Applications of μ-bubbles to studying GRS

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Abstract

In this proceeding, we will discuss two applicaitons of μ-bubbles to studying gradient Ricci solitons (GRS for short). One is to estimating the scalar curvature decay of nonparabolic steady GRS. Another is to estimating the diameter upper bound of a Riemannian manifold whose ∞-Bakry-Emery Ricci tensor is bounded below by a positive constant. (Although this diameter bound is not the best, it will be presented as an application of μ-bubble technique.)

1 Introduction

A triple (M, g, f) consisting of a Riemannian manifold (M, g) and a function f on M is called *steady gradient Ricci soliton* (steady GRS for short) if

$$\mathrm{Ric}_g + \mathrm{Hess}_g f = 0,$$

where Ric_g and $\mathrm{Hess}_g f$ denote respectively the Ricci tensor of g and the Hessian of f with respect to g. We will discuss two applicaitons of μ-bubble technique to studying GRS in this proceeding. One is to estimating the scalar curvature decay of nonparabolic steady GRS. First of all, we have the following fundamental properties of steady GRS.

Proposition 1.1. *Let (M, g, f) be a complete steady GRS. Then the following things are known.*

(a) Hamilton [11] proved that

$$R_g + |\nabla f|_g^2 = C_0 \tag{1}$$

for a constant C_0. Here, R_g, $|\nabla f|_g$ denote the scalar curvature of g and the square norm of the gradient of f with respect to g respectively.

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(b) *Zhang [32] proved that (for general complete ancient solution of the Ricci flow, not necessarily steady GRS,) $R_g \geq 0$. In particular, the constant C_0 in (1) is always nonnegative.*

(c) $\inf_M R_g = 0$ *([21, Theorem 1.7] and [12, Theorem 2.2], [8, 9] or [30])*

(d) *Steady Ricci solitons appear as Type II singularity models of Ricci flows with nonnegative curvature operator and positive Ricci curvature (see [10, 11] and [4, Theorem 3.4])*

The property (c) above has some similarities in mean curvature flows (see [20, 26, 31]). Beyond the property (c), we will investigate the decay of the scalar curvature by using μ-bubbles. Note that Wu [30, Corollary 1.3] has investigated the order of decay of $\inf_M R_g$. On the one hand, we will give another type of decay estimate of $\inf_M R_g$ in the following our main theorem. Recall that a complete Riemannian manifold (M, g) is *nonparabolic* if it admits a positive symmetric Green's function. All the information we need about Green's function is written in the Appendix (Section 4).

Theorem 1.1. *Let $n \geq 2$ and $0 < \alpha \leq 1$. Suppose that (M^n, g, f) is an n-dimensional complete non-compact nonparabolic steady gradient Ricci soliton. Assume that*

$$\lim_{d_g(p,x)\to\infty} G(x) \cdot d_g^{\alpha}(p, x) = 0 \tag{2}$$

where $G(\cdot)$ is the minimal positive Green's function with the pole at $p \in M$ and $d_g(p, x)$ is the distance from p to x with respect to g. Moreover we assume that there is a constant $C \geq 0$ such that

$$\mathrm{Ric}_g(x) \geq -C d_g(p, x)^{-2} \cdot G(x) \tag{3}$$

for all $x \in M$ with $d_g(p, x) >> 1$. Then there is a positive constant $C = C(n) > 0$ such that

$$\liminf_{x\to\infty} R_g(x)\, d_g^{\alpha}(x, p) \begin{cases} \leq C C_0^{1/2} & \text{if } \alpha = 1, \\ = 0 & \text{if } 0 < \alpha < 1. \end{cases}$$

Here C_0 denotes the constant in the equation (1).

In the next section, we firstly provide a proof of a weaker version of (c) in Proposition 1.1 under certain much stronger condition (Proposition 2.1) via using μ-bubbles as a template of the proof of Theorem 1.1. One of the reasons this method can be used for proofs is that the scalar curvature R_g appears in the second variation of the μ-bubble functional. Then, by carefully selecting the datum for the functional, a contradiction is derived. See Section 4 and Section 2 for more detail.

Remark 1.1.

- The assumption (3) can be replaced with

$$\mathrm{Ric}_g(x) \geq -C d_g^{-2}(p, x) \;\text{ and }\; \lim_{d_g(p,x)\to+\infty} d_g^{2-\alpha}(p, x) \cdot G(x) = +\infty.$$

  See Remark 2.2 below.

- Suppose that (M^n, g, f) is an n-dimensional complete non-compact steady gradient Ricci soliton with $\mathrm{Ric}_g \geq 0$. If there is a point $p \in M$ such that $\mathrm{Ric}_g(p) = 0$, then (M, g) is Ricci flat [7, Remark 6.58].

- Munteanu–Wang have recently proven [22, Lemma 2.4] that C can be taken to be zero in dimension three without assuming that (M, g, f) is a steady gradient Ricci soliton and the assumption (2).

2 Proofs

Firstly, we give a proof of a weaker version of the property (c) in Proposition 1.1 under an additional assumption (Proposition 2.1) as a prelude to proving Theorem 1.1. The main tools of the proofs are gradient estimate (Proposition 4.1) and the properties of μ-bubbles (Propositions 4.3, 4.4 and 4.5), which are all given in Appendix (Section 4).

Proposition 2.1. *Let $n \geq 2$. Suppose that (M^n, g, f) is an n-dimensional complete non-compact steady gradient Ricci soliton. Assume that there is a constant $C \geq 0$ and a point $p \in M$ such that*

$$\mathrm{Ric}_g(x) \geq -C d_g(p, x)^{-\alpha} \tag{4}$$

for all $x \in M$ with $d_g(p, x) >> 1$ for some positive constant $\alpha > 0$. Then

$$C_0^{-1} \inf_M R_g \leq A(n),$$

where

$$A(n) := \frac{\frac{4}{3} + \frac{(n-3)(n+1)}{4(n-1)}}{\frac{7}{3} + \frac{(n-3)(n+1)}{4(n-1)}}.$$

Remark 2.1.

