Time-dependent subdifferential evolution equations and applications to quasi-subdifferential equations

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Abstract

We review the theory of time-dependent subdifferential evolution equations and its applications to quasi-subdifferential evolution equations of parabolic and elliptic-parabolic types. The key concept, for the theory and the applications as well, is the energy inequality satisfied by the solution.

1 Introduction

We study some classes of abstract nonlinear evolution equations in a real Hilbert space H with norm $|\cdot|_H$ and inner product $(\cdot,\cdot)$. We consider the following classes.

Time-dependent subdifferential evolution equation:

$$u'(t) + \partial\varphi(t; u(t)) \ni 0. \tag{TDSE}$$

Quasi-subdifferential evolution equation:

$$u'(t) + \partial\varphi(t, u; u(t)) \ni 0. \tag{QSE}$$

Also we consider the following equation.

Elliptic-parabolic quasi-subdifferential evolution equation: (B is a monotone Lipschitz continuous operator on H)

$$(Bu)'(t) + \partial\varphi(t, Bu; u(t)) \ni 0. \tag{EP}$$

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All of these equations are associated with the subdifferential $\partial\varphi$ of a proper ($\not\equiv +\infty$), l.s.c. (lower-semicontinuous) and convex function $\varphi : H \to \mathbf{R} \cup \{+\infty\}$. The subdifferential $\partial\varphi$ is a possibly multi-valued operator defined by $z^* \in \partial\varphi(z)$ if and only if $z \in D(\varphi) := \{x \in H|\ \varphi(x) < +\infty\}$ (the effective domain of φ) and

$$(z^*, w - z) \leq \varphi(w) - \varphi(z) \text{ for all } w \in H.$$

We put $D(\partial\varphi) := \{x \in H|\ \partial\varphi(x) \neq \emptyset\}$ (the domain of the operator $\partial\varphi$). In (TDSE), we are given time-dependent subdifferentials $\partial\varphi(t;\cdot)$ of a time-dependent family $\varphi : [0,T] \times H \to \mathbf{R} \cup \{+\infty\}$ of convex functions. In (QSE) and (EP), we consider the subdifferential of a family $\varphi : [0,T] \times \mathcal{K} \times H \to \mathbf{R} \cup \{+\infty\}$, where $\mathcal{K} \subset L^2(0,T;H)$ is a set of H-valued functions.

Idea between (TDSE) and (QSE)

We are concerned with strong solutions of the equations, that is, solutions having strong H-derivatives, with a prescribed initial condition $u(0) = u_0$. To obtain a solution of (QSE), we employ the theory of (TDSE). For this, the large steps are explained as follows:

Step 1 Let a function $v : [0,T] \to H$ be given.

Step 2 Solve by the theory of (TDSE) the problem:

$$\begin{cases} u'(t) + \partial\varphi(t, v; u(t)) \ni 0, \\ u(0) = u_0. \end{cases}$$

Step 3 Find a fixed point of $v \mapsto u$ by choosing a sufficiently small time iterval $[0, T_0]$ (a local solution).

Step 4 Prolong the local solution to obtain a solution on the whole interval $[0,T]$.

The key role in Step 3 is played by the energy inequality satisfied by the solution of (TDSE). In fact, we will see in the next section that the energy inequality itself is the essence of the solvability theory of (TDSE) (see Section 2.3).

In the next section, we review the theory of time-dependent subdifferential evolution equations featuring in particular the papers by Kenmoch [26], Yamada [71] and Kubo [43]. In Section 3, (QSE) is treated by following the results of Kano, Kenmochi and Murase [21] and Kubo and Yamazaki [47, 48]. The results of elliptic-parabolic problems related to (EPQVE) by Kenmochi, Kubo, Pawłow, Shirakawa and Yamazaki [31, 34, 35, 45, 46, 49, 73] are reviewed in Section 4.

2 Time-dependent subdifferential evolution equations

2.1 Subdifferentials and their evolution equations

If the convex function φ is Fréchet or Gataux differentiable, then its Fréchet or Gataux differential coincides with the subdifferential $\partial\varphi$. In the calculus of variations, the subdifferential operator is the Euler-Lagrange operator of a convex functional integral. We give two typical examples in the following.

