{r} \frac{dM}{dr}(r) = 0 \quad \text{ for } r > 0.$$

Putting $s = \log r$ $(r > 0)$ and $\mathfrak{M}(s) = M(r)$, we see that

$$\frac{dM}{dr}(r) = e^{-s} \frac{d\mathfrak{M}}{ds}(s), \quad \frac{d^2 M}{dr^2}(r) = e^{-2s} \frac{d^2 \mathfrak{M}}{ds^2}(s) - e^{-2s} \frac{d\mathfrak{M}}{ds}(s),$$

whence we have that

$$\begin{aligned} &\frac{d^2 \mathfrak{M}}{ds^2} - n \frac{d\mathfrak{M}}{ds} + \mathfrak{M}^{1/(n-1)} \frac{d\mathfrak{M}}{ds} = 0 && \text{in } \mathbf{R}^n, \\ &\frac{d}{ds} \left\{ \frac{d\mathfrak{M}}{ds} - n\mathfrak{M} + \frac{n-1}{n} \mathfrak{M}^{n/(n-1)} \right\} = 0 && \text{in } \mathbf{R}^n. \end{aligned}$$

Since $\lim_{s \to -\infty} \mathfrak{M}(s) = \lim_{r \to 0} M(r) = 0$, we further derive

$$\begin{aligned} \frac{d\mathfrak{M}}{ds} &= \frac{n-1}{n} \mathfrak{M} \left\{ \frac{n^2}{n-1} - \mathfrak{M}^{1/(n-1)} \right\}, \\ \frac{d\mathfrak{M}}{ds} &= (n-1) \mathfrak{M}^{(n-2)/(n-1)} \frac{d\mathfrak{M}^{1/(n-1)}}{ds} \end{aligned}$$

and that

$$\frac{d\mathfrak{M}^{1/(n-1)}}{ds} = \frac{1}{n} \mathfrak{M}^{1/(n-1)} \left\{ \frac{n^2}{n-1} - \mathfrak{M}^{1/(n-1)} \right\}$$

in $\mathbf{R}_+$. Putting $\mathfrak{N} = \mathfrak{M}^{1/(n-1)}$, we obtain

$$
\begin{aligned}
&\frac{d\mathfrak{N}}{ds} = \frac{1}{n}\mathfrak{N}\left\{\frac{n^2}{n-1} - \mathfrak{N}\right\} \qquad \text{in } \mathbf{R}, \\
&\mathfrak{N},\ \frac{d\mathfrak{N}}{ds} > 0 \ \text{ in } \mathbf{R}, \qquad \lim_{s\to-\infty} \mathfrak{N}(s) = 0.
\end{aligned}
$$

A simple calculation leads us to

$$
\begin{aligned}
\mathfrak{N}(s) &= \frac{n^2}{n-1}\frac{e^{ns/(n-1)}}{1+e^{ns/(n-1)}}, \\
\mathfrak{M}(s) &= \left\{\frac{n^2}{n-1}\right\}^{n-1}\left\{\frac{e^{ns/(n-1)}}{1+e^{ns/(n-1)}}\right\}^{n-1}
\end{aligned}
$$

for $s \in \mathbf{R}_+$, whence the definitions of the functions M and $\mathfrak{M}$ yield

$$
\begin{aligned}
M(r) &= \left\{\frac{n^2}{n-1}\right\}^{n-1}\frac{r^n}{(1+r^{n/(n-1)})^{n-1}} \qquad \text{for } r > 0, \\
U(x) &= \frac{n^{2n-1}}{(n-1)^{(n-1)}}\frac{\lambda^n}{(1+|\lambda x|^{n/(n-1)})^n} \qquad \text{for } x \in \mathbf{R}^n, \\
V(x) &= C - \left(\frac{n^2}{n-1}\right)^{n-1}\lambda^{n-2}\int_0^{|\lambda x|}\frac{\tau}{(1+\tau^{n/(n-1)})^{n-1}}\,d\tau \qquad \text{for } x \in \mathbf{R}^n
\end{aligned}
$$

with $C \in \mathbf{R}$ and $\lambda > 0$. Furthermore, the explicit form of the function U tells us