- It turns out that $C_0^{-1} \inf_M R_g$ (C_0 is the constant in (1)) is a scale invariant non-negative constant. Here, "scale invariant" means that this quantity is invariant under scaling: $g \mapsto c \cdot g$, where $c > 0$ is some positive constant.
- Munteanu–Wang have recently proven a much stronger statement for dimension 3 without assuming that (M, g, f) is a steady gradient Ricci soliton [22, Theorem 3.5]. Their proof is based on their analysis of certain harmonic functions and it relies on the fact that the manifold is three dimensional. Our proof is instead relies on μ-bubbles introduced by Gromov.
- A complete non-compact Riemannian manifold is nonparabolic if and only if it has at least one nonparabolic end. (See immediately after [16, Definition 20.5].) Munteanu–Sesum proved in [21, Theorem 1.5] that any steady gradient Ricci soliton has at most one nonparabolic end.

2.1 Proof of Proposition 2.1

We give a proof of Proposition 2.1. We prove this separately for the case where $n \leq 7$ and for the general case. In the case of $n \geq 8$, we will follow the argument in [3, Section 4].

Proof of Proposition 2.1. **Step 1: smooth case ($\mathbf{n \leq 7}$)**

Fix a point $p \in M$. Let $0 < L < L'$, and ρ_0 be a smoothing of the distance function $d_g(p, \cdot)|_{\overline{B_g(p,L')} \setminus B(p,L)}$ from a fixed point $p \in M$ such that $\rho_0(x) = d(p, x)$ for all $x \in$

$\partial\left(\overline{B_g(p,L')}\setminus B(p,L)\right)$ and $|\nabla\rho_0|_g \leq 2$. From the Sard's theorem, one can assume that $\partial\left(\overline{B_g(p,L')}\setminus B(p,L)\right)$ is smooth compact $(n-1)$-dimensional submanifold of M. Let X be one of the connected components of $\overline{\{x\in M \mid \rho(x)\leq L'\}}\setminus\{x\in M \mid \rho(x)<L\}$. Set $\partial_- X := \partial\{x\in M \mid \rho(x)\leq L'\}$. Take some reference Caccioppoli set Ω_0 with $\partial_- X \Subset \Omega_0 \Subset X\setminus\partial_+ X$, and in the setting of Appendix, take $\psi=\phi\equiv 1$. Then, we obtain from (19) that

$$\begin{aligned} 0 \leq & -\int_\Sigma uR_g + CL^{-\alpha}u + \frac{4}{3}\int_\Sigma u|\nabla^X f|_g^2 - \frac{n+1}{n-1}ug(\nabla^X f,\nu)^2 \\ & - \frac{1}{n-1}\int_\Sigma u\left(h^2-(n+1)hg(\nabla^X f,\nu)\right) + \int_\Sigma u|\nabla^X h|_g. \end{aligned} \tag{5}$$

We have also used the assumption (4) here. Using Young's inequality,

$$(n+1)g(\nabla^X f,\nu)h \leq \frac{n+1}{2\varepsilon}g(\nabla^X f,\nu)^2 + \frac{\varepsilon(n+1)}{2}h^2.$$

Taking $\varepsilon = \frac{2}{n+1}\delta$ ($\delta\in(0,1)$) here, we get

$$(n+1)g(\nabla^X f,\nu)h \leq \left(\frac{2}{n+1}\right)^2\delta^{-1}g(\nabla^X f,\nu)^2 + \delta h^2,$$

and the terms containing $g(\nabla^X f,\nu)^2$ in the integrand can be estimated as

$$\begin{aligned} \left(\frac{1}{n-1}\left(\frac{n+1}{2}\right)^2\delta^{-1} - \frac{n+1}{n-1}\right)g(\nabla^X f,\nu)^2 &= \frac{(n+1-4\delta)(n+1)}{4\delta(n-1)}g(\nabla^X f,\nu)^2 \\ &\leq \frac{(n+1-4\delta)(n+1)}{4\delta(n-1)}|\nabla^X f|_g^2. \end{aligned}$$

Putting them together into the above stability inequality (5), we obtain that

$$\begin{aligned} 0 \leq & -\int_\Sigma uR_g + CL^{-\alpha}u + \left(\frac{4}{3}+\frac{(n+1-4\delta)(n+1)}{4\delta(n-1)}\right)\int_\Sigma u|\nabla^X f|_g^2 \\ & - \frac{1-\delta}{n-1}\int_\Sigma uh^2 + \int_\Sigma u|\nabla^X h|_g. \end{aligned} \tag{6}$$

If $\inf_M R(g)=0$, then the desired assertion is trivial, hence we can assume that $\inf_M R(g)>0$. In order to obtain a contradiction, we suppose that

$$-\left(\frac{7}{3}+\frac{(n+1-4\delta)(n+1)}{4\delta(n-1)}\right)\inf_M R(g) + \left(\frac{4}{3}+\frac{(n+1-4\delta)(n+1)}{4\delta(n-1)}\right)C_0 < 0. \tag{7}$$

Let L_0 be an arbitrary positive real constant. Since the diameter of (M,g) is infinite, we can take L' so that $L_0 < L'-L$ and neither of the sets $\partial_- X, \partial_+ X$ are non-empty. Let $\rho : X \to \left(-\frac{L'-L}{2},\frac{L'-L}{2}\right)$ be a smoothing of the signed distance function of the set $\{x\in M \mid d_g(\partial_- X,x)=d_g(\partial_+ X,x)\}$ with $|\mathrm{Lip}\,\rho|\leq 2$ and $\rho\equiv\pm\frac{L'-L}{2}$ on $\partial_\pm X$. Hence, in particular,

$$\rho \to \begin{cases} -\frac{L'-L}{2} & \text{at } \partial_- X, \\ \frac{L'-L}{2} & \text{at } \partial_+ X. \end{cases}$$

Take a smooth function h defined as

$$h(x) := -\frac{2\pi A}{L' - L} \tan\left(\frac{\pi}{L' - L}\rho(x)\right),$$

where A is a real positive constant, which will be taken as $A = \frac{n-1}{1-\delta}$. Then it satisfies

$$-\frac{1}{A}h^2(x) + |\nabla^X h|(x) \leq 4A\left(\frac{\pi}{L' - L}\right)^2. \tag{8}$$