Dirichlet problem

Let $\Omega \subset \mathbf{R}^N$ $(N \geq 1)$ be a bounded domain with smooth boundary $\partial\Omega$ and let $\varphi : H\,(= L^2(\Omega)) \to \mathbf{R} \cup \{+\infty\}$ be defined by

$$\varphi(z) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla z|^2 dx, & z \in H_0^1(\Omega), \\ +\infty, & z \in L^2(\Omega) \setminus H_0^1(\Omega). \end{cases}$$

Then we have

$D(\partial\varphi) = H^2(\Omega) \cap H_0^1(\Omega)$ and

$$\partial\varphi(z) = -\Delta z \quad \text{for } z \in D(\partial\varphi).$$

Therefore the evolution equation

$$u'(t) + \partial\varphi(u(t)) = 0$$

is equivalent to the boundary value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = 0 & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases}$$

A unilateral or obstacle problem

Let $K := \{z \in H_0^1(\Omega) | \; z \geq g \text{ in } \Omega\}$ (g is a given obstacle with $g|_{\partial\Omega} \leq 0$) and put

$$\varphi(z) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla z|^2 dx, & z \in K, \\ \quad +\infty, & z \in L^2(\Omega) \setminus K. \end{cases}$$

Then $z^* \in \partial\varphi(z)$ if and only if $z \in H^2(\Omega) \cap H_0^1(\Omega)$ and there holds:

$$z^* = -\Delta z + w^*, \; z \geq g, \; w^* \leq 0, \; (z - g)w^* = 0 \quad \text{in } \Omega.$$

Therefore the evolution equation $u'(t) + \partial\varphi(u(t)) \ni 0$ is equivalent to the unilateral (or complementarity) problem:

$$\begin{cases} u \geq g, \; \dfrac{\partial u}{\partial t} - \Delta u \geq 0, \; (u - g)\left(\dfrac{\partial u}{\partial t} - \Delta u\right) = 0 & \text{in } \Omega, \\ \qquad u = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus, the subdifferential evolution equation

$$u'(t) + \partial\varphi(u(t)) \ni 0 \tag{SE}$$

is an abstract formulation of parabolic PDEs and their unilateral problems or variational inequlities as founded by J.-L. Lions and G. Stampacchia [52]. It is also a special case of nonlinear evolution equations as founded by the nonlinear Hille-Yosida theory of Y. Kōmura [40]. We notice that a solution of (SE) satisfies, when $\partial\varphi$ is single-valued,

$$\frac{d}{dt}\varphi(u(t)) = (\partial\varphi(u(t)), u'(t)) = -|u'(t)|_H^2,$$

hence,

$$\frac{d}{dt}\varphi(u(t)) + |u'(t)|_H^2 = 0.$$

This identity will be generalized for time-dependent problem (TDSE) as an energy inequality (see Section 2.3).

Among the inventions of Kōmura, we would emphasize the concept of Yosida-regularization of the generator of a nonlinear semi-group:

$$A_\lambda = \frac{I - (I + \lambda A)^{-1}}{\lambda} \quad (\lambda > 0).$$

We also notice that an important concept of the Yosida-regularization φ_λ of a convex function φ was introduced by Moreau [56]:

$$\varphi_\lambda(z) := \inf_{w \in H} \left\{ \frac{1}{2\lambda}|w - z|_H^2 + \varphi(w) \right\},$$

which is Fréchet differentiable and satisfies [15, Proposition 2.11]

$$\partial\left(\varphi_\lambda\right) = \left(\partial\varphi\right)_\lambda,$$

that is, the subdifferential of the Yosida-regularization is the Yosida-regularization of the subdifferential.

The linear semi-group theory by K. Yosida [75] (cf. Remark 2.1) and E, Hille [19] as well as its nonlinear version in a Hilbert space by Y. Kōmura [40] are both generalized to the case of time-dependent generator by T. Kato [23, 24] in the name of *evolution equations* (or the equation of evolution) in an abstract Banach space. When the generator of the nonlinear semi-group or the evolution equation is a subdifferential of a convex function, the equation is called a subdifferential evolution equation that forms a well-developed class of evolution equations because of its relation to convex analysis (see Rockafellar [65]) and of its range of applications such as variational inequalities. We refer, for instance, to Brézis [13] for early applications to problems of PDEs.

Remark 2.1. *The motivation of K. Yosida to study semigroups of operators was to investigate an infinite dimensional analogue of a Lie group and its Lie algebra. In fact, in Yosida [74], a locally compact Banach algebra analogue of von Neumann's [59] had been proved.*

2.2 Time-dependence condition for the solvability of (TDSE)

Now we shall review the study of time-dependent subdifferential evolution equation in a real Hilbert space H:

$$\begin{cases} u'(t) + \partial\varphi(t; u(t)) \ni 0, & 0 < t < T, \\ u(0) = u_0, \end{cases} \tag{CP}$$

where $\partial\varphi(t; \cdot)$ is the subdifferential of a family $\varphi(t; \cdot)$ of proper l.s.c. convex functions dependent on t, and u_0 is a given initial value.

Various approaches to the solvability[1] of (CP) were proposed around mid-1970s. We will compare some of them later (see Section 2.3) by featuring the solvability condition due to Kenmochi [26] in 1975 and Yamada [71] in 1976. Although their researches appear to be essentially independent, Yamada acknowledged Kenmochi in [71, p.514]. Also, we refer to a still earlier result by Kenmochi [25] and the priority of this type of condition belongs to Kenmochi.