$$
\begin{aligned}
&U \in C^1(\mathbf{R}^n) \cap C^2(\mathbf{R}^n \setminus \{0\}) \cap W^{2,p}(\mathbf{R}^n) \qquad \text{for } p \in \left[1, \frac{n(n-1)}{n-2}\right), \\
&D^2U(x) \sim \frac{O(1)}{|x|^{(n-2)/(n-1)}} \qquad \text{as } |x| \to 0
\end{aligned}
$$

and (U, V) solves (1) in a classical sense in $\mathbf{R}^n$. In fact, U belongs to $C(\mathbf{R}^n) \cap L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$. Recalling $\overline{U}(|x|) = U(x)$, we see that

$$
\begin{aligned}
\frac{d\overline{U}}{dr}(r) &= \left(-\frac{n^2}{n-1}\right)\frac{n^{2n-1}}{(n-1)^{(n-1)}}\frac{(\lambda r)^{1/(n-1)}}{(1+(\lambda r)^{n/(n-1)})^{n+1}} = -\frac{n^{2n+1}}{(n-1)^n}\frac{(\lambda r)^{1/(n-1)}}{(1+(\lambda r)^{n/(n-1)})^{n+1}}, \\
\frac{d^2\overline{U}}{dr^2}(r) &= -\frac{1}{n-1}\frac{n^{2n+1}}{(n-1)^n}\frac{1}{(\lambda r)^{(n-2)/(n-1)}(1+(\lambda r)^{n/(n-1)})^{n+1}} \\
&\quad + \frac{(n+1)n}{n-1}\frac{n^{2n+1}}{(n-1)^n}\frac{(\lambda r)^{2/(n-1)}}{(1+(\lambda r)^{n/(n-1)})^{n+2}},
\end{aligned}
$$

which means that $DU \in C(\mathbf{R}^n) \cap L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$ and that $D^2U \in L^p(|x| < 1) \cap L^\infty(|x| > 1) \cap L^1(\mathbf{R}^n) \cap C(\mathbf{R}^n \setminus \{0\})$, where $p \in [1, n(n-1)/(n-2))$. Then, the stationary solution U

belongs to $C^1(\mathbf{R}^n) \cap C^2(\mathbf{R}^n \setminus \{0\}) \cap W^{2,p}(\mathbf{R}^n)$, where $p \in [1, n(n-1)/(n-2))$ and ΔU has asymptotic behavior

$$\Delta U(x) = -\frac{n^2 - 2n + 2}{n-1} \frac{n^{2n+1}}{(n-1)^n} \frac{1 + o(1)}{|x|^{(n-2)/(n-1)}} \quad \text{as } |x| \to 0.$$

□

Remark 2. *In the case of $n = 2$, we see that $\alpha_c = 0$ and that the steady states mentioned in Proposition 1 is the following function*

$$U_\lambda(x) = \frac{8\lambda^2}{(1 + \lambda^2 |x|^2)^2}$$

satisfying $M^* = 8\pi$.

For $L > 0$, put $B(L) = \{x \in \mathbf{R}^n : |x| < L\}$.
The L^1-norm M^* of those steady states is the critical number in the following sense;

$$\begin{cases} U_t = \Delta U - \nabla \left(\dfrac{U}{|\nabla V|^\alpha} \nabla V \right) & \text{in } B(L) \times (0, T_m), \\ 0 = \Delta V - \dfrac{\|U^I\|_{L^1}}{|B(L)|} + U & \text{in } B(L) \times (0, T_m), \\ \dfrac{\partial U}{\partial n} = \dfrac{\partial V}{\partial n} = 0 & \text{on } \partial B(L) \times (0, T_m), \\ U(\cdot, 0) = U^I & \text{on } \Omega. \end{cases} \tag{5}$$

has the following properties;

Theorem 3 ([4])**.** *Suppose that U^I is a radial, non-negative and continuous function on $B(L)$. Then, there exists a unique mild solution U to (5) in $B(L) \times (0, T_m)$ satisfying $\|U(\cdot, t)\|_{L^1} = \|U^I\|_{L^1}$ for $t \in [0, T_m)$. Here, $T_m \in (0, \infty]$ is the maximal existence time of the mild solution to (5). Furthermore, the following hold;*

(i) If $\|U^I\|_{L^1} < M^* |S^{n-1}|$, the solution U to (5) exists globally in time and is uniformly bounded;
(ii) There exist radial solutions U to (5) such that those L^1-norm is bigger than $M^* |S^{n-1}|$ and that $T_m < \infty$, that is to say, those solutions blow up in finite time.