Thus, from (6) and (8) with taking L and L_0 large enough, we finally obtain that

$$0 \leq \left.\frac{d^2}{dt^2}\right|_{t=0} \mathcal{A}_{u,h}(\Phi_t(\Omega)) < 0.$$

This is impossible. Therefore our supposition (7) was not correct and hence

$$-\left(\frac{7}{3} + \frac{(n+1-4\delta)(n+1)}{4\delta(n-1)}\right)\inf_M R(g) + \left(\frac{4}{3} + \frac{(n+1-4\delta)(n+1)}{4\delta(n-1)}\right) C_0 \geq 0.$$

Sorting this out, we finally obtain that

$$\frac{\inf_M R(g)}{C_0} \leq A(\delta, n) \quad \text{for all } \delta \in (0,1),$$

where

$$A(\delta, n) := \frac{\frac{4}{3} + \frac{(n+1-4\delta)(n+1)}{4\delta(n-1)}}{\frac{7}{3} + \frac{(n+1-4\delta)(n+1)}{4\delta(n-1)}} \quad (< 1).$$

And it easily turns out that

- $A(\delta, n)$ is non-increasing with respect to δ, and
- $A(1, n)$ is increasing with respect to n.

Hence $\frac{\inf_M R(g)}{C_0} \leq A(1,n) = A(n)$ and $A(\delta, n) > A(1,n) \geq A(1,2) = \frac{7}{19}$.

Step 2: general case ($\mathbf{n \geq 8}$)

We will prove the general case following the argument in [3, Section 4] (cf. [1, Appendix], [33, Proof of Theorem 2.2]). Let $\mathcal{S}$ be the singular set of Σ, then it is compact. From Proposition 4.3, $\mathcal{S}$ has Hausdorff dimension at most $n-8$. In particular, $\mathcal{H}^{n-7}(\mathcal{S}) = 0$. Hence, for any $\delta > 0$, there exist a finite collection of balls $\{B_{r_i}(x_i)\}$ such that $r_i \leq \delta$, $\mathcal{S} \subset \cup_i B_{r_i}(x_i) =: S_\delta$, and $\sum_i r_i^{n-7} \leq 1$. For each i, one can find a smooth function $\eta_i : M \to [0,1]$ such that

$$\eta_i|_{B_{r_i}}(x_i) \equiv 0, \quad \eta_i|_{M \setminus B_{2r_i}(x_i)} \equiv 1, \quad |\nabla^M \eta_i| \leq 2r_i^{-1}.$$

Setting $\tilde{\eta}(x) := \min_i\{\eta_i(x)\}$ and regularizing this, we can obtain a smooth function $\eta : M \to [0,1]$ such that

$$\eta|_{\cup_i B_{r_i(x_i)}} \equiv 0, \quad \eta|_{\cup_i B_{M \setminus B_{2r_i}(x_i)}} \equiv 1, \quad |\nabla^M \eta| \leq 2|\nabla^M \tilde{\eta}|, \quad \mathcal{H}^n - a.e.$$

Hence, we have

$$|\nabla^M \eta| \leq \sum_i 2|\nabla^M \eta_i| \leq 2\sum_i r_i^{-1}, \quad \mathcal{H}^n - a.e. \tag{9}$$

Put $S'_\delta := \cup_i B_{2r_i}(x_i)$ and let $W_\delta := (S'_\delta \setminus S_\delta) \cap \Sigma$. Since Σ is contained in the interior of X and the function h is bounded there, from the first variation formula (17), the generalized mean curvature H of Σ is bounded. Therefore, according to the monotonicity formula for varifolds ([25, Theorem 17.6]), we have, for all i,

$$\mathcal{H}^{n-1}(B_{2r_i}(x_i) \cap \Sigma) \leq C r_i^{n-1} \tag{10}$$

for some positive constant C depending only on L, L' and g. Since $W_\delta \subset S'_\delta \cap \Sigma$, from (10) we have

$$\mathcal{H}^{n-1}(W_\delta) \leq \mathcal{H}^{n-1}(S'_\delta \cap \Sigma) \leq \sum_i \mathcal{H}^{n-1}(B_{2r_i}(x_i) \cap \Sigma) \leq C\sum_i r_i^{n-1} \leq C\delta^6. \tag{11}$$

By replacing ϕ with $\eta\phi$ in Step 1 above, from (19), we have

$$\begin{aligned}
0 \leq &- \int_U uR_g + CL^{-\alpha}u + \left(\frac{4}{3} + \frac{(n+1-4\delta)(n+1)}{4\delta(n-1)}\right) \int_U u|\nabla^X f|_g^2 \\
&- \frac{1-\delta}{n-1} \int_U uh^2 + \int_U u|\nabla^X h|_g \\
&+ C_1 \mathcal{H}^{n-1}(W_\delta) + \frac{4}{3} \int_{W_\delta} \eta\phi^2 u\, g(\nabla^\Sigma \eta, \nabla^\Sigma f) + \frac{4}{3} \int_{W_\delta} u|\nabla^\Sigma \eta|_g^2.
\end{aligned}$$

Here, the constant C_1 depends only on L, L', g and f and $U := (\Sigma \setminus \mathcal{S}) \setminus S'_\delta$. Then, using (9) and (11) to estimate the last three terms in the right-hand side, we have

$$\begin{aligned}
0 \leq &- \int_U uR_g + CL^{-\alpha}u + \left(\frac{4}{3} + \frac{(n+1-4\delta)(n+1)}{4\delta(n-1)}\right) \int_U u|\nabla^X f|_g^2 \\
&- \frac{1-\delta}{n-1} \int_U uh^2 + \int_U u|\nabla^X h|_g \\
&+ C_1\delta^6 + C_2 r_i^{-1} \mathcal{H}^{n-1}(W_\delta) + C_3 r_i^{-2} \mathcal{H}^{n-1}(W_\delta) \\
\leq &- \int_U uR_g + CL^{-\alpha}u + \left(\frac{4}{3} + \frac{(n+1-4\delta)(n+1)}{4\delta(n-1)}\right) \int_U u|\nabla^X f|_g^2 \\
&- \frac{1-\delta}{n-1} \int_U uh^2 + \int_U u|\nabla^X h|_g + C_1\delta^6 + CC_2\delta^5 + CC_3\delta^4.
\end{aligned}$$

Therefore, by taking $\delta > 0$ small enough, one can argue in the same way as in Step 1 and get the same conclusion in dimensions ≥ 8. □

2.2 Proof of Theorem 1.1

Now, let's give an outline of a proof of Theorem 1.1 using the same strategy as the proof above.