In [70, p.530 Remark], Watanabe quoted Kenmochi's idea as 'of great originality'. The weight $(\varphi_\lambda(t; z))^{1/2}$ (see Section 2.3) introduced by Yamada enables wider applications to, for instance, problems in non-cylindrical domains and free boundary problems. Kenmochi used a time-descretization while Yamada employed the Yosida-regularization which clarifies the mechanism of solvability by the energy inequality.

Since the abstract result was establiched, it has found to date various applications to problems of PDEs in a variety of manners. We refer, for instance, to Yamada [72], Ôtani and Yamada [60], Kenmochi [28], Kenmochi and Pawłow [35], Kubo and Kumazaki [44], Kumazaki, Aiki and Muntean [51].

A typical concrete time-dependent convex function is given by a time-dependent convex set $K(t)$ as below

$$\varphi(t; z) := \begin{cases} \frac{1}{2}\int_\Omega |\nabla z|^2 dx, & z \in K(t), \\ +\infty, & z \in L^2(\Omega) \setminus K(t). \end{cases}$$

More specifially, $K(t)$ is given, for instance, as follows.

[1] the uniqueness of a strong solution is a direct consequence of the monotonicity of the subdifferentials

A time-dependent obstacle problem:

$$K(t) = \{z \in H_0^1(\Omega) | \ z \geq g(t) \text{ in } \Omega\}.$$

A non-cylindrical domain problem:

$$K(t) = \{z \in H_0^1(\Omega) | \ z = 0 \text{ in } \Omega \setminus \Omega(t)\} \ (\overline{\Omega(t)} \subset \Omega).$$

The first approach to (CP) was due to J. Watanabe [69] for time-independent effective domain $D(\varphi(t)) \equiv D : \quad 0 \leq s < t \leq T, \ \forall z \in D :$

$$\varphi(t; z) - \varphi(s; z) \leq C|t - s| \left(|\varphi(s; z)| + 1\right).$$

Notice that the domain $D(\partial\varphi(t))$ can depend on t (see Remark 2.2) although in general there holds $\overline{D(\varphi)} = \overline{D(\partial\varphi)}$ ([15, Proposition 2.11]).

Remark 2.2. *Let $\Omega \subset \mathbf{R}^N$ be a bounded domain with a smooth bondary Γ and let $2 \leq r \leq 2(N-1)/(N-2)$ if $N > 2$ and $2 \leq r < \infty$ if $N = 2$ (see [1, Theorem 5.36] for the boundary trace $H^1(\Omega) \to L^r(\Gamma)$ for such r). And define $\varphi^r : H \ (:= L^2(\Omega)) \to \mathbf{R} \cup \{+\infty\}$ by*

$$\varphi^r(z) := \begin{cases} \dfrac{1}{2} \displaystyle\int_\Omega |\nabla z|^2 dx + \dfrac{1}{r} \displaystyle\int_\Gamma |z|^r d\Gamma, & z \in H^1(\Omega), \\ \qquad\qquad +\infty, & z \in L^2(\Omega) \setminus H^1(\Omega). \end{cases}$$

Then we have for all such r

$$D(\varphi^r) = H^1(\Omega).$$

On the other hand we have $z^* \in \partial\varphi^r(z)$ if and only if

$$\partial\varphi^r(z) = -\Delta z \qquad \text{and} \qquad -\frac{\partial z}{\partial n} = |z|^{r-2} z \ \text{ in } H^{-1/2}(\Gamma).$$

Therefore we have

$$D(\partial\varphi^r) = \left\{ z \in H \,\middle|\, -\Delta z \in L^2(\Omega) \text{ and } -\frac{\partial z}{\partial n} = |z|^{r-2} z \text{ in } H^{-1/2}(\Gamma) \right\},$$

which varies with r.

Watanabe's condition was weakened by Maruo [54] and Attouch and Damlamian [10]. Also, Péralba [64] introduced another time-dependence condition expressed by the dual convex function of $\partial\varphi$ (it seems not easy to verify this type of condition in applications).

A different approach was due to Attouch, Bénilan, Damlamian and Picard [9] which introduced a differential condition for Yosida-regularization:

$$\frac{d}{dt}\varphi_\lambda(t;z) \leq a(t)\left(\varphi_\lambda(t;z)\right)^{1/2}|\partial\varphi_\lambda(t;z)|_H + b(t)\varphi_\lambda(t;z).$$

As for the case of variable domain $D(\varphi(t))$, where $D(\varphi(t)) \cap D(\varphi(s)) = \emptyset$ $(t \neq s)$ can occur, Kenmochi [26] and Yamada [71] introduced a powerful condition given in Section 2.3 for the solvability of the problem. After that, Kenmochi [27, Theorem 1.5.1] (see Kenmochi [26, p.310]) proved that conditions of Kenmochi-Yamada type are (quite non-trivially) sufficient for conditions of Attouch et al type to hold.

The Kenmochi-Yamada condition was weakened by Yotsutani [77] and Ôtani [62] and was characterized by its energy inequality (Section 2.3) by K. [43].