Proof of Theorem 3. Step 1. We find finite-time blowup solutions whose L^1-norm is bigger than M^*. Suppose that U^I is a radial, non-negative and continuous function satisfying $\|U(\cdot, t)\|_{L^1} = \|U^I\|_{L^1} > M^* |S^{n-1}|$ and that

$$\int_0^r \overline{U^I}(\xi) \xi^{n-1} d\xi \geq \frac{\|U^I\|_{L^1}}{n|B(L)|} r^n = \frac{\|U^I\|_{L^1}}{|S^{n-1}|} \left(\frac{r}{L}\right)^n = M_{L1} \left(\frac{r}{L}\right)^n \quad \text{for } r \in [0, L], \tag{6}$$

where $\overline{U^I}(|x|) = U^I(x)$. Let U be a mild solution to (5) with $U(\cdot,0) = U^I$ and let us put

$$\begin{aligned}\overline{U}(|x|,t) &= U(x,t) \quad \text{for } (r,t) \in [0,L] \times (0,T_m),\\ M(r,t) &= \int_0^r \overline{U}(\xi,t)\xi^{n-1}d\xi \quad \text{for } (r,t) \in [0,L] \times (0,T_m).\end{aligned} \tag{7}$$

Then, M and $M_{L1}(r/L)^n$ satisfy that

$$\begin{cases} M_t = M_{rr} - \dfrac{n-1}{r}M_r + \dfrac{M_r}{r}\dfrac{M - M_{L1}(r/L)^n}{|M - M_{L1}(r/L)^n|^\alpha} & \text{in } (0,L)\times(0,T_m),\\ \qquad M(0,t) = 0, \quad M(L,t) = M_{L1} & \text{for } t \in (0,T_m),\end{cases} \tag{8}$$

which together with the comparison theorem we imply that

$$M(r,t) \geq M_{L1}\left(\frac{r}{L}\right)^n \quad \text{for } (r,t) \in [0,L]\times[0,T_m). \tag{9}$$

Multiplying $|x|^2$ by the first equation of (5), integrating over $B(L)$ and using the boundary conditions, we obtain

$$\frac{d}{dt}\int_{B(L)} |x|^2 U(x,t)dx = -\int_{B(L)} 2x\cdot\nabla U(x,t)dx + \int_{B(L)} U(x,t)\frac{2x\cdot\nabla V(x,t)}{|\nabla V(x,t)|^\alpha}dx.$$

Since U is positive in $B(L)\times(0,T_m)$, then we see that

$$\begin{aligned}-\int_{B(L)} 2x\cdot\nabla U(x,t)dx &= -2|S^{n-1}|\int_0^L r^n\frac{d\overline{U}}{dr}(r,t)dr\\ &= -2|S^{n-1}|L^n\overline{U}(L,t) + 2n|S^{n-1}|\int_0^L \overline{U}(r,t)r^{n-1}dr\\ &\leq 2n|S^{n-1}|\int_0^L \overline{U}(r,t)r^{n-1}dr.\end{aligned}$$

Since (9) and the second equation of (5) leads us to

$$\begin{aligned}\frac{x}{|x|}\cdot\nabla V(x,t) = \frac{d\overline{V}}{dr}(r,t) &= \frac{1}{r^{n-1}}\int_0^r\left(\frac{\|U^I\|_{L^1}}{|B(L)|} - U(x,t)\right)r^{n-1}dr\\ &= \frac{1}{r^{n-1}}\left\{M_{L1}\left(\frac{r}{L}\right)^n - M(r,t)\right\} \leq 0 \quad \text{for } (r,t)\in(0,L)\times(0,T_m),\end{aligned}$$