Outline of a proof of Theorem 1.1. The general case can be proved in the same way as Step 2 in the proof of Proposition 2.1, so below we will prove it for the case where Σ is smooth.

In the setting of Appendix, we take $\phi \equiv 1$ and $\psi(\cdot) := G(p, \cdot)$, where G is the minimal positive Green's function with the pole at p. And, as in Appendix, we set $u := e^{\psi f}$. Take X as a component of the set $\{x \in M \mid a \leq \psi(x) \leq 2a\}$ $(0 < a \leq 1)$. By Sard's theorem and our assumption: $G(x) \to 0$ as $d_g(p,x) \to \infty$, one can take a so that X is smooth compact connected manifold with boundary. From the assumption, we can assume that $X \subset M \setminus \overline{B_g(p, L)}$ for sufficiently large $L > 0$, which is determined later. Take some reference Caccioppoli set Ω_0 with $\partial_- X \Subset \Omega_0 \Subset X \setminus \partial_+ X$. From the identity (1), $|\nabla^X f| \leq C_0^{1/2}$ and

$$|f(x)| \leq C_0^{1/2} d_g(p, x) + |f(p)|. \tag{12}$$

Using these, Proposition 4.1 and (3), then we obtain from (19) that

$$\begin{aligned} 0 \leq &\int_\Sigma u\psi d_g(p,x)^{-\alpha} \left(-R_g d_g^\alpha(p,x) + 2\psi^{-1} d_g^\alpha(p,x) \cdot g(\nabla^X \psi, \nabla^X f)\right) \\ &+ \int_\Sigma u\psi d_g(p,x)^{-\alpha} d_g^\alpha(p,x)(\text{other terms}) + B(h, |\nabla h|_g). \end{aligned}$$

Here, by taking L large enough and using (2) and (3), $d_g^\alpha(p,x)$(other terms) can be arbitrary small. Note here that we have used the assumption (3) to estimate the $\mathrm{Ric}_g(\nu,\nu)$-term. Here $B(h, |\nabla h|_g)$ is the terms containing h or $|\nabla h|$ and this is explicitly expressed as

$$B(h, |\nabla h|_g) = -\frac{1}{n-1} u h^2 + \frac{n+1}{n-1}\left(u\psi g(\nabla^X f, \nu) h + u f g(\nabla^X \psi, \nu) h\right) + u|\nabla^X h|.$$

Using Young's inequality, we can estimate this term $B(h, |\nabla h|_g)$ as

$$B(h, |\nabla h|_g) \leq \left(-A\psi^{-1}h^2 + \psi^{-1}|\nabla h|_g + C_1 \psi g(\nabla^X f, \nu)^2 + C_2 f^2 g(\nabla^X \psi, \nu)^2\right) u\psi \tag{13}$$

for some positive constants A, C_1 and C_2. The third and fourth term of the right hand side of (13) are taken to be sufficiently small by using (2), Proposition 4.1 and (12) by taking L large enough. Now take a function h defined as

$$h(x) := \frac{A^{-1}}{1 - a^{-1}\psi(x)} - \frac{A^{-1}}{1 - (2a)^{-1}\psi(x)}.$$

Then, we have

$$\begin{aligned} |\nabla^X h|_g &\leq \frac{(a^{-1} - (2a)^{-1})|\nabla^X \psi|\psi^2 + (a^{-1} - (2a)^{-1})|\nabla^X \psi|(a^{-1}\psi^2 + (2a)^{-1}\psi^2)}{A(1 - a^{-1}\psi)^2(1 - (2a)^{-1}\psi)^2} \\ &\leq \frac{(2a)^{-1}\frac{C}{L}\psi^2}{A(1 - a^{-1}\psi)^2(1 - (2a)^{-1}\psi)^2}. \end{aligned}$$

for some positive constant $C = C(n) > 0$. Here, we have used Proposition 4.1 and $0 < a \leq 1$ in the last inequality. Thus, taking L large enough so that $C/L \leq 1$, we finally obtain that

$$|\nabla^X h|_g \leq \frac{(2a)^{-1}\psi^2}{A(1 - a^{-1}\psi)^2(1 - (2a)^{-1}\psi)^2}.$$

On the other hand, since $a \le 1$,

$$-Ah^2 = -\frac{(2a)^{-2}\psi^2}{A(1-a^{-1}\psi)^2(1-(2a)^{-1}\psi)^2} \le -\frac{(2a)^{-1}\psi^2}{A(1-a^{-1}\psi)^2(1-(2a)^{-1}\psi)^2}.$$

Therefore $-A\psi^{-1}h^2 + \psi^{-1}|\nabla^X h|_g \le 0$. From Proposition 4.1 and (1), the second term can be estimated as

$$2\psi^{-1} d_g^\alpha(p,x) \cdot g(\nabla^X\psi, \nabla^X f) \begin{cases} \le CC_0^{1/2} & \text{if } \alpha = 1, \\ \le CL^{\alpha-1} & \text{if } 0 < \alpha < 1 \end{cases}$$

for some positive constant $0 < C = C(n)$. Therefore, if it was true that

$$\liminf_{x\to\infty} R_g d_g^\alpha(p,x) > \begin{cases} CC_0^{1/2} & \text{if } \alpha = 1, \\ 0 & \text{if } 0 < \alpha < 1 \end{cases}$$

respectively, then we could obtain in both cases that

$$0 \le \left.\frac{d^2}{dt^2}\right|_{t=0} \mathcal{A}_{u,h}(\Phi_t(\Omega)) < 0.$$

This is impossible. □

Remark 2.2. From the proof above, it can be noticed that the assumption (3) can be replaced with

$$\mathrm{Ric}_g(x) \ge -Cd_g^{-2}(p,x) \quad \text{and} \quad \lim_{d_g(p,x)\to+\infty} d_g^{2-\alpha}(p,x)\cdot G(x) = +\infty.$$

3 Myers-type theorem

In [3], Bray–Gui–Liu–Zhang found a new proof of Bishop's volume comparison theorem. They used "singular soap bubble", and the key idea was investigating the second derivative of the isoperimetric profile function. Here, we will give a proof of another theorem involving Ricci lower bound: Myers theorem, using warped μ-bubbles.