After the establishment of the solvability, various aspects of (TDSE) have been studied. For instance, Ôtani [61] studied the smoothing effects and non-monotone perturbations. Large time behaviour of the solution was discussed by Furuya, Miyashiba and Kenmochi [18] and Kenmochi and Ôtani [32, 33]. Akagi and Ôtani [4] studied $V-V^*$ formalism. Doubly nonlinear equations associated with time-dependent subdifferentials were studied by Akagi [3] and Kenmochi, Shirakawa and Yamazaki [36].

2.3 Relation of time-dependence conditions and the energy inequality

For simplicity, we suppose $\varphi(t;z) \geq c > 0$ for a constant c and all the conditions below are given in a simplified form.

The condition introduced by Kenmochi and Yamada is as follows:

$$a = |\alpha'| \in L^2(0,T), b = |\beta'| \in L^1(0,T): \ 0 \leq s < t \leq T$$
$$\forall z \in D(\varphi(s))\ \exists \tilde{z} \in D(\varphi(t))$$
$$\begin{cases} |\tilde{z} - z|_H \leq |\alpha(t) - \alpha(s)|\left(\varphi(s;z)\right)^{1/2} \\ \varphi(t;\tilde{z}) - \varphi(s;z) \leq |\beta(t) - \beta(s)|\varphi(s;z) \end{cases}$$

Under this condition the existence of a strong solution of (CP) satisfying a class of energy inequality was proved.

Kenmochi [27] discussed the relation of the solvability conditions as below.

(A) Kenmochi-Yamada condition

$$\forall z \in D(\varphi(s))\ \exists \tilde{z} \in D(\varphi(t)) \quad \begin{cases} |\tilde{z} - z|_H \le |\alpha(t) - \alpha(s)| \left(\varphi(s; z)\right)^{1/2} \\ \varphi(t; \tilde{z}) - \varphi(s; z) \le |\beta(t) - \beta(s)| \varphi(s; z) \end{cases}$$

$\Longrightarrow$ (B) Attouch-Bénilan-Damlamian-Picard condition

$$\frac{d}{dt}\varphi_\lambda(t; z) \le a(t) \left(\varphi_\lambda(t; z)\right)^{1/2} |\partial\varphi_\lambda(t; z)|_H + b(t)\varphi_\lambda(t; z)$$

$\Longrightarrow$ (C) Energy inequality (differential form)

$$\frac{d}{dt}\Big(\varphi_\lambda(t; u)\Big) - (\partial\varphi_\lambda(t; u), u') \le a(t) \left(\varphi_\lambda(t; u)\right)^{1/2} |\partial\varphi_\lambda(t; u)|_H + b(t)\varphi_\lambda(t; u)$$

$$\left(\text{cf.}\ \ \frac{d}{dt}\Big(\varphi_\lambda(t; u)\Big) = \left(\frac{\partial\varphi_\lambda}{\partial t}\right)(t; u) + (\partial\varphi_\lambda(t; u), u') \ \ \text{chain rule}\right)$$

$\Longrightarrow$ (D) $\exists$ a strong solution of (CP) with an energy inequality: $0 \le s < t \le T$

$$\varphi(t; u(t)) - \varphi(s; u(s)) + \int_s^t |u'(\tau)|_H^2 d\tau$$

$$\le \int_s^t \Big\{a(\tau) \left(\varphi(\tau; u(\tau))\right)^{1/2} |u'(\tau)|_H + b(\tau)\varphi(\tau; u(\tau))\Big\} d\tau$$

$\Longrightarrow$ (E) $\exists$ a strong solution of (CP) with an energy inequality: $0 \le s < t \le T$

$$\varphi(t; u(t)) - \varphi(s; u(s)) + (1 - \varepsilon) \int_s^t |u'(\tau)|_H^2 d\tau \le \int_s^t c(\tau)\varphi(\tau; u(\tau))d\tau$$

$(c := C(\varepsilon)a^2 + b)$

In (D) and (E) above, the initial condition $u(s) = u_0$ can be imposed for any $0 \le s < T$ and $u_0 \in D(\varphi(s))$. By K. [43, Theorems 1 and 3, Lemma 4.1] with $u(s) = u_0 = z, u(t) = \tilde{z}$, we see that all these conditions are (essentially) equivalent, that is, the Kenmochi-Yamada condition is characterized by the energy inequality satisfied by the solution (see also Kubo-Kumazaki [44, Remark 4.2] and Kenmochi [29, Vol. 1, p. 279]).