then we see that

$$
\begin{aligned}
&\int_{B(L)} U(x,t)\frac{x\cdot\nabla V(x,t)}{|\nabla V(x,t)|^{\alpha}}dx = \int_{B(L)} U(x,t)\left\{(M(|x|,t) - M_{L1}\left(\frac{|x|}{L}\right)^{n}\right\}^{1/(n-1)} dx\\
&= |S^{n-1}|\int_0^L \overline{U}(r,t)r^{n-1}\left\{M(r,t) - M_{L1}\left(\frac{r}{L}\right)^{n}\right\}^{1/(n-1)} dr\\
&\geq |S^{n-1}|\int_0^L \overline{U}(r,t)r^{n-1}M(r,t)^{1/(n-1)}dr - |S^{n-1}|\int_0^L \overline{U}(r,t)r^{n-1}\left(\frac{M_{L1}}{L^n}\right)^{1/(n-1)} r^{n/(n-1)}dr\\
&= |S^{n-1}|\frac{(n-1)}{n}M_{L1}^{n/(n-1)} - |S^{n-1}|M_{L1}^{(n-2)/(2n-2)}\left(\frac{M_{L1}}{L^n}\right)^{1/(n-1)}\left\{\int_0^L \overline{U}(r,t)r^{n-1}r^2dr\right\}^{n/(2n-2)}.
\end{aligned}
$$

Putting

$$
\begin{aligned}
I(t) &= \int_0^L \overline{U}(r,t)r^{n-1}r^2dr,\\
A &= 2M_{L1}^{(n-2)/(2n-2)}\left(\frac{M_{L1}}{L^n}\right)^{1/(n-1)},
\end{aligned}
$$

we obtain that

$$
\begin{aligned}
\frac{dI}{dt} &\leq 2nM_{L1} - \frac{2(n-1)}{n}M_{L1}^{n/(n-1)} + AI\\
&= \frac{2(n-1)}{n}M_{L1}\left\{(M^*)^{1/(n-1)} - M_{L1}^{1/(n-1)}\right\} + AI.
\end{aligned}
\tag{10}
$$

We can choose the initial condition U satisfying that $M_{L1} > M^*$ and that

$$
I(0) < \frac{1}{A}\frac{2(n-1)}{n}M_{L1}\left\{M_{L1}^{1/(n-1)} - (M^*)^{1/(n-1)}\right\}.
$$

Since this means that I is decreasing with respect to t, then the solution blows up in finite time. In fact, if the solution exists globally in time, the solution U is positive in $B(L)\times(0,\infty)$. However, (10) ensure the existence of $\tilde{T}\in(0,\infty)$ such that $I(\tilde{T}) = 0$. It contradicts the positivity of the solution. Then, the solution blows up in finite time, if $M_{L1} > M^*$ and $I(0)$ is sufficiently small. Therefore, (ii) holds.

Step 2. Next, we show (i). For any $\lambda > 0$, we define M_λ as

$$
M_\lambda(r) = \frac{1}{|S^{n-1}|}\int_{|x|<r} U_\lambda(x)dx.
$$

Since $M_{L1} < M^*$ and $\lim_{\lambda\to\infty} M_\lambda = M^*$, then we can find a positive constant Λ such that $M^I < M_\Lambda$ on $(0,L]$. Then, the mass function M defined as (7) satisfies $M \leq M_\Lambda)$ in $[0,L]\times(0,T_m)$

by using the comparison theorem, which together with the parabolic regularity argument we imply that the solution exists globally in time and that the solution is uniformly bounded in time. Thus, the proof is complete. □

This theorem says that the constant M^* is the threshold number for blowup and time-global existence of solutions. Then, it is worth investigating behavior of solution whose L^1-norm is equal to the threshold number M^*.
Behavior of solutions whose L^1-norm is equal to M^ast is related to stability of steady states. In [2, 1], stability of steady states is shown in the case of $n = 2$ and $\alpha = \alpha_c = 0$. We also show the following stability of steady states in the case where $n \geq 3$ and $\alpha = \alpha_c$.