Proposition 3.1. *Let $n \ge 2$ and (M^n, g, f) be a tuple that consists of an n-dimensional complete Riemannian manifold (M, g) and a function f on M satisfying*

$$\mathrm{Ric}_g + \mathrm{Hess}_g f \ge \frac{\lambda}{2} g \tag{14}$$

for some positive constant λ. Assume that $0 < \sup_M |\nabla f|_g < \infty$. Then M is compact and

$$\mathrm{diam}(M,g) \le \pi\sqrt{\frac{2(n-1)}{\lambda B}} \cdot \frac{B(1+B)^{1/4}}{\left((2+B)\sqrt{1+B} - 2(B+1)\right)^{1/2}}, \tag{15}$$

where

$$B := \frac{(n-1)\lambda}{2\sup_M |\nabla f|_g^2}.$$

Remark 3.1. The right hand side of the estimate diverges to $+\infty$ as $B \to 0$ and converges to $\pi\sqrt{\frac{2(n-1)}{\lambda}}$ as $B \to +\infty$ respectively. In particular, the case of $B = +\infty$ (which corresponds to the case of $f \equiv \text{const}$) recovers Myers' theorem [23].

Proof. The general case can be proved in the same way as Step 2 in the proof of Proposition 2.1, so below we will prove it for the case where Σ is smooth.

Firstly, note that the 2nd variation of $\mathcal{A}$ can also be written as the following alternative form (see [6, Proof of Theorem 4.3]):

$$\begin{aligned} 0 \leq \frac{d^2}{dt^2}\mathcal{A}(\Omega_t)\Big|_{t=0} &= \int_\Sigma \psi^2 \mathrm{Hess}_g u(\nu,\nu) - \psi^2 g(\nabla^\Sigma u, \nabla^\Sigma \psi) \\ &+ \int_\Sigma \psi^2 g(\nabla^M u, \nu) H \\ &- \int_\Sigma u\left(\psi\Delta^\Sigma\psi + (|A^\Sigma|^2 + \mathrm{Ric}(\nu,\nu))\psi^2\right) \\ &- \int_\Sigma \psi^2 g(\nabla^M u,\nu)h + \psi^2 u g(\nabla^M h,\nu). \end{aligned}$$

We will prove the proposition by contradiction. Suppose that

$$\mathrm{diam}(M,g) > (1+\tilde{\delta})\sqrt{\frac{2(n-1)(1+a)^2\pi^2}{\lambda\varepsilon_\delta(1-\delta)}} =: L$$

for fixed arbitrary constants $\tilde{\delta}, a > 0$ Here, we set

$$(0,1) \ni \varepsilon_\delta := \left(1 + \frac{(n-1)\lambda\delta}{2\sup_M |\nabla f|_g^2}\right)^{-1} \frac{(n-1)\lambda\delta}{2\sup_M |\nabla f|_g^2} \quad (\delta \in (0,1)).$$

Then there exist two points $p, q \in M$ such that $d_g(p,q) = L$. Consider the set $M_{p,q} := \{x \in M \mid d_g(x,p) = d_g(x,q)\}$, and take $X \subset M$ as one of the components of the set $\{x \in M \mid d_g(M_{p,q}, x) \leq L/2\}$. (More precisely, in order to construct such an X, we need to take a smoothing of the distance function and perturb the boundaries by using Sard's theorem (see the proof of Proposition 2.1). But, since we can perform such smoothing with arbitrary small error, we can still discuss without loss of generality.) By Proposition 4.3, we can find a warped μ-bubble Ω of the functional $\mathcal{A}_{u,h}$ with the choice $u = e^{-f}$ and $\psi = \text{const}$ and some reference Caccioppoli set Ω_0. Set $\Sigma := \partial\Omega \setminus \partial_- X$. Then, by Proposition 4.5, (18) and our assumption (14), we obtain that

$$\begin{aligned} 0 &\leq \int_\Sigma u\left[\psi^2|\nabla^M h| - \frac{\psi^2}{n-1}|A^\Sigma|^2 - \frac{\lambda}{2}\psi^2\right] \\ &\leq \int_\Sigma u\left[\psi^2|\nabla^M h| - \frac{\psi^2}{n-1}\left(h^2 + 2g(\nabla^M f,\nu)h + g(\nabla^M f,\nu)^2\right) - \frac{\lambda}{2}\psi^2\right]. \end{aligned}$$

Using Young's inequality, we obtain that

$$2g(\nabla^M f,\nu)h \leq \frac{1}{1-\varepsilon}g(\nabla^M f,\nu)^2 + (1-\varepsilon)h^2$$

for all $\varepsilon \in (0,1)$. So,

$$0 \leq \int_\Sigma u \left[\psi^2 \left(|\nabla^M h| - \frac{\varepsilon}{n-1}h^2\right) + \frac{\psi^2}{n-1} \cdot \frac{\varepsilon}{1-\varepsilon} g(\nabla^M f, \nu)^2 - \frac{\lambda}{2}\psi^2\right].$$

Thus we take h as

$$h(x) := -\frac{(n-1)(1+a)\pi}{\varepsilon L} \tan\left(\frac{\pi}{L}\rho(x)\right),$$

where $\rho(x)$ is a smoothing of the signed distance function from $M_{p,q}$ with $|\mathrm{Lip}\,\rho| \leq 1+a$. Then it holds that

$$|\nabla^M h| - \frac{\varepsilon}{n-1}h^2 \leq \frac{(n-1)(1+a)^2\pi^2}{\varepsilon L^2}.$$

Then, we get

$$0 \leq \int_\Sigma u\psi^2 \left[\frac{(n-1)(1+a)^2\pi^2}{\varepsilon L^2} + \frac{1}{n-1} \cdot \frac{\varepsilon}{1-\varepsilon} \sup_M |\nabla^M f|^2 - \frac{\lambda}{2}\right].$$