Some remarks

We reviewed above the results for (TDSE) from the view point of abstract theory of evolution equations. Parabolic variational inequalities with time-dependent constraint have been formulated also by the equation

$$u'(t) + Au(t) + \partial I_{K(t)}(u(t)) \ni 0.$$

J.-L. Lions noticed that its initial-value problem is solvable if the convex set $K(t)$ increases as t does ([53, p.271 Exemple 9.3]). Then, Brézis [14] and Biroli [12] proved the exisitence of a solution to by supposing some time-dependence conditions on $t \mapsto K(t)$. The condition of Biroli is expressed by the *retraction* of a time-dependent family of convex sets introduced by Moreau [57]. The retraction of $t \mapsto K(t)$ measures the rate of decreasing of the convex sets as t increases. Later, Kenmochi and Ôtani [33] introduced a notion of topology of the set of convex functions which measures largely the Hausdorff distance of the epigraph of the convex funtions (see Attouch [8] for an apparently equivalent notion of topology). We note that the Kenmochi-Yamada condition largely measures the Hausdorff semi-distance, which is equal to the retraction of convex sets when the convex functions are the indicators thereof, of the epigraphs of a time-dependent family of convex functions $\varphi(t;\cdot)$. We refer to Moreau [58] for more about the notion of retraction and its relation to nonlinear evolution equations.

Another important subject is evolution equations in a general Banach space. For this we here only refer to the book by Yosida [76] for linear evolution equations and Crandall-Liggett theory of nonlinear semigroups, and to Kobayashi, Kobayashi and Oharu [39] and the references therein for problems with time-dependent generators.

3 Quasi-subdifferential evolution equations

Here we consider quasi-subdifferential evolution equations (QSE).

First we consider evolutionary variational inequalities introduced by Lions and Stampacchia [52]. This problem is exemplified by

$$u_t - \Delta u = f \qquad \& \qquad u \in K \text{ (a closed convex set in } L^2(\Omega))$$

or in a weak form

$$u(t) \in K, \quad (u', u - z) + \int_\Omega \nabla u \cdot \nabla (u - z) dx \le \int_\Omega f(u - z) dx \quad \forall z \in K.$$

Thus, a variational inequality is a problem for a PDE, e.g. the heat equation as above, with a constraint imposed by a closed convex set.

Now a quasi-variational inequality is a problem for a PDE with a constraint itself dependent on the unknown function. Such a problem was first introduced by Benssousan and Lions [11] and has a vast body of literature. In short:

$$\text{quasi-variational inequality} = \text{PDE \& constraint depending on } u$$

or for example,

$$u_t - \Delta u = f \qquad \& \qquad u \in K(u)$$

or in a weak form,

$$u(t) \in K(u), \quad (u', u-z) + \int_\Omega \nabla u \cdot \nabla(u-z)dx \leq \int_\Omega f(u-z)dx \quad \forall z \in K(u)$$

We refer, for instance, to Mignot and Puel [55] for another early study of parabolic quasi-variational inequality.

More recently, quasi-variational evolution equations were studied in view of problems with non-local effect like *hysteresis* (cf. Visintin [68], Kenmochi, Koyama and Meyer [30]), *phase-transitions* (cf. Colli, Kenmochi and Kubo [16]), *shape memory alloys* (cf. Aiki [2]), and so on.

In an abstract form (QSE), the quasi-variational evolution equation was studied by Stefanelli [67] and Stefanelli and Kenmochi [38]. They proved the existence of a weak solution, that is, a solution without a strong time-derivative.

On the other hand, Kano, Kenmochi and Murase [21] proved the existence of a strong solution of (QSE), that is, a solution with a strong time-derivative. Then, Kubo and Yamazaki [47] generalized the result of Kano et al.

Other types of quasi-variational problems have been sudied by, for instance, Kadoya, Kenmochi and Niezgódka [22], Ito [20], Fukao and Kenmochi [17], Rodrigues and Santos [66], Kenmochi, Shirakawa and Yamazaki [37].

We give some typical problems that can be treated by the abstract results of Kano et al [21] and Kubo and Yamazaki [47] in the form (QSE) with a given right hand $f(t)$:

$$u'(t) + \partial\varphi(t, u; u(t)) \ni f(t). \tag{QSE f}$$

Kano-Kenmochi-Murase (2009):

$$\partial_t u - \nabla \cdot \boldsymbol{a}(\nabla u) = f \quad \text{with a constraint: } u \in K(t,u),$$

$$\varphi(t,u;z) := \begin{cases} \displaystyle\int_\Omega \hat{a}(\nabla z)dx, & \text{if } z \in K(t,u), \\ +\infty, & \text{otherwize,} \end{cases}$$

Kubo-Yamazaki (2018):

$$\partial_t u - \nabla \cdot \boldsymbol{a}(u, \nabla u) = f \quad \text{with a constraint: } u \in K(t,u),$$

$$\varphi(t,u;z) := \begin{cases} \displaystyle\int_\Omega \hat{a}(u(t), \nabla z)dx, & \text{if } z \in K(t,u), \\ +\infty, & \text{otherwize,} \end{cases}$$

where $K(t,u)$ is a closed convex set in $L^2(\Omega)$ depending on time t and the function u. A typical example of $K(t,u)$ is given as follows:

$$K(t,u) := \left\{ z \in H \;\middle|\; z \geq g(t) + \int_0^t \int_\Omega \rho(t,x;\tau,\xi;u(\tau,\xi))d\xi d\tau \right\}.$$

Idea between (QSE) and (TDSE) （revisited）

We give once again the fundamental strategy of showing the existence of a solution to (QSE) with a prescribed initial value u_0 by using the theory of (TDSE).

