Isometries between function spaces consisting of metrics

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1 Introduction

This article is a résumé of the paper [11] with some additional questions. Throughout the article, let $\mathrm{C}(Z)$ be the space of continuous bounded real-valued functions on a topological space Z with the sup-norm $\|\cdot\|$, which induces the sup-metric on $\mathrm{C}(Z)$. The study on isometries[1] between function spaces plays an important role in functional analysis, and traces its history back to the Banach-Stone theorem [1, 13].

Theorem 1.1. *Suppose that X and Y are compact Hausdorff spaces. Then X and Y are homeomorphic if and only if $\mathrm{C}(X)$ and $\mathrm{C}(Y)$ are isometric. Moreover, for every isometry $T : \mathrm{C}(X) \to \mathrm{C}(Y)$, there are a homeomorphism $\phi : Y \to X$ and a map $\alpha : Y \to \{-1, 1\}$ such that*

$$T(f)(y) = \alpha(y)f(\phi(y))$$

for all $f \in \mathrm{C}(X)$ and $y \in Y$.

Its developments have been obtained until now (see [3] as a historical note). Recently, D. Hirota, I. Matsuzaki and T. Miura [6] (cf. [14]) showed a Banach-Stone type theorem on positive cones $\mathrm{C}_0^+(Z) \subset \mathrm{C}(Z)$ vanishing at ∞ as follows:

Theorem 1.2. *Let X and Y be locally compact Hausdorff spaces. Then X and Y are homeomorphic if and only if $\mathrm{C}_0^+(X)$ and $\mathrm{C}_0^+(Y)$ are isometric. Furthermore, for any isometry $T : \mathrm{C}_0^+(X) \to \mathrm{C}_0^+(Y)$, there exists a homeomorphism $\phi : Y \to X$ such that*

$$T(f)(y) = f(\phi(y))$$

for every $f \in \mathrm{C}_0^+(X)$ and every $y \in Y$.

It is natural to ask on what function spaces the similar results hold. In this paper, we shall establish certain Banach-Stone type theorem on spaces of metrics. From now on, spaces are assumed to be metrizable. Let $\mathrm{PM}(Z) \subset \mathrm{C}(Z^2)$ be the space consisting of pseudometrics

[1] In this article, an isometry means a surjective isometry.

on a space Z, and let $\mathrm{AM}(Z) \subset \mathrm{PM}(Z)$ be the subspace consisting of admissible metrics, that is, metrics inducing the topology on Z. M.E. Shanks [12] proved a Banach-Stone type theorem on spaces of metrics as follows:

Theorem 1.3. *If X and Y are compact spaces, then the following are equivalent:*

(1) *X and Y are homeomorphic;*

(2) *$\mathrm{PM}(X)$ and $\mathrm{PM}(Y)$ are isometric;*

(3) *$\mathrm{AM}(X)$ and $\mathrm{AM}(Y)$ are isometric.*

The method of Shanks was different from those of Banach and Stone, and it was not shown that the isometries are composition operators by homeomorphisms as $(*)$ in Main Theorem. Set

$$\mathrm{Pc}(Z) = \{d \in \mathrm{AM}(Z) \mid \text{ there exists a non-empty compact subset } K \subset Z \text{ such that } \|d|_{K^2}\| = \|d\| \text{ and if } d(x,y) = \|d\|, \text{ then } x, y \in K\}$$

and

$$\mathrm{Pp}(Z) = \{d \in \mathrm{AM}(Z) \mid \text{ there uniquely exists } \{z, w\} \subset Z \text{ such that } d(z,w) = \|d\|\}.$$

Remark that $\mathrm{Pp}(Z) \subset \mathrm{Pc}(Z)$ and that $\mathrm{AM}(Z) = \mathrm{Pc}(Z)$ if Z is compact. Let $\mathrm{M}(Z)$ be $\mathrm{Pc}(Z)$ or $\mathrm{Pp}(Z)$. We will generalize Shanks' result and describe isometries by using homeomorphisms as follows:

Main Theorem. *Let X and Y be spaces. The following are equivalent:*

(1) *X and Y are homeomorphic;*

(2) *there exists an isometry $T : \mathrm{PM}(X) \to \mathrm{PM}(Y)$ with $T(\mathrm{M}(X)) = \mathrm{M}(Y)$;*