From the definition of ε_δ, we have

$$\frac{1}{n-1} \cdot \frac{\varepsilon_\delta}{1-\varepsilon_\delta} \sup_M |\nabla^M f|^2 = \frac{\lambda}{2}\delta.$$

Then it satisfies that

$$\frac{1}{n-1} \cdot \frac{\varepsilon_\delta}{1-\varepsilon_\delta} \sup_M |\nabla^M f|^2 - \frac{\lambda}{2} = -\frac{\lambda}{2}(1-\delta).$$

Summing up with this and the definition of L, we get a contradiction:

$$0 \leq \int_\Sigma u\psi^2 \left[\frac{(n-1)(1+a)^2\pi^2}{\varepsilon_\delta} \cdot \frac{1}{L^2} + \frac{1}{n-1} \cdot \frac{\varepsilon_\delta}{1-\varepsilon_\delta} \sup_M |\nabla^M f|^2 - \frac{\lambda}{2}\right] < 0.$$

Hence we obtain that

$$\mathrm{diam}(M,g) \leq (1+\tilde{\delta})\sqrt{\frac{2(n-1)(1+a)^2\pi^2}{\lambda\varepsilon_\delta(1-\delta)}}$$

for all $\tilde{\delta} > 0$ and $a > 0$ by contradiction. Therefore we finally reach the conclusion that

$$\mathrm{diam}(M,g) \leq \sqrt{\frac{2(n-1)\pi^2}{\lambda\varepsilon_\delta(1-\delta)}} \quad \text{for all } \delta \in (0,1).$$

In particular, M is compact.

The right hand side of the previous estimate can be written as

$$\pi\sqrt{\frac{2(n-1)}{\lambda B}}\sqrt{\frac{1+B\delta}{\delta(1-\delta)}} =: \pi\sqrt{\frac{2(n-1)}{\lambda B}} \cdot F(\delta).$$

An easy calculation implies that $F(\delta)$ attains its minimum

$$\frac{B(1+B)^{1/4}}{\left((2+B)\sqrt{1+B} - 2(B+1)\right)^{1/2}}$$

at $\delta = -B^{-1} + B^{-1}\sqrt{1+B}$ ($\in (0,1)$). Hence we obtain the desired estimate. $\square$

Remark 3.2. This type of estimate is also given by [19, 27, 29]. Among these, [29, Theorem 1.3] is the sharpest estimate of this form. We note here that our estimate (15) is not better than that of [29, Theorem 1.3]. Moreover Limoncu [19] and Tadano [27] have proven this type of estimate under more general setting (i.e., not only for the case of (M^n, g) is a steady **gradient** Ricci soliton but also for the case of (M^n, g, X) is a tuple that consists of a Riemannian n-manifold (M^n, g) and a vector field X on M satisfying $\mathrm{Ric}_g + \frac{1}{2}\mathcal{L}_X g \geq \frac{\lambda}{2} g$ for some positive constant λ with $\sup_M |X|_g < \infty$).

Remark 3.3. In [1], Antonelli–Xu gave Bishop–Gromov volume comparison and Myers-type theorem under a spectral assumption. It may be worth asking whether the method used in [1] ("unequally warped μ-bubbles") can be used to prove Myers-type theorem (and a kind of Cheng's maximal diameter theorem) when (14) is satisfied in a spectral sense(, which is weaker than (14)).

4 Appendix

4.1 Nonparabolicity and steady GRS

Example 4.1. From the recent result of Bamler–Chan–Ma–Zhang [2, Theorem 1.1] and the criterion (16) below, if (M^n, g, f) is a complete steady GRS with $n \geq 4$, $\mathrm{Ric}_g \geq 0$ and the corresponding Ricci flow $(M, g_t)_{t \in \mathbb{R}}$ has a uniformly bounded Nash entropy (see the condition (1.4) in [2]), which is either

- $(M, g_t)_{t \in \mathbb{R}}$ arises as a singularity model, or
- (M, g) has bounded curvature,

then (M, g) is nonparabolic.

Example 4.2. The Cigar soliton $(\mathrm{Cigar}, g_{\mathrm{Cigar}})$ is the unique two dimensional complete steady GRS with positive Gaussian curvature. Moreover, since the Cigar soliton has linear volume growth, it is parabolic (i.e., it is not nonparabolic).

Example 4.3. $(\mathrm{Cigar} \times \mathbb{R}, g_{\mathrm{Cigar}} + g_{eucl})$ is a three dimensional steady GRS with nonnegative curvature. Since the Cigar soliton has linear volume growth, it turns out that $\mathrm{Cigar} \times \mathbb{R}$ has quadratic volume growth (by the coarea formula). Hence it is parabolic.

Example 4.4. The n-dimensional Bryant solitons on $\mathbb{R}^n$ $(n \geq 3)$ are rotationally symmetric and positive sectional curvature. The volume of geodesic balls $B_r(o)$ grow on the order $r^{\frac{n+1}{2}}$, and the curvature decay is of $O(r^{-1})$. In particular, the n-dimensional Bryant soliton satisfies $\lim_{d(p,x) \to +\infty} R(x) = 0$, and it is parabolic if $n = 3$ and nonparabolic if $n \geq 4$.

Example 4.5. Lai [13] recently found a family of n-dimensional $(n \geq 3)$ steady GRS which is $\mathbb{Z}_2 \times O(n-1)$-symmetric but not rotationally symmetric with positive curvature operator. Moreover, she [14] has also proven that there exists a $\mathbb{Z}_2 \times O(2)$-symmetric three dimensional flying wing (M_θ, g_θ) which is asymptotic to a sector with angle θ for all $\theta \in (0, \pi)$. (Note that $\theta = 0, \pi$ respectively corresponds to the Bryant soliton and $\mathrm{Cigar} \times \mathbb{R}$.) To the authors' knowledge, it is not known whether these GRSs are parabolic or not. From the general result (c) of Proposition 1.1, we know that $\inf_{M_\theta} R_g = 0$ holds in this case as well. However, in contrast to the Bryant soliton $(\theta = 0)$ case, the limit of the scalar curvature along a geodesic Γ fixed by the $O(2)$-symmetry is positive. Furthermore, there is a quantitative relation between this limit along Γ and the asymptotic cone angle θ (see [15, Theorem 1.6]).