Step 1 Given $v : [0,T] \to H$.

Step 2 Solve by (TDSE)-theory the following Cauchy problem:

$$\begin{cases} u'(t) + \partial\varphi(t,v;u(t)) \ni f(t), & 0 < t < T, \\ u(0) = u_0. \end{cases}$$

Step 3 Find a fixed point of $v \mapsto u$ by choosing a sufficiently small time iterval $[0,T_0]$ (a local solution) by the energy inequality of (TDSE) (Section 2.3).

Step 4 Prolong the local solution to obtain a solution on the whole interval $[0, T]$.

We remark that in Step 4 above we need a subtle consideration on the domain of definition of $\varphi(t, v; \cdot)$ for v (see [47, Remarks 2.2 and 2.8]).

3.1 T-periodic solutions

When we have T-periodic data:

$$\varphi(t+T, v; \cdot) = \varphi(t, v; \cdot), \; f(t+T) = f(t), \tag{T1}$$

we are concerned with priodic solutions with period T:

$$u(t+T) = u(t). \tag{T2}$$

For (TDSE), we had a detailed analysis by Kenmochi and Ôtani [32]. While for (QSE), we have the following result.

Theorem (K.-Yamazaki [48]). *In addition to the conditions for existence result of* [47]*, assume* (T1)*, then we have a solution of* (QSE f) *with* (T2).

Outline of proof

(1) Given a w with $w(t+T) = w(t)$.

(2) Consider

$$\begin{cases} u'_{\nu,w} + \partial\varphi(t, w; u_{\nu,w}) + \nu u_{\nu,w}(t) = f(t) \\ u_{\nu,w}(t+T) = u_{\nu,w}(t). \end{cases}$$

(3) Find a fixed point of $w \mapsto u_{\nu,w}$ and take limits $\nu \to 0$.

The main idea is to derive an apriori bound:

$$\sup_{t \in \mathbf{R}} |u_{\nu,w}(t)|_H \le M_0$$

for some $M_0 > 0$.

4 Elliptic-parabolic problems

4.1 Elliptic-parabolic variational inequalities

Let $B : H \to H$ be a nonlinear monotone Lipschitz continuous operator. Kenmochi and Pawłow [34] studied an abstract problem in the following form:

$$(Bu)'(t) + \partial\varphi(t; u(t)) \ni f(t), \quad 0 < t < T$$

and applied to free boundary problems, e.g. a filtration problem without a gravitational effect, [35]. Then, Kenmochi and Kubo [31] studied a broblem of the following form for the case where the gravitational force is considered:

$$(Bu)'(t) + \partial\varphi(t, Bu(t); u(t)) \ni 0, \quad 0 < t < T.$$

This is the first approach where the energy inequality of (TDSE) is applied to more gereral problem by a fixed point analysis. Recently Kubo and Yamazaki [49] consider an abstract elliptic-parabolic quasi-variational evolution equation (EP) a special case of which has the above type of equation:

$$\varphi(t, Bu; u(t)) = \varphi(t, Bu(t); u(t)) \quad \text{(local dependence on } Bu\text{)}\,.$$

Let us explain the development of these problems below.

Elliptic-Parabolic PDE

A systematic mathematical analysis of the following ellitic-parabolic PDE which is modeled on flows in porous media in a spatial domain $\Omega \subset \mathbf{R}^N (N \geq 1)$ was initiated by Alt and Luckhaus [6]:

$$\partial_t b(u) - \nabla \cdot \boldsymbol{a}(b(u), \nabla u) = f(t, x) \quad \text{in } (0, T) \times \Omega.$$

This PDE is called of an elliptic-parabolic type since it is elliptic (with t as a parameter) in the region $\{b'(u) = 0\}$ and parabolic in $\{b'(u) > 0\}$ (notice $\partial_t b(u) = b'(u)\partial_t u$). Here $b : \mathbf{R} \to \mathbf{R}$ is a non-decreasing function and $\boldsymbol{a}(v, \boldsymbol{p}) = \partial_{\boldsymbol{p}} \hat{a}(v, \boldsymbol{p})$ is an elliptic vector with the potential $\hat{a}(v, \boldsymbol{p})$ that is convex in $\boldsymbol{p} \in \mathbf{R}^N$.