Remark 4. *Any radial solution (U, V) to (1) blows up at a finite time in the case where $\|U^I\|_{L^1} > M^*$. This is shown by using an argument similar to the one in [5].*

Theorem 5. *Suppose that the radial and positive initial condition U^I satisfying*

$$\begin{aligned} &\|U^I\|_{L^1} = M^*, \\ &\int_{\mathbf{R}^n} (1 + |x|^2)|U^I(x) - U_\lambda(x)|dx < \infty \quad \textit{for some } \lambda > 0. \end{aligned}$$

Then, the solution to (1) satisfies

$$\lim_{t \to \infty} \|U(\cdot, t) - U_\lambda\|_{L^\infty} = 0.$$

Remark 6. *For $\tilde{\lambda} > \lambda > 0$, we see that*

$$\int_{\mathbf{R}^n} (1 + |x|^2)|U_{\tilde{\lambda}}(x) - U_\lambda(x)|dx = \infty.$$

We will describe a sketch of the proof of Theorem 5.
We define $\mathcal{B}$ as

$$\mathcal{B} = \{f \in L^1(\mathbf{R}^n); f \text{ is radial and non-negative on } \mathbf{R}^n,\ \|f\|_{L^1} = M^*\}.$$

Lemma 7. *There exists a functional W on $\mathcal{B} \times \mathbf{R}_+$ satisfying the following;*

(i) *For $f \in \mathbf{B}$ and $\lambda > 0$, $W(f, \lambda) > -\infty$;*

(ii) *If $f \in \mathcal{B}$ satisfies*

$$\int_{\mathbf{R}^n} (1 + |x|^2)|f(x) - U_\lambda(x)|dx < \infty \quad \textit{for some } \lambda > 0,$$

then $W(f, \lambda) < \infty$;

(iii) *For any pair λ and μ with $\lambda > 0$, $\mu > 0$ and $\lambda \neq \mu$, $W(U_\mu, \lambda) = \infty$;*

(iv) *If U^I is an element of $\mathbf{R}$ and satisfies $W(U^I, \lambda) < \infty$ with some $\lambda > 0$, the corresponding solution (U, V) to (1) satisfies*

$$\frac{d}{dt} W(U(\cdot, t), \lambda) \leq 0 \quad \textit{for } t \in (0, T_m).$$

Proof of sketch of Theorem 5. Since we see that $M(r,t) \leq M_\lambda(r)$ for any sufficiently large $\mu > 0$, then it follows from the comparison theorem that

$$M(r,t) \leq M_\mu(r) \quad \text{for } r \geq 0,\ t \geq 0,$$

form which together with the parabolic regularity theorem we obtain boundedness of the solution (U, V) and its differentiation. Then, this guarantees that the solution exists globally in time and that there exist a sequence $\{t_m\}$ and a constant $\tilde{\lambda} > 0$ such that $\lim_{m\to\infty} t_m = \infty$, $\lim_{m\to\infty} U(\cdot, t_m) = U_{\tilde{\lambda}}$. This and (iv) of Lemma 7 mean that $W(U_{\tilde{\lambda}}, \lambda)$ is bounded. Combining this with (iii) of Lemma 7 we imply that $\tilde{\lambda} = \lambda$. Then, we see that $\lim_{t\to\infty} U(\cdot, t) = U_\lambda$. Thus, the proof is complete. □

References

[1] J. López-Gómez, T. Nagai and T. Yamada, The basin of attraction of the steady-states for a chemotaxis model in $\mathbf{R}^2$ with critical mass. Arch. Rational Mech. Anal., 207: 159-184, 2013.

[2] P. Biler, G. Karch, P. Laurençot and T. Nadzieja, The 8π-problem for radially symmetric solutions of a chemotaxis model in the plane. Math. Meth. Appl. Sci., 29:1563–1583, 2006.

[3] S. Kohatsu and T. Senba, Self-similar solutions to a flux-limited Keller-Segel system, Nonlinear Analysis: Real World Applications, 84:104308, 2025.

[4] S. Kohatsu and T. Senba, Critical mass and stability of radial steady states for aflux-limited Keller–Segel system in the critical case, preprint.

[5] M. Kurokiba and T. Ogawa, Finite time blow-up of the solution for a nonlinear parabolic equation of drift-diffusion type, Differential Integral Equations, 16 (2003), 427-452.

[6] X. Mao and Y. Li, Instability of homogeneous steady state in chomotaxis systems with flux limitation. Nonlinear Anal., 243: Paper No. 113527, 18 pages, 2024.

[7] M. Winkler, A critical blow-up exponent for flux limitation in a Keller–Segel system. Indiana Univ. Math. J., 71(4):1437–1465, 2022.