(3) *there exists an isometry $T : \mathrm{AM}(X) \to \mathrm{AM}(Y)$ with $T(\mathrm{M}(X)) = \mathrm{M}(Y)$;*

(4) *there exists an isometry $T : \mathrm{M}(X) \to \mathrm{M}(Y)$.*

In this case, for each isometry $T : \mathrm{PM}(X) \to \mathrm{PM}(Y)$ with $T(\mathrm{M}(X)) = \mathrm{M}(Y)$, there exists a homeomorphism $\phi : Y \to X$ such that for any $d \in \mathrm{PM}(X)$ and for any $x, y \in Y$,

$$T(d)(x,y) = d(\phi(x), \phi(y)). \qquad (*)$$

Except for the case that X or Y is a doubleton, the homeomorphism ϕ can be chosen uniquely.

2 The peaking function argument

As is easily observed, if a map $\phi : Y \to X$ from a space Y to a space X is a homeomorphism, then the operator $T : \mathrm{PM}(X) \to \mathrm{PM}(Y)$ defined by $(*)$ in Main Theorem is an isometry such that $T(\mathrm{AM}(X)) = \mathrm{AM}(Y)$ and $T(\mathrm{M}(X)) = \mathrm{M}(Y)$. This means that the implications (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3) and (1) $\Rightarrow$ (4) hold in Main Theorem. Recall that for a space Z, $\mathrm{C}(Z^2)$ is a Banach space and $\mathrm{PM}(Z)$ is a closed subset of $\mathrm{C}(Z^2)$, and hence the sup-metric induced by $\|\cdot\|$ is also complete on $\mathrm{PM}(Z)$ (cf. [10]). The subspace $\mathrm{AM}(Z)$ is dense in $\mathrm{PM}(Z)$ (refer to [9, Proposition 5]). Furthermore, we have the following:

Proposition 2.1. *For every space Z, the subset $\mathrm{Pp}(Z)$ is dense in $\mathrm{PM}(Z)$.*

By the completeness of the sup-metric on $\mathrm{PM}(Z)$ and the density of $\mathrm{Pp}(Z) \subset \mathrm{PM}(Z)$, each isometry in (3) and (4) of Main Theorem can be extended to the one in (2), which implies that (3) $\Rightarrow$ (2) and (4) $\Rightarrow$ (2) are valid. It remains to prove (2) $\Rightarrow$ (1) and the description $(*)$ in Main Theorem. We shall adopt the peaking function argument, which is based on [6] and Stone's method. In the rest of this article, assume that X and Y are non-degenerate spaces and $T : \mathrm{PM}(X) \to \mathrm{PM}(Y)$ is an isometry.

Proposition 2.2. *The isometry T is norm-preserving, that is, $\|T(d)\| = \|d\|$ for all $d \in \mathrm{PM}(X)$ and $\|T^{-1}(\rho)\| = \|\rho\|$ for all $\rho \in \mathrm{PM}(Y)$.*

On addition of metrics, we have the following (refer to [8, Lemma 2.1]).

Lemma 2.3. *Let Z be a space. For each $d \in \mathrm{PM}(Z)$ and each $\rho \in \mathrm{AM}(Z)$, the sum $d + \rho \in \mathrm{AM}(Z)$.*

Urysohn's lemma plays a key role in proving the Banach-Stone theorem. F. Hausdorff [4] showed the metric extension theorem, which states that for every space Z and its closed set $A \subset Z$, any $d \in \mathrm{AM}(A)$ can be extended over Z. We may obtain the pseudometric version of it as follows (see [7] for example).