The mathematical model of saturated-unsaturated flows is given by a variational inequality as follows. Let $s := b(u)$ and u be, respectively, the saturation and the pressure of the fluid in a domain $\Omega \subset \mathbf{R}^N (N \geq 1)$ ocupied with a porous media. Suppose that the water level of the reservor changes in time t and hence the boundary of Ω is time-dependently decomposed as $\partial\Omega = \Gamma_A(t) \cup \Gamma_R(t) \cup \Gamma_I$, where $\Gamma_A(t)$, $\Gamma_R(t)$ and Γ_I are recpectively the boundary portion in touch with the air, the reservor and the impervious layer. Then, the problem is formulated by the following boundary value problem:

$$\partial_t b(u) - \nabla \cdot [\nabla u + \boldsymbol{k}(b(u))] = f(t,x) \quad \text{in } (0,T) \times \Omega,$$

$$u \le p_A, \nu \cdot [\nabla u + \boldsymbol{k}(b(u))] \le 0, (u - p_A)\nu \cdot [\nabla u + \boldsymbol{k}(b(u))] = 0 \text{ on } \Gamma_A(t),$$

$$u = p_R \quad \text{on } \Gamma_R(t), \qquad \nu \cdot [\nabla u + \boldsymbol{k}(b(u))] = 0 \quad \text{on } \Gamma_I,$$

where p_A and p_R are recpectively given pressures on the boundary portion in touch with the air and the reservor, and $f(t,x)$ is the external supply of the fluid. Finally, the initial value is imposed on the saturation:

$$b(u)\big|_{t=0} = b_0 \quad \text{in } \Omega.$$

Notice that the given vector function $\boldsymbol{k}(\cdot)$ refers to the gravitational force depending on the saturation.

For a weak solution (without a strong $L^2(\Omega)$-derivative) of the above initial boundary value problem, the existence was shown by Alt, Luckhaus and Visintin [7], and then the uniqueness by Otto [63]. For a strong solution (with a strong $L^2(\Omega)$-derivative) by the subdifferential operator approach, the existence and uniqueness was proved by Kenmochi and Pawłow [34] in the case where the gravitation is negligible $\boldsymbol{k} \equiv 0$, and by Kenmochi and Kubo [31] when the gravitation is considered $\boldsymbol{k} \not\equiv 0$. An abstract evolution equation for this problem and problems with general elliptic vector $\boldsymbol{a}$ and constraint $K(t)$ were studied by Yamazaki [73] and Kubo and Yamazaki [46], recpectively.

Variational inequalities for elliptic-parabolic systems:

$$\partial_t \boldsymbol{b}(\boldsymbol{u}) - \nabla \cdot \boldsymbol{a}(\boldsymbol{b}(\boldsymbol{u}), \nabla \boldsymbol{u}) = \boldsymbol{f}(t,x) \quad \text{in } (0,T) \times \Omega,$$

where $\boldsymbol{u}$, $\boldsymbol{f} \in \mathbf{R}^m$, $\boldsymbol{b} = \partial j : \mathbf{R}^m \to \mathbf{R}^m$, $j : \mathbf{R}^m \to \mathbf{R}$ convex $(m \ge 1)$, are modeled on, for example, oil-water flows $(m = 2)$ and were studied by Kröner and Luckhaus [41] and Alt and DiBenedetto [5] for the existence of a weak solution, and by Kubo, Shirakawa and Yamazaki [45] for a strong solution.

We notice that for the problem of a system the uniquness seems in general unknown even for the strong solution. For a classical solution, we refer to Kružkov and Sukorjanskiĭ [42].

Quasi-subdifferential operator approach for $\boldsymbol{k} \not\equiv 0$

We shall give a brief review of Kenmochi and Kubo [31] below. This idea is the origin of all our study of (QSE) and (EP) via (TDSE).

The convex function is given by (for given t and $w \in H := L^2(\Omega)$)

$$\partial\varphi(t, w; z) := \begin{cases} \displaystyle\int_\Omega \left(\frac{1}{2}|\nabla z|^2 + \boldsymbol{k}(w) \cdot \nabla z \right) dx, & z \in K(t) \\ +\infty, & z \in L^2(\Omega) \setminus K(t). \end{cases}$$

Here $K(t)$ is an appropriately defined convex set.

The strategy is given below.

(1) Given v, put $\Phi(t; z) := \varphi(t, v(t); z)$.

(2) Solve $\begin{cases} (Bu)' + \partial\Phi(t; u) \ni f \\ Bu(0) = b_0 \end{cases}$ by Kenmochi-Pawłow [34, 35].

(3) Find a fixed point of $v \mapsto Bu$ by using the energy inequality:

$(a,\ b,\ C > 0:$ structure data of $\varphi(t, v(t); u))$

$$\frac{d}{dt}\Phi_\lambda(t; u) - (\partial\Phi_\lambda(t; u), u')$$
$$\leq a(t) \left(\Phi_\lambda(t, u)\right)^{1/2} |\partial\Phi_\lambda(t; u)|_H + b(t)\Phi_\lambda(t; u) + C|v'(t)|_H \left(\Phi_\lambda(t; u)\right)^{1/2}$$

by choosing a small interval $[0, T_0]$ (a local solution).

(4) Prolong the local solution to a global one.