4.2 Green's functions on a Riemannian manifold with Ricci lower bound

Let (M^n, g) be a smooth complete non-compact Riemannian manifold. Recall that it is called *nonparabolic* if it admits a positive symmetric Green's function. It is well known that in this case the minimal positive Green's function $G(x, y)$ may be obtained as the limit of the Dirichlet Green's function of a sequence of compact exhaustive domains of the manifold. Then,

$$\Delta_x G(x, y) = -\delta_x(y), \quad G(x, y) = G(y, x) > 0.$$

The first key tool used to prove our main theorems is the following gradient estimate for positive harmonic functions by Li–Wang [17] (see also [16]).

Proposition 4.1 ([17, Lemma 2.1] or [16, Theorem 6.1])**.** *Let (M^n, g) be a complete Riemannian manifold. Suppose that h is a positive harmonic function defined on the geodesic ball $B_g(p, 2R) \subset M$ of radius $2R$ centered at p and $B_g(p, 2R) \cap \partial M = \emptyset$, and*

$$\mathrm{Ric}_g \geq -(n-1)\rho^2$$

for some constant $\rho \in \mathbb{R}$. Then there is a positive constant $C = C(n) > 0$ such that

$$\frac{|\nabla h|_g^2(x)}{h(x)^2} \leq C(1 + \varepsilon^{-1})R^{-2} + \frac{(4(n-1)^2 + 2\varepsilon)\rho^2}{4 - 2\varepsilon}$$

for all $x \in B_g(p, R)$ and for any $\varepsilon < 2$.

By [28, Corollary of Theorem 2], a complete non-compact Riemannian manifold (M^n, g) with $\mathrm{Ric}_g \geq 0$ is nonparabolic if and only if

$$\int_1^\infty \frac{t}{\mathrm{Vol}_g(B_g(p, t))}\, dt < +\infty, \tag{16}$$

where $\mathrm{Vol}_g(B_g(p, t))$ is the volume of geodesic ball $B_g(p, t)$ of radius t centered at p with respect to g. The second key tool is the following estimate of the minimal positive Green's function $G(x, y)$ by Li–Yau [18, Theorem 5.2].

Proposition 4.2 ([18, Theorem 5.2])**.** *Suppose that (M^n, g) is a complete nonparabolic Riemannian manifold with $\mathrm{Ric}_g \geq 0$. Then the minimal positive Green's function $G(x, y)$ satisfies that there is a positive constant $C = C(n) > 0$ such that*

$$C^{-1} \int_{d_g(x,y)}^\infty \frac{t}{\mathrm{Vol}_g(B_g(x, t))}\, dt \leq G(x, y) \leq C \int_{d_g(x,y)}^\infty \frac{t}{\mathrm{Vol}_g(B_g(x, t))}\, dt$$

for all $x \neq y$. In particular, $G(x, y) \to 0$ as $d_g(x, y) \to \infty$.

4.3 Warped μ-bubbles

Let (X, g) be an oriented connected Riemannian manifold together with a decomposition $\partial X = \partial_- X \sqcup \partial_+ X$, where $\partial_\pm X$ are (non-empty) unions of boundary components. Fix a smooth function $u > 0$ on X and a smooth function h on the interior $\mathring{X}$. Choose a Caccioppoli set Ω_0 with smooth boundary, which contains an open neighborhood of $\partial_- X$ and is disjoint from $\partial_+ X$. Consider the following functional

$$\mathcal{A}_{u,h}(\Omega) = \int_{\partial^*\Omega} u \, d\mathcal{H}^{n-1} - \int_X (\chi_\Omega - \chi_{\Omega_0}) hu \, d\mathcal{H}^n$$

for all Caccioppoli sets Ω with $\Omega \Delta \Omega_0 \Subset \mathring{X}$. Here, $\mathcal{H}^k$ denotes the k-Hausdorff measure with respect to the distance d_g induced from g. $\mathcal{C}(X)$ denotes the set of all Caccioppoli sets Ω such that $\Omega \Delta \Omega_0 \Subset \mathring{X}$. If $\Omega \in \mathcal{C}(X)$, Ω contains an open neighborhood of $\partial_- X$ and is disjoint from $\partial_+ X$. A Caccioppoli set minimizing $\mathcal{A}_{u,h}$ in this class $\mathcal{C}(X)$ is called a *warped μ-bubble*. The existence and regularity of a minimizer of $\mathcal{A}_{u,h}$ was given in [5, Proposition 12] (see also [34, Proposition 2.1]) and [33, Theorem 2.2].

Proposition 4.3 ([5, Proposition 12] and [33, Theorem 2.2]). *Let $n \geq 2$. Suppose that $h(x) \to \pm\infty$ as $x \to \partial_\mp X$. Then there is a warped μ-bubble Ω such that $\Omega \Delta \Omega_0 \Subset \mathring{X}$. Moreover, the regular part $\partial^{\mathrm{reg}}\Omega$ of $\partial\Omega$ is a smooth hypersurface and the singular part $\partial^{\mathrm{sing}}\Omega$ of $\partial\Omega$ has Hausdorff dimension at most $n-8$.*

The first and second variation formula are given in [5, Lemma 13] and [5, Lemma 14] respectively (see also [24, 4.1, 4.3]). We use the the second variation formula in the form of [6, Theorem 4.3].

Proposition 4.4 (First variation [24, Lemma 4.10]). *Suppose $\Omega \in \mathcal{C}(X)$ is a smooth and let Σ be a connected component of $\partial\Omega \setminus \partial_- X$. For any smooth function ϕ on Σ let V_ϕ be a vector field on X, which vanishes outside a small neighborhood of Σ and agree with $\phi\nu$ on Σ. Here, ν is the outwards pointing unit normal of Σ. Let Φ_t be the flow generated by V_ϕ with $\Phi_0 = \mathrm{id}$. Then*

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{A}_{u,h}(\Phi_t(\Omega)) = \int_\Sigma (Hu + g(\nabla^X u, \nu) - hu)\phi \, d\mathcal{H}^{n-1}.$$

Here, $\nabla^X u \in \mathfrak{X}(X)$ denotes the gradient vector field of u with respect to g and H is the scalar mean curvature of Σ. In particular, a smooth μ-bubble Ω satisfies

$$H = -u^{-1} g(\nabla^X u, \nu) + h \tag{17}$$

along $\partial\Omega$.