Theorem 2.4. *Suppose that Z is a space and A is a closed set in Z. For each $d \in \mathrm{PM}(A)$, there exists $\tilde{d} \in \mathrm{PM}(Z)$ such that $\tilde{d}|_{A^2} = d$ and $\|\tilde{d}\| = \|d\|$.*

Given a space Z, we denote by $\mathrm{Fin}_2(Z)$ the hyperspace consisting of non-empty subspaces in Z of cardinality less than or equal to 2, that is equipped with the Vietoris topology, and let $\mathrm{D}(Z) \subset \mathrm{Fin}_2(Z)$ be the subspace consisting of doubletons. For each $\{x, y\} \in \mathrm{Fin}_2(Z)$, put

$$\mathcal{P}(Z, \{x, y\}) = \{d \in \mathrm{PM}(Z) \mid d(x, y) = \|d\|\},$$

and for each $d \in \mathrm{PM}(Z)$, put

$$\mathcal{F}(Z, d) = \{\{x, y\} \in \mathrm{Fin}_2(Z) \mid d(x, y) = \|d\|\}.$$

For a pseudometric $d \in \mathrm{PM}(Z)$, a point $z \in Z$, and a positive number $r > 0$, let

$$B_d(z, r) = \{w \in Z \mid d(z, w) < r\} \text{ and } \overline{B}_d(z, r) = \{w \in Z \mid d(z, w) \leq r\}.$$

The following is a key lemma in our strategy.

Lemma 2.5. *Let Z be a non-degenerate space. For each $d \in \mathrm{PM}(Z)$ and each $\{x, y\} \in \mathrm{D}(Z)$, there exists $\rho \in \mathcal{P}(Z, \{x, y\})$ such that*

$$d + \rho \in \mathcal{P}(Z, \{x, y\}) \cap \mathrm{M}(Z).$$

Sketch of Proof. We can easily construct an admissible metric $e \in \mathcal{P}(Z, \{x, y\}) \cap \mathrm{AM}(Z)$ with $\|e\| = 1$ by virtue of Hausdorff's metric extension theorem. Note that $d' = d + e \in \mathrm{AM}(Z)$ according to Lemma 2.3, and let $a = d'(x, y) > 0$ and

$$b = \min \left\{ \sup_{z \in Z} d'(x, z), \sup_{z \in Z} d'(y, z) \right\}.$$

Using Theorem 2.4, we can take $\rho_n \in \mathrm{PM}(Z)$ for each positive integer n so that

(i) $\rho_n(z, w) = 4b$ if $z \in \overline{B}_{d'}(x, a/2^{n+1})$ and $w \in \overline{B}_{d'}(y, a/2^{n+1})$;

(ii) $\rho_n(z, w) = 2b$ if $z \in Z \setminus (B_{d'}(x, a/2^n) \cup B_{d'}(y, a/2^n))$ and $w \in \overline{B}_{d'}(x, a/2^{n+1}) \cup \overline{B}_{d'}(y, a/2^{n+1})$;

(iii) $\rho_n(z, w) = 0$ if $z, w \in Z \setminus (B_{d'}(x, a/2^n) \cup B_{d'}(y, a/2^n))$, if $z, w \in \overline{B}_{d'}(x, a/2^{n+1})$, or if $z, w \in \overline{B}_{d'}(y, a/2^{n+1})$;

(iv) $\rho_n(z, w) \leq 4b$ if otherwise.

Then $\rho = e + \sum_{n=1}^{\infty} \rho_n / 2^n$ is the desired metric. □

Henceforth suppose that $T(\mathrm{M}(X)) \subset \mathrm{M}(Y)$. We can prove the following:

Lemma 2.6. *For every $\{x, y\} \in \mathrm{Fin}_2(X)$ and every $d_i \in \mathcal{P}(X, \{x, y\}) \cap \mathrm{M}(X)$, $1 \leq i \leq n$, $\bigcap_{i=1}^{n} \mathcal{F}(Y, T(d_i)) \neq \emptyset$.*

Applying the finite intersection property in compact spaces (cf. [2, Theorem 3.1.1]), we can obtain the following:

Lemma 2.7. *For each $\{x, y\} \in \mathrm{D}(X)$,*

$$\bigcap_{d \in \mathcal{P}(X, \{x, y\}) \cap \mathrm{M}(X)} \mathcal{F}(Y, T(d)) \neq \emptyset.$$