4.2 Elliptic-parabolic quasi-subdifferential evolution equations

Here, we present the result of Kubo and Yamazaki [49] and the outline of the proof for the sake of a completeness of the argument given so far.

The data and conditions are given below.

$V \subset H \subset V^*$ a triplet of Hilbert spaces with compact embeddings
$F : V \to V^*$ the duality map
$B : H \to H$ bounded, Lipschitz, $B = \partial j,\ j : H \to \mathbf{R}$
$\exists C > 0\ (Bu_1 - Bu_2, u_1 - u_2) \geq C|Bu_1 - Bu_2|_H^2$
$\mathcal{K} := \{v : [0,T] \to H$ right continuous$\}$
$\varphi : [0,T] \times \mathcal{K} \times H \to \mathbf{R} \cup \{+\infty\}$ proper, l.s.c., convex in $z \in H$

(Φ1) $\exists C_1 > 0\ \forall(t,v,z) :\ \varphi(t,v;z) \geq C_1|z|_V^2$

(Φ2) $v_1 = v_2$ on $[0,t] \implies \varphi(t,v_1;\cdot) = \varphi(t,v_2;\cdot)$

(Φ3) $\exists \alpha \in W^{1,2}(0,T)\ \exists \beta \in W^{1,1}(0,T)\ \exists C_2 > 0$
$\forall v \in \mathcal{K},\ 0 \leq s < t \leq T,\ z \in D(\varphi(s,v;\cdot))\ \exists \tilde{z} \in D(\varphi(t,v;\cdot))$

$$|\tilde{z} - z|_H \leq |\alpha(t) - \alpha(s)|\big(\varphi(s,v;z)\big)^{1/2},$$
$$\varphi(t,v;\tilde{z}) - \varphi(s,v;z)$$
$$\leq |\beta(t) - \beta(s)|\varphi(s,v;z) + C_2|v(t) - v(s)|_H\big(\varphi(s,v;z)\big)^{1/2}$$

(Φ4) $(t,v) \mapsto \varphi(t,v;\cdot)$ Mosco continuous

The result is stated as follows.

Theorem (K.-Yamazaki [49]).

$$(\Phi1) - (\Phi4), \quad b_0 = Bu_0, \quad \varphi(0,b_0;u_0) < +\infty$$

$$\implies \exists u \in L^\infty(0,T;H), \quad Bu \in W^{1,2}(0,T;H)$$

$$\begin{cases} (Bu)'(t) + \partial\varphi(t,Bu;u(t)) \ni 0, \quad 0 < t < T \\ Bu(0) = b_0 \end{cases}$$

The proof is outlined below.

(1) Given v, put $\Phi(t;u) := \varphi(t,v;u)$.

(2) Put $B_\varepsilon := B + \varepsilon I,\ L := F|_{D(L)},\ D(L) := \{z \in H |\ Fz \in H\}$ and solve

$$\begin{cases} (B_\varepsilon u)' + \lambda L u + \partial \Phi_\lambda(t;u) = 0, \\ B_\varepsilon u(0) = b_0 + \varepsilon u_0. \end{cases}$$

(3) Find a local solution (a fixed point of $v \mapsto Bu$) by the energy inequality:

$$\frac{d}{dt}\Phi^t_\lambda(u) - (\partial\Phi_\lambda(t;u), u')$$
$$\leq \alpha'(t)\left(\Phi_\lambda(t;u)\right)^{1/2} |\partial\Phi_\lambda(t;u)|_H + \beta'(t)\Phi_\lambda(t;u) + C_2|v'(t)|_H \left(\Phi_\lambda(t;u)\right)^{1/2}.$$

(4) Prolong the local solution $\big($cf. the definiton of $\mathcal{K}$ (right continuity)$\big)$.

Also, by applying the idea of [48] (see Theorem in §3.1) with appropriate modifications, we can show the existence of a T-periodic solution ($Bu(t+T) = Bu(t)$) under T-perodic data (T1). The idea of the proof is the following.

(1) Given a w with $w(t+T) = w(t)$

(2) Put $B_\varepsilon := B + \varepsilon I,\ L := F|_{D(L)},\ D(L) := \{z \in H |\ Fz \in H\}$
and consider

$$\begin{cases} (B_\varepsilon u_{\varepsilon,\lambda,\nu})' + \lambda L u_{\varepsilon,\lambda,\nu} + \partial\varphi_\lambda(t,w;u_{\varepsilon,\lambda,\nu}) + \nu B_\varepsilon u_{\varepsilon,\lambda,\nu}(t) = f(t) \\ B_\varepsilon u_{\varepsilon,\lambda,\nu}(t+T) = B_\varepsilon u_{\varepsilon,\lambda,\nu}(t). \end{cases}$$

(3) Find a fixed point of $w \mapsto B_\varepsilon u_{\varepsilon,\lambda,\nu}$ and take limits $\varepsilon, \lambda, \nu \to 0$.

See [50] for the detail.

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