Proposition 4.5 (Second variation [6, In the proof of Theorem 4.3]). *Suppose $\Omega \in \mathcal{C}(X)$ is a smooth μ-bubble and let Σ be a connected component of $\partial\Omega \setminus \partial_- X$. For any smooth function $\phi \in C_0^\infty(\Sigma)$ on Σ let V_ϕ be a vector field on X, which vanishes outside a small neighborhood of Σ and agree with $\phi\nu$ on Σ. Here, ν is the outwards pointing unit normal* *of Σ. Let Φ_t be the flow generated by V_ϕ with $\Phi_0 = \mathrm{id}$. Then*

$$\frac{d^2}{dt^2}\bigg|_{t=0} \mathcal{A}_{u,h}(\Phi_t(\Omega)) = \int_\Sigma \phi^2(\Delta^X u - \Delta^\Sigma u) - 2\phi^2 u^{-1} g(\nabla^X u, \nu)^2 \, d\mathcal{H}^{n-1} + \int_\Sigma u(|\nabla^\Sigma \phi|_g^2 - (|A_\Sigma|^2 + \mathrm{Ric}_g(\nu,\nu))\phi^2) \, d\mathcal{H}^{n-1} + \int_\Sigma \phi^2 g(\nabla^X u, \nu)h - \phi^2 u g(\nabla^X h, \nu) \, d\mathcal{H}^{n-1}. \tag{18}$$

Here, $\Delta^X u, \nabla^\Sigma \phi$ and A_Σ denote respectively the Laplacian of u with respect to g, the gradient of ϕ with respect to the induced metric $g|_\Sigma$ and the second fundamental form of Σ with respect to g.

In our proofs, we consider the smooth connected compact Riemannian manifold $(X, g|_X)$ for some $X \subset M$, and warped μ-bubbles on it. Using Proposition 4.3 we can find a warped μ-bubble Ω minimizing

$$\mathcal{A}_{u,h}(\Omega) = \int_{\partial^*\Omega} u \, d\mathcal{H}^{n-1} - \int_X (\chi_\Omega - \chi_{\Omega_0}) hu \, d\mathcal{H}^n$$

for all Caccioppoli sets Ω with $\Omega \Delta \Omega_0 \Subset \mathring{X}$ for some reference Caccioppoli set Ω_0. And set Σ be a component of $\partial\Omega$ contained in $\mathring{X}$. Then, from Proposition 4.3, Σ is compact and smooth. Moreover, since Ω is a minimizer of $\mathcal{A}_{u,h}$, its second derivative at Ω (i.e., the right hand side of (18)) is nonnegative.

Take $u := e^{\psi f}$ where ψ is a smooth function on X and f is the potential function of the Ricci soliton. Then one can easily compute as follows.

$$\begin{aligned}
\nabla^X u &= u\nabla^X(\psi f) = u\psi \nabla^X f + uf\nabla^X \psi, \\
\Delta^X u &= u\psi\Delta^X f + 2ug(\nabla^X\psi, \nabla^X f) + uf\Delta^X\psi + u\psi^2|\nabla^X f|_g^2 \\
&\quad + uf^2|\nabla^X\psi|_g^2 + 2u\psi f g(\nabla^X\psi, \nabla^X f) \\
&= -u\psi R_g + 2ug(\nabla^X\psi, \nabla^X f) + uf\Delta^X\psi + u\psi^2|\nabla^X f|_g^2 \\
&\quad + uf^2|\nabla^X\psi|_g^2 + 2u\psi f g(\nabla^X\psi, \nabla^X f).
\end{aligned}$$

We have used the traced soliton equation: $R_g + \Delta^X f = 0$ in the last equality. Moreover, from the first variation formula (17), we have

$$|A_\Sigma|^2 \geq \frac{1}{n-1} H^2 = \frac{1}{n-1}|h - u^{-1}g(\nabla^X u, \nu)|^2.$$

Putting these together into (18) and [6, p. 13], we obtain that

$$
\begin{aligned}
0 \leq & \int_\Sigma -\phi^2 u\psi R_g + \phi^2 u\psi^2 |\nabla^X f|_g^2 + \phi^2 u f^2 |\nabla^X \psi|_g^2 + 2\phi^2 u\psi f g(\nabla^X \psi, \nabla^X f) \\
& + \int_\Sigma -2\phi^2 u\psi^2 g(\nabla^X f, \nu)^2 - 2\phi^2 u f^2 g(\nabla^X \psi, \nu)^2 \\
& + \int_\Sigma -4\phi^2 u\psi f g(\nabla^X f, \nu) \cdot g(\nabla^X \psi, \nu) + \int_\Sigma \phi^2 u f \Delta^X \psi + 2\phi^2 u g(\nabla^X \psi, \nabla^X f) \\
& - \int_\Sigma \frac{\phi^2 u}{n-1} |h - \psi g(\nabla^X f, \nu) - f g(\nabla^X \psi, \nu)|^2 - \mathrm{Ric}_g(\nu, \nu)\phi^2 u \\
& + \int_\Sigma \phi^2 u\psi g(\nabla^X f, \nu) h + \phi^2 u f g(\nabla^X \psi, \nu) h - \int_\Sigma \phi^2 u g(\nabla^X h, \nu) \\
& + \frac{1}{3} \int_\Sigma \phi^2 u\psi^2 |\nabla^\Sigma f|_g^2 + \phi^2 u f^2 |\nabla^\Sigma \psi|_g^2 + \frac{2}{3} \int_\Sigma \phi^2 u\psi f g(\nabla^\Sigma \psi, \nabla^\Sigma f) \\
& + \frac{4}{3} \int_\Sigma \phi u\psi g(\nabla^\Sigma f, \nabla^\Sigma \phi) + \phi u f g(\nabla^\Sigma \psi, \nabla^\Sigma \phi) + \frac{4}{3} \int_\Sigma u |\nabla^\Sigma \phi|_g^2.
\end{aligned}
\tag{19}
$$

We mainly use this inequality to prove our claims.

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