Sketch of Proof. We shall show the case where $\mathrm{M}(X) = \mathrm{Pc}(X)$ and $\mathrm{M}(Y) = \mathrm{Pc}(Y)$. Fix any $d_0 \in \mathcal{P}(X, \{x, y\}) \cap \mathrm{Pc}(X) \neq \emptyset$ due to Lemma 2.5, so there are $\{z_0, w_0\} \subset Y$ and a compact set $L \subset Y$ such that $T(d_0)(z_0, w_0) = \|T(d_0)\|$ and for all $z, w \in Y$ with $T(d_0)(z, w) = \|T(d_0)\|$, $z, w \in L$. Then $\mathcal{F}(Y, T(d_0)) \subset \mathrm{Fin}_2(L)$ is compact, and according to Lemma 2.6,

$$\{\mathcal{F}(Y, T(d)) \cap \mathcal{F}(Y, T(d_0)) \mid d \in \mathcal{P}(X, \{x, y\})\}$$

has the finite intersection property. Therefore

$$\bigcap_{d \in \mathcal{P}(X, \{x, y\}) \cap \mathrm{Pc}(X)} \mathcal{F}(Y, T(d)) = \bigcap_{d \in \mathcal{P}(X, \{x, y\}) \cap \mathrm{Pc}(X)} (\mathcal{F}(Y, T(d)) \cap \mathcal{F}(Y, T(d_0))) \neq \emptyset.$$

□

Moreover, assume that $T(\mathrm{M}(X)) = \mathrm{M}(Y)$. By virtue of Lemma 2.7, a bijection $\Phi : \mathrm{D}(Y) \to \mathrm{D}(X)$ can be given by

$$\{\Phi(\{z,w\})\} = \bigcap_{\rho \in \mathcal{P}(Y,\{z,w\}) \cap \mathrm{M}(Y)} \mathcal{F}(X, T^{-1}(\rho))$$

for each $\{z,w\} \in \mathrm{D}(Y)$.

Proposition 2.8. *The bijection Φ satisfies that*

$$T(\mathcal{P}(X,\{x,y\}) \cap \mathrm{M}(X)) = \mathcal{P}(Y, \Phi^{-1}(\{x,y\})) \cap \mathrm{M}(Y)$$

for any $\{x,y\} \in \mathrm{D}(X)$ and that

$$T^{-1}(\mathcal{P}(Y,\{z,w\}) \cap \mathrm{M}(Y)) = \mathcal{P}(X, \Phi(\{z,w\})) \cap \mathrm{M}(X)$$

for any $\{z,w\} \in \mathrm{D}(Y)$.

The map Φ induces the following equality.

Proposition 2.9. *The equality $T(d)(x,y) = d(z,w)$ holds for all $d \in \mathrm{PM}(X)$ and all $\{x,y\} \in \mathrm{D}(Y)$, where $\Phi(\{x,y\}) = \{z,w\}$.*

If the cardinality of Y is greater than 2, we can define a bijection $\phi : Y \to X$ by

$$\{\phi(y)\} = \bigcap_{z \in Y \setminus \{y\}} \Phi(\{y,z\})$$

for each $y \in Y$. The map ϕ is compatible with Φ.

Proposition 2.10. *The following equalities hold: $\Phi(\{x,y\}) = \{\phi(x), \phi(y)\}$ for all $\{x,y\} \in \mathrm{D}(Y)$ and $\Phi^{-1}(\{z,w\}) = \{\phi^{-1}(z), \phi^{-1}(w)\}$ for all $\{z,w\} \in \mathrm{D}(X)$.*

The bijection ϕ is the desired homeomorphism and satisfies $(*)$ in Main Theorem by Propositions 2.9 and 2.10.

3 Problems

Many Banach-Stone type theorems are known. For example, isometries on function spaces consisting of uniformly continuous functions and lipschitz functions have been researched (refer to [5, 15]). We can expect that similar results on subspaces of metrics hold. Let $Z = (Z, d_Z)$ be a metric space, where d_Z is an admissible metric on Z. It is said that $d \in \mathrm{AM}(Z)$ is uniformly continuous with respect to d_X if for every $\epsilon > 0$, there is $\delta > 0$ such that for any $x, y \in Z$ with $d_Z(x,y) < \delta$, $d(x,y) < \epsilon$. A metric $d \in \mathrm{AM}(Z)$ is called to be lipschitz provided that there exists $k > 0$ such that $d(x,y) \leq k d_Z(x,y)$ for all $x, y \in Z$.

Problem. Establish Banach-Stone type theorems on spaces consisting of uniformly continuous metrics and lipschitz metrics.

4 Acknowledgements

This article was partially supